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INEQUALITIES FOR POSITIVE LINEAR MAPS ON HERMITIAN MATRICES

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Abstract. The aim of this work is to generalize the main inequalities in [9] as follows: Let A be a Hermitian matrix, let Φ be a normalized positive linear map, let f and g be real valued continuous functions and let $F(u,v)$ be a real valued function matrix non-decreasing in its first variable. Real constants α and β such that

$$\alpha I \leqslant F\left[\Phi\left(f\left(A\right)\right), g\left(\Phi(A)\right)\right] \leqslant \beta I$$

are determined. If f is a concave (resp. convex) function then the determination of β (resp. α) is reduced to solving a single variable maximization (resp. minimization) problem. Some applications of these results to the power function, the means and the Hadamard product are also given.

1. Introduction

Let A be a Hermitian matrix with spectrum in $[m,M]$, let $f(t)$ be a real valued continuous function on $[m,M]$ and let Φ be a normalized positive linear map. In [9], Li and Mathias determined real constants α_i and β_i $(i=1,2)$ such that

$$\alpha_1 \Phi\left(f\left(A\right)\right) \leqslant f\left(\Phi(A)\right) \leqslant \beta_1 \Phi\left(f\left(A\right)\right), \tag{1.1}$$

$$\alpha_2 I \leqslant f\left(\Phi(A)\right) - \Phi\left(f\left(A\right)\right) \leqslant \beta_2 I. \tag{1.2}$$

Also, Mond and Pečarić [11, 13] showed the following theorem for operator convex functions, which is an extension of the converses of Jensen's inequality: Let A_i be positive operators on a Hilbert space H satisfying $mI \leqslant A_i \leqslant MI$ $(i=1,2,\ldots,n)$, where $0<m<M$, ϕ_i $(i=1,2,\ldots,n)$ normalized positive linear maps and ω_i $(i=1,2,\ldots,n)$ positive numbers such that $\sum_{i=1}^n \omega_i = 1$. Let f be a operator convex function on $[m,M]$. If $F(u,v)$ is a real valued function operator monotone in its first variable, then

$$F\left[\sum_{i=1}^n \omega_i \phi_i\left(f\left(A_i\right)\right), f\left(\sum_{i=1}^n \omega_i \phi_i\left(A_i\right)\right)\right] \leqslant \left\{\max_{m\leqslant t\leqslant M} F\left[f(m) + \frac{f(M)-f(m)}{M-m}(t-m), f(t)\right]\right\} I. \tag{1.3}$$

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As a result of it, if we put $F(u,v) = v^{-\frac{1}{2}}uv^{-\frac{1}{2}}$ or $F(u,v) = u-v$, then the upper bounds in the ratio and difference inequalities were given for the power function $f(t) = t^p$. Similarly, Mond and Pečarić [12, 14] gave the upper bounds in the ratio and difference inequalities of means: If $r < s$ and either $s \notin (-1,1)$ or $r \notin (-1,1)$, then

$$M_n^{[s]}(A;\phi,w) \leqslant \tilde{\Delta} M_n^{[r]}(A;\phi,w) \tag{1.4}$$

and

$$M_n^{[s]}(A;\phi,w) - M_n^{[r]}(A;\phi,w) \leqslant \Delta I, \tag{1.5}$$

where

$$M_n^{[r]}(A;\phi,w) = \left(\frac{\sum_{i=1}^n \omega_i \phi_i \left(A_i^r\right)}{\sum_{i=1}^n \omega_i} \right)^{1/r} \quad (r \neq 0,\ r \in \mathbb{R}),$$

and the upper bounds $\tilde{\Delta}$ and Δ are explicitly determined. Moreover, in [15] they gave some other extensions of Ando's results [1] to self-adjoint operators.

In [16, 17] Mond and Pečarić considered the following mean and estimated the difference and ratio inequalities analogous to (1.4) and (1.5):

$$M_k^{[r]}(A;U) = \left(\sum_{i=1}^k U_i A_i^r U_i^* \right)^{1/r} \quad (r \neq 0),$$

where $A_i (i = 1,\ldots,k)$ are positive definite Hermitian matrices of order n, with eigenvalues contained in the interval $[m,M]$ ($0 < m < M$) and U_i are $t \times n$ matrices such that $\sum_{i=1}^k U_i U_i^* = I$.

On the other hand, Mond and Pečarić showed a general inequality for positive operators including real valued convex functions. Furthermore, Furuta [7, 8] gave extensions of inequalities due to Ky Fan, Mond and Pečarić which are associated with Hölder-McCarthy and Kantorovich type inequalities. Inspired by Furuta's ideas, Mićić, Seo, Takahasi and Tominaga in [19] generalized a theorem by Mond and Pečarić on the converses of Jensen's inequality: Let A_i be positive operators on a Hilbert space H satisfying $mI \leqslant A_i \leqslant MI$ ($i = 1,2,\cdots n$), where $0 < m < M$. Let $f(t)$ be a real valued continuous convex function on $[m,M]$ and let $x_1, x_2, \cdots, x_n$ be any finite number of vectors in H such that $\sum_{i=1}^n \|x_i\|^2 = 1$. If $g(t)$ is a real valued continuous function and $F(u,v)$ is a real valued function non-decreasing in u, then

$$F\left[\sum_{i=1}^n \left(f\left(A_i\right)x_i, x_i\right), g\left(\sum_{i=1}^n \left(A_i x_i, x_i\right) \right) \right] \leqslant \left\{ \max_{m \leqslant t \leqslant M} F\left[f(m) + \frac{f(M)-f(m)}{M-m}(t-m), g(t) \right] \right\} I. \tag{1.6}$$

In this paper, based on ideas due to Mond-Pečarić and Furuta, we shall generalize a theorem of Li-Mathias, that is, we shall determine real constants α and β such that the following inequality holds

$$\alpha I \leqslant F\left[\Phi\left(f(A)\right), g\left(\Phi(A)\right)\right] \leqslant \beta I. \tag{1.7}$$

Similarly to [9], the problem of bounds determination in (1.7) is reduced to a problem of approximation of a function f by a matrix convex or a matrix concave function (Section 3). In the case f is a concave (resp. a convex) function this problem is reduced to solving a single variable maximization (resp. minimization) problem (Section 4). In these cases the inequality is sharp and then a non-trivial positive linear map attaining the equality is given. As applications we shall show general difference and ratio sharp inequalities, and inequalities related to the power functions, the means of operators and the Hadamard product.

2. Preliminaries

We use H_n to denote the space of $n \times n$ Hermitian matrices and use $\geqslant$ to denote the positive semi-definite partial order, so that $A \geqslant B$ means that $A - B$ is positive semi-definite, i.e. $X^*(A-B)X \geqslant 0$ for all n-vectors X.

If $A \in H_n$, then there exists a unitary matrix U such that

$$A = U^* \Lambda U,$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the λ_i are the eigenvalues of A. Assume now that $f(\lambda_i) \in C, i \in \{1, 2, \ldots, n\}$ is well defined. Then $f(A)$ may be defined by

$$f(A) = U^* \operatorname{diag}(f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n))U.$$

$f \mid H_n$ is called a matrix function.

A matrix function f is called matrix monotone on an interval $[m,M]$ if for $n = 1, 2, \ldots$ and for all $A, B \in H_n$ with spectra in $[m,M]$

$$A \geqslant B \Rightarrow f(A) \geqslant f(B)$$

and it is called matrix convex on $[m,M]$ if for $n = 1, 2, \ldots$ and for all $A, B \in H_n$ with spectra in $[m,M]$

$$f((1-t)A + tB)) \leqslant (1-t)f(A) + tf(B) \qquad 0 \leqslant t \leqslant 1. \tag{2.1}$$

We say that f is matrix concave if the reverse inequality in (2.1) holds for $n = 1, 2, \ldots$ See [2] or [10, Part V] for more information about matrix monotone and matrix convex (concave) functions.

A linear map Φ from H_n to $H_{\tilde{n}}$ is said to be positive if it transforms H_n^+ to $H_{\tilde{n}}^+$, where H_n^+ is the open cone of positive definite matrices. It follows that a positive linear map Φ is monotone in the sense that $A \geqslant B$ implies $\Phi(A) \geqslant \Phi(B)$. A positive linear map is said to be normalized if it maps the identity matrix I_n to the identity matrix I_m. See [9] for some common examples of normalized positive linear maps.

If $A \in H_n$ has a spectrum in $[m,M]$ then so does $\Phi(A)$ for any normalized positive linear map Φ and so matrix function $f(\Phi(A))$ is also defined.

For the sake of convenience, we prepare some notations. We denote

$$\mu_f = \frac{f(M) - f(m)}{M - m}, \quad \nu_f = \frac{Mf(m) - mf(M)}{M - m}$$

for a real valued continuous function f on the interval $[m,M]$ and particularly

$$\mu_{t^p} = \frac{M^p - m^p}{M-m}, \quad \nu_{t^p} = \frac{Mm^p - mM^p}{M-m}$$

for the power function $f(t) = t^p$. Also, we introduce the following constant by Furuta (see [8] and [20]):

$$C_f(m,M;q) = \frac{\nu_f}{1-q}\left(\frac{1-q}{q}\frac{\mu_f}{\nu_f}\right)^q,$$

where q is a real number such that $q > 1$ or $q < 0$.

We denote $\{conx.\}$ (resp. $\{conc.\}$) the set of continuous matrix convex (resp. concave) functions defined on $[m,M]$.

We shall use the following Jensen's inequality (cf. [1, Theorem 4], [3, 4], [9, Theorem 2.1]):

JENSEN'S INEQUALITY. *Let f be a matrix concave function on $[m,M]$ and let $A \in H_n$ with spectrum in $[m,M]$. If Φ is a normalized positive linear map, then*

$$f(\Phi(A)) \geqslant \Phi(f(A)).$$

3. A general theorem

We shall first generalize a theorem of Li-Mathias, which is based on the ideas due to Mond-Pečarić and Furuta.

THEOREM 3.1. *Let A be a Hermitian matrix with spectrum contained in $[m,M]$. Let Φ be a normalized positive linear map from H_n to $H_{\tilde{n}}$. Let f and g be real valued continuous functions on $[m,M]$. Let $F(u,v)$ be a real valued function defined on $U \times V$, matrix non-decreasing in u, where U and V are intervals such that $U \supset f[m,M]$ and $V \supset g[m,M]$. Then the following inequalities hold*

$$\left\{ \max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \min_{m\leqslant t\leqslant M} F[k(t),g(t)] \right\} I \leqslant F[\Phi(f(A)), g(\Phi(A))]$$
$$\leqslant \left\{ \min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{m\leqslant t\leqslant M} F[k(t),g(t)] \right\} I. \tag{3.1}$$

Proof. We prove the right-hand side of the inequality (3.1). Let k be a matrix concave function on $[m,M]$ such that $f(t) \leqslant k(t)$ for all $t \in [m,M]$. It follows from the spectral theorem that $f(A) \leqslant k(A)$. Using the positivity of Φ we have $\Phi(f(A)) \leqslant$

$\Phi(k(A))$. Furthermore, by Jensen's inequality, we have $\Phi(k(A)) \leqslant k(\Phi(A))$ and it follows that $\Phi(f(A)) \leqslant k(\Phi(A))$. Using the matrix non-decreasing character of $F(\cdot, v)$, we have

$$\begin{aligned} F[\Phi(f(A)), g(\Phi(A))] &\leqslant F[k(\Phi(A)), g(\Phi(A))] \\ &\leqslant \left\{ \max_{\mathrm{t}\in\sigma(\Phi(\mathrm{A}))} F[k(t), g(t)] \right\} I \\ &\leqslant \left\{ \max_{\mathrm{m}\leqslant \mathrm{t}\leqslant \mathrm{M}} F[k(t), g(t)] \right\} I. \end{aligned}$$

Therefore, we minimize this bound over all matrix concave functions k to obtain the upper bound in the inequality (3.1).

As a complementary result, we cite the following theorem:

THEOREM 3.2. *Under the same hypothesis as in Theorem 3.1, except that F is matrix non-increasing in its first variable, the following inequalities hold*

$$\begin{aligned} \left\{ \max_{\substack{k\in\{conc.\}\\ k\geqslant f}} \min_{\mathrm{m}\leqslant \mathrm{t}\leqslant \mathrm{M}} F[k(t), g(t)] \right\} I &\leqslant F[\Phi(f(A)), g(\Phi(A))] \\ &\leqslant \left\{ \min_{\substack{k\in\{conx.\}\\ k\leqslant f}} \max_{\mathrm{m}\leqslant \mathrm{t}\leqslant \mathrm{M}} F[k(t), g(t)] \right\} I. \end{aligned} \tag{3.2}$$

Proof. We prove this theorem by replacing F by $-F$ in Theorem 3.1.

If we put $g = f$ in Theorem 3.1, then we have the following corollary:

COROLLARY 3.3. *Let A, Φ, f and F be as in Theorem 3.1, the following inequalities hold*

$$\begin{aligned} \left\{ \max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \min_{\mathrm{m}\leqslant \mathrm{t}\leqslant \mathrm{M}} F[k(t), f(t)] \right\} I &\leqslant F[\Phi(f(A)), f(\Phi(A))] \\ &\leqslant \left\{ \min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{\mathrm{m}\leqslant \mathrm{t}\leqslant \mathrm{M}} F[k(t), f(t)] \right\} I. \end{aligned} \tag{3.3}$$

REMARK 3.4. Notice that the constant function $k(t) = \max_{m\leqslant s\leqslant M} f(s)$ for all $t \in [m, M]$ is a matrix concave function that bounds the function f from above. Since we are optimizing over the right-hand side of (3.1) matrix concave functions, we can show that there are indeed a function k that attains the extreme.

THEOREM 3.5. *Let the hypothesis of Theorem 3.1 be satisfied. If f is a real valued continuous convex function on $[m,M]$ then the following inequality holds*

$$F\left[\Phi\left(f\left(A\right)\right),g\left(\Phi(A)\right)\right]\leqslant\left\{\max_{\mathrm{m}\leqslant\mathrm{t}\leqslant\mathrm{M}}F\left[\mu_f\cdot t+\nu_f,g(t)\right]\right\}I. \tag{3.4}$$

If f is a concave function on $[m,M]$ then the following inequality holds

$$F\left[\Phi\left(f\left(A\right)\right),g\left(\Phi(A)\right)\right]\geqslant\left\{\min_{\mathrm{m}\leqslant\mathrm{t}\leqslant\mathrm{M}}F\left[\mu_f\cdot t+\nu_f,g(t)\right]\right\}I. \tag{3.5}$$

Proof. We prove only inequality (3.4). If we put $h(t)=\mu_f\cdot t+\nu_f$, then h is matrix concave function. The convexity of f ensures that $f(t)\leqslant h(t)$ for all $t\in[m,M]$. If k is a matrix concave function and $f(t)\leqslant k(t)$ for all $t\in[m,M]$ then $h(m)=f(m)\leqslant k(m)$ and $h(M)=f(M)\leqslant k(M)$. Since a matrix concave function is necessarily concave, we have $h(t)\leqslant k(t)$ for all $t\in[m,M]$. Using the matrix non-decreasing character of $F(\cdot,v)$, we have

$$F\left[h(t),g(t)\right]\leqslant F\left[k(t),g(t)\right]\quad\text{for all } t\in[m,M].$$

It follows that the minimum in the right-hand side of (3.1) is attained at h. Thus we proved the inequality (3.4).

REMARK 3.6. The bounds in Theorem 3.1 are rather hard to be evaluated in general. One may consider only linear function k instead of all matrix concave or matrix convex functions. This simplifies the evaluation of the bounds at the cost of the possibility to weaken them. Obviously, if $f=g$ is a concave (resp. convex) function on $[m,M]$ then the graph of the linear function k which satisfies $k\geqslant f$ (resp. $k\leqslant f$) and

$$\min_{\substack{k\in\{conc.\}\\ k\geqslant f}}\ \max_{m\leqslant t\leqslant M}F\left[k(t),f(t)\right]\leqslant\max_{m\leqslant t\leqslant M}F\left[k_r(t),f(t)\right]$$

$$\left(\text{resp. }\min_{m\leqslant t\leqslant M}F\left[k_r(t),f(t)\right]\leqslant\max_{\substack{k\in\{conx.\}\\ k\leqslant f}}\ \min_{m\leqslant t\leqslant M}F\left[k(t),f(t)\right]\right)$$

is tangent to the graph of $y=f(t)$ passing through a point $(r,f(r))$ with $m\leqslant r\leqslant M$. For such a function

$$k_r(t)=f(r)+f'(r)(t-r),$$

the value $\max_{m\leqslant t\leqslant M}F\left[k_r(t),f(t)\right]$ (resp. $\min_{m\leqslant t\leqslant M}F\left[k_r(t),f(t)\right]$) occurs at $t=m$ or M. It follows that the optimal solution of this maximization (resp. minimization) problem occurs at the function k_r such that

$$F\left[k_r(m),f(m)\right]=F\left[k_r(M),f(M)\right]. \tag{3.6}$$

We do not know for sure that the result is optimal in the following sense: The left-hand side or the right-hand side of the inequality (3.1) is sharp in the sense that for any real valued continuous functions f and g and for any matrix A with spectrum in $[m,M]$ there is a non-trivial normalized positive linear map Φ for which the bound is attained.

4. Bounds for converses of Jensen's inequality

As application of Theorem 3.1, we discuss an extension of [9, Lemma 2.2], which give us a unified view to bounds in a theorem by Li-Mathias. Moreover, we shall consider the optimality of our results by using a technique of [9].

THEOREM 4.1. *Let A be a Hermitian matrix with spectrum contained in $[m,M]$. Let Φ be a normalized positive linear map from H_n to $H_{\tilde{n}}$. Let f and g be real valued continuous functions on $[m,M]$. Then for a given real number α*

$$\alpha g(\Phi(A)) + \beta_1 I \leqslant \Phi(f(A)) \leqslant \alpha g(\Phi(A)) + \beta_2 I \tag{4.1}$$

holds for

$$\beta_1 = \max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \min_{m\leqslant t\leqslant M} \{k(t) - \alpha g(t)\},$$

$$\beta_2 = \min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{m\leqslant t\leqslant M} \{k(t) - \alpha g(t)\}.$$

Proof. Let us put $F(u,v) = u - \alpha v$ in Theorem 3.1. Then it follows from the right-hand side of (3.1) that

$$\Phi(f(A)) - \alpha g(\Phi(A)) \leqslant \left[\min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{m\leqslant t\leqslant M} F(k(t), g(t))\right] I$$

$$= \left[\min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{m\leqslant t\leqslant M} \{k(t) - \alpha g(t)\}\right] I.$$

The left-hand side of (4.1) is proved in the same way.

We have Corollary 4.2 if we put $\alpha = 1$ in Theorem 4.1 and Corollary 4.4 if we choose α such that $\beta = 0$ in Theorem 4.1. We frequently use them in the case that the function k is explicitly defined.

COROLLARY 4.2. *Let the hypothesis of Theorem 4.1 be satisfied. Then*

$$\left[\max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \min_{m\leqslant t\leqslant M} \{k(t) - g(t)\}\right] I \leqslant \Phi(f(A)) - g(\Phi(A))$$

$$\leqslant \left[\min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{m\leqslant t\leqslant M} \{k(t) - g(t)\}\right] I. \tag{4.2}$$

Furthermore, if g is a strictly convex differentiable function on $[m,M]$, then for every matrix strictly concave differentiable function k in the right-hand side of (4.2) we have $\max_{\mathrm{m}\leqslant\mathrm{t}\leqslant\mathrm{M}} \{k(t) - g(t)\} = k(t_0) - g(t_0)$ where $t_0 \in [m,M]$ may be determined as follows:

- *if* $k'(m) > g'(m)$ *and* $k'(M) < g'(M)$, *then let* t_0 *be the unique solution in* (m,M) *of the equation* $k'(t) = g'(t)$,
- *if* $k'(m) \leqslant g'(m)$, *then let* $t_0 = m$,
- *if* $k'(M) \geqslant g'(M)$, *then let* $t_0 = M$.

Similarly, if g *is a strictly concave differentiable function on* $[m,M]$, *then for any matrix strictly convex differentiable function* k *in the left-hand side of* (4.2) *we have* $\min_{m\leqslant t\leqslant M}\{k(t)-g(t)\} = k(t_0)-g(t_0)$ *where* $t_0 \in [m,M]$ *may be determined as follows:*

- *if* $k'(m) < g'(m)$ *and* $k'(M) > g'(M)$, *then let* t_0 *be the unique solution in* (m,M) *of the equation* $k'(t) = g'(t)$
- *if* $k'(m) \geqslant g'(m)$, *then let* $t_0 = m$,
- *if* $k'(M) \leqslant g'(M)$, *then let* $t_0 = M$.

Proof. If we put $\alpha = 1$ in Theorem 4.1, then we have (4.2).

Let g be a strictly convex and k (matrix) strictly concave both differentiable functions. We denote $h(t) = k(t) - g(t)$. Then $h'(t)$ is strictly decreasing on $[m,M]$. If $h'(m) > 0$ and $h'(M) < 0$, then the equation $h'(t) = 0$ has exactly one solution $t_0 \in (m,M)$ and the maximum value on $[m,M]$ of the function h is attained for $t = t_0$. If $h'(m) \leqslant 0$, then we have $h' \leqslant 0$ on [m,M]. Thus h is a decreasing function on $[m,M]$ and the maximum value on $[m,M]$ of this function is attained for $t = m$. Similarly, if $h'(M) \geqslant 0$ then $h' \geqslant 0$ on [m,M], i.e. h is increasing on $[m,M]$ and the maximum value on $[m,M]$ of the function h is attained for $t = M$.

The case when g is a strictly concave and k (matrix) strictly convex both differentiable functions is proved in the same way.

REMARK 4.3. We obtain the inequality (1.2) (i.e. [9, Lemma 2.2, ineq. (2.1)]) if we put $g = f$ in (4.2).

COROLLARY 4.4. *Let the hypothesis of Theorem 4.1 be satisfied. Suppose in addition that either of the following conditions holds* (i) $g(t) > 0$ *for all* $t \in [m,M]$ *or* (ii) $g(t) < 0$ *for all* $t \in [m,M]$. *Then the following inequality*

$$\left[\max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \min_{m\leqslant t\leqslant M}\left\{\frac{k(t)}{g(t)}\right\}\right] g\left(\Phi(A)\right) \leqslant \Phi\left(f\left(A\right)\right) \leqslant \left[\min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{m\leqslant t\leqslant M}\left\{\frac{k(t)}{g(t)}\right\}\right] g\left(\Phi(A)\right) \tag{4.3}$$

holds in case (i), *or*

$$\left[\max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \max_{m\leqslant t\leqslant M}\left\{\frac{k(t)}{g(t)}\right\}\right] g\left(\Phi(A)\right) \leqslant \Phi\left(f\left(A\right)\right) \leqslant \left[\min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \min_{m\leqslant t\leqslant M}\left\{\frac{k(t)}{g(t)}\right\}\right] g\left(\Phi(A)\right) \tag{4.4}$$

holds in case (ii).

Furthermore, if g *is a strictly convex twice differentiable function on* $[m,M]$ *and if* $f(t)/g(t) > 0$ *for all* $t \in [m,M]$, *then for any matrix strictly concave twice differentiable function* k *in the right-hand side of* (4.3) *(resp.* (4.4)*) we have*

$$\max_{m\leqslant t\leqslant M}\{k(t)/g(t)\} = k(t_0)/g(t_0) \qquad \left(\text{resp. } \min_{m\leqslant t\leqslant M}\{k(t)/g(t)\} = k(t_0)/g(t_0) \right)$$

where $t_0 \in [m,M]$ *may be determined as follows:*

- *if* $k'(m) > k(m)\frac{g'(m)}{g(m)}$ *and* $k'(M) < k(M)\frac{g'(M)}{g(M)}$, *then let* t_0 *be the unique solution in* (m,M) *of the equation* $k'(t)g(t) = k(t)g'(t)$
- *if* $k'(m) \leqslant k(m)\frac{g'(m)}{g(m)}$, *then let* $t_0 = m$
- *if* $k'(M) \geqslant k(M)\frac{g'(M)}{g(M)}$, *then let* $t_0 = M$.

Similarly, if g *is a strictly concave twice differentiable function on* $[m,M]$ *and if* $f(t)/g(t) < 0$ *for all* $t \in [m,M]$, *then for any matrix strictly convex twice differentiable function* k *in the left-hand side of* (4.3) *(resp.* (4.4)*) we have*

$$\min_{m\leqslant t\leqslant M}\{k(t)/g(t)\} = k(t_0)/g(t_0) \qquad \left(\text{resp. } \max_{m\leqslant t\leqslant M}\{k(t)/g(t)\} = k(t_0)/g(t_0) \right)$$

where $t_0 \in [m,M]$ *may be determined as follows:*

- *if* $k'(m) < k(m)\frac{g'(m)}{g(m)}$ *and* $k'(M) > k(M)\frac{g'(M)}{g(M)}$, *then let* t_0 *be the unique solution in* (m,M) *of the equation* $k'(t)g(t) = k(t)g'(t)$
- *if* $k'(m) \geqslant k(m)\frac{g'(m)}{g(m)}$, *then let* $t_0 = m$
- *if* $k'(M) \leqslant k(M)\frac{g'(M)}{g(M)}$, *then let* $t_0 = M$.

Proof. The inequality (4.3) for case (i) and (4.4) for case (ii) follows from Theorem 4.1 if we choose α_i such that $\beta_i = 0$ ($i = 1,2$).

Further we only prove it for the case of (i). Suppose that g is strictly convex and k is matrix strictly concave. Put $h(t) = k(t)/g(t)$. Now $h'(t) = H(t)/g^2(t)$, where $H(t) = k'(t)g(t) - k(t)g'(t)$. Since $k'' < 0$, $k \geqslant f$, $g'' > 0$, $f/g > 0$ on $[m,M]$, we have $H'(t) = k''(t)g(t) - k(t)g''(t) < 0$, so that H is strictly decreasing on $[m,M]$. If $H(m) > 0$ and $H(M) < 0$, the equation $H(t) \equiv k'(t)g(t) - k(t)g'(t) = 0$ has exactly one solution $t_0 \in (m,M)$. Hence, the maximum value on $[m,M]$ of the function h is attained for $t = t_0$. If $H(m) \leqslant 0$ then we have $H \leqslant 0$ on [m,M] since H is a strictly decreasing function on [m,M], so that h is strictly decreasing on $[m,M]$. Hence, the maximum value on $[m,M]$ of the function h is attained for $t = m$. Similarly, if $H(M) \geqslant 0$ then $H \geqslant 0$ on [m,M], i.e. h is strictly increasing on $[m,M]$ and the maximum value on $[m,M]$ of the function h is attained for $t = M$.

REMARK 4.5. We obtain the inequality (1.1) (i.e. [9, Lemma 2.2, ineq. (2.2)]) if we put $g = f$ in (4.3).

Next we observe sharp inequalities in the sense that was described in Remark 3.6. For these results we need the following lemma as an extension of [9, Lemma 2.3].

LEMMA 4.6. *Let A be a Hermitian matrix with spectrum contained in $[m,M]$ and let $\lambda_{min}(A)=m$, $\lambda_{max}(A)=M$. Let f,g be real valued continuous functions on $[m,M]$. Then for any $t^*\in[m,M]$ there is a real valued normalized positive linear map Φ such that*

$$F\left[\Phi\left(f\left(A\right)\right),g\left(\Phi(A)\right)\right]=F\left[\mu_f\cdot t^*+\nu_f,g(t^*)\right].$$

Proof. Let U be a unitary matrix such that

$$U^*AU=diag(\lambda_1,\lambda_2,\ldots,\lambda_n),$$

where $\lambda_1=m$ and $\lambda_2=M$. For $t^*\in[m,M]$, we denote $\theta=(M-t^*)/(M-m)$. We define map $\Phi:H_n\to\mathbb{C}$ by

$$\Phi(X)=\left(\sqrt{\theta}e_1+\sqrt{1-\theta}e_2\right)^*X\left(\sqrt{\theta}e_1+\sqrt{1-\theta}e_2\right),$$

where e_1 and e_2 are unit eigenvectors of A corresponding to λ_1 and λ_2, respectively. One can check that Φ is a normalized positive linear map. Now we have

$$\begin{aligned}g\left(\Phi(A)\right)&=g\left(\left(\sqrt{\theta}e_1+\sqrt{1-\theta}e_2\right)^*A\left(\sqrt{\theta}e_1+\sqrt{1-\theta}e_2\right)\right)\\&=g\left(\theta\lambda_1+(1-\theta)\lambda_2\right)=g\left(\theta m+(1-\theta)M\right)=g(t^*)\end{aligned}$$

and

$$\begin{aligned}\Phi\left(f\left(A\right)\right)&=\left(\sqrt{\theta}e_1+\sqrt{1-\theta}e_2\right)^*f\left(A\right)\left(\sqrt{\theta}e_1+\sqrt{1-\theta}e_2\right)\\&=\theta f\left(m\right)+(1-\theta)f\left(M\right)=\frac{M-t^*}{M-m}f\left(m\right)+\frac{t^*-m}{M-m}f\left(M\right)\\&=\mu_f\cdot t^*+\nu_f.\end{aligned}$$

Thus we have

$$F\left[\Phi\left(f\left(A\right)\right),g\left(\Phi(A)\right)\right]=F\left[\mu_f\cdot t^*+\nu_f,g(t^*)\right],$$

as required.

Adding to some conditions in Theorem 4.1, we obtain the explicit estimations of the bounds for the ratio and difference inequalities.

THEOREM 4.7. *Let A be a Hermitian matrix with spectrum contained in $[m,M]$. Let Φ be a normalized positive linear map from H_n to $H_{\tilde{n}}$. Let f and g be real valued continuous functions on $[m,M]$. Moreover, if f is a convex function (resp. a concave function) on $[m,M]$, then for a given real number α*

$$\Phi(f(A))\leqslant\alpha g(\Phi(A))+\beta I\qquad(resp.\quad\Phi(f(A))\geqslant\alpha g(\Phi(A))+\beta I)\tag{4.5}$$

holds for

$$\beta=\max_{m\leqslant t\leqslant M}\{\mu_f\cdot t+\nu_f-\alpha g(t)\}\qquad(resp.\quad\beta=\min_{m\leqslant t\leqslant M}\{\mu_f\cdot t+\nu_f-\alpha g(t)\}).$$

Proof. We only prove it for the convex case. Let us put $F(u,v) = u - \alpha v$ in Theorem 3.5. Then it follows from the inequality (3.4) that

$$\begin{aligned}\Phi(f(A)) - \alpha g(\Phi(A)) &\leqslant \max_{m\leqslant t\leqslant M} F[\mu_f \cdot t + \nu_f, g(t)]I \\ &= \max_{m\leqslant t\leqslant M} \{\mu_f \cdot t + \nu_f - \alpha g(t)\}I,\end{aligned}$$

which gives the desired inequality.

Next we shall show three corollaries of Theorem 4.7, which frequently are used in the cases that g is a convex or concave function.

COROLLARY 4.8. *Let the hypothesis of Theorem 4.7 be satisfied. Let $\alpha \in \mathbb{R}$ a given real number. If f is a convex and αg is a concave function (resp. f is a concave function and αg is a convex) on $[m,M]$, then*

$$\Phi(f(A)) \leqslant \alpha g(\Phi(A)) + \beta I \qquad (resp. \quad \Phi(f(A)) \geqslant \alpha g(\Phi(A)) + \beta I)$$

holds for $\beta = \mu_f \cdot t_0 + \nu_f - \alpha g(t_0)$ where

$$t_0 = \begin{cases} M & if \quad \mu_f \geqslant \alpha\mu_g \quad (resp.\ \mu_f \leqslant \alpha\mu_g) \\ m & if \quad \mu_f < \alpha\mu_g \quad (resp.\ \mu_f > \alpha\mu_g) \end{cases} \tag{4.6}$$

This inequality is sharp in the sense that for any Hermitian matrix A there is a real valued normalized positive linear map Φ such that $\Phi(f(A)) - \alpha g(\Phi(A))$ is equal to the upper bound (resp. the lower bound).

Proof. We prove only the case when f is a convex and αg a concave function on $[m,M]$. Then $h(t) = \mu_f \cdot t + \nu_f - \alpha g(t)$ is a convex function and so $\max_{m\leqslant t\leqslant M} h(t) = \max\{h(m), h(M)\}$. If $h(M) \geqslant h(m)$ then $\max_{m\leqslant t\leqslant M} h(t) = h(M)$, otherwise $\max_{m\leqslant t\leqslant M} h(t) = h(m)$. Put $t^* = t_0$ and denote by $\Phi : H_n \to \mathbb{C}$ a normalized positive linear map defined as

$$\Phi(X) = \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right)^* X \left(\sqrt{\theta}e_1 + \sqrt{1-\theta}e_2\right), \tag{4.7}$$

where $\theta = (M - t_0)/(M - m)$, e_1 and e_2 are unit eigenvectors of A corresponding to $\lambda_{\min}(A) = m$ and $\lambda_{\max}(A) = M$ respectively. Then it follows from Lemma 4.6 that $\Phi(f(A)) - \alpha g(\Phi(A)) = [\mu_f t_0 + \nu_f - \alpha g(t_0)]I$.

If we put 1 in Theorem 4.7, then we have the following:

COROLLARY 4.9. *Let the hypothesis of Theorem 4.7 be satisfied. If f is a convex (resp. a concave) function on $[m,M]$ then the following inequality holds*

$$\Phi(f(A)) - g(\Phi(A)) \leqslant [\max_{m\leqslant t\leqslant M}\{\mu_f \cdot t + \nu_f - g(t)\}]\, I \tag{4.8}$$

$$(resp.\ \ \Phi(f(A)) - g(\Phi(A)) \geqslant [\min_{m\leqslant t\leqslant M}\{\mu_f \cdot t + \nu_f - g(t)\}]\, I). \tag{4.9}$$

Suppose in addition that g is a strictly convex (resp. a strictly concave) differentiable function on $[m,M]$, then the inequality (4.8) (resp. (4.9)) is sharp and the equality *is attained for a real valued normalized positive linear map* $\Phi : H_n \to \mathbb{C}$ *defined by* (4.7) *and*

$$t_0 = \begin{cases} g'^{-1}(\mu_f) & \text{if} \quad g'(m) < \mu_f < g'(M) \quad (\text{resp.} \quad g'(M) < \mu_f < g'(m)), \\ m & \text{if} \quad g'(m) \geqslant \mu_f \quad (\text{resp.} \quad g'(m) \leqslant \mu_f), \\ M & \text{if} \quad g'(M) \leqslant \mu_f \quad (\text{resp.} \quad g'(M) \geqslant \mu_f). \end{cases} \tag{4.10}$$

REMARK 4.10. We can obtain the opposite inequality of (4.8) in Corollary 4.9. Instead of maximizing over all matrix convex functions we took the easier route over the favourable chosen linear functions: Let f and g be two differentiable functions on $[m,M]$ and let f be convex. If $f'(m) \leqslant \mu_g \leqslant f'(M)$ then

$$\{f(r) - \mu_g r - \nu_g\}I \leqslant \Phi(f(A)) - g(\Phi(A)),$$

when g is a strictly convex function and

$$\{f(r) - g(t_0) + \mu_g(t_0 - r)\}I \leqslant \Phi(f(A)) - g(\Phi(A)),$$

when g is a strictly concave, where $r = f'^{-1}(\mu_g)$ and $t_0 = g'^{-1}(\mu_g)$. Otherwise, if $\mu_g < f'(m)$ or $f'(M) < \mu_g$ then

$$\begin{aligned} \max\Big\{ &\min_{m\leqslant t\leqslant M} \{f(m) + f'(m)(t-m) - g(t)\}, \\ &\min_{m\leqslant t\leqslant M} \{f(M) + f'(M)(t-M) - g(t)\}\Big\} I \\ &\leqslant \Phi(f(A)) - g(\Phi(A)). \end{aligned}$$

Indeed, because f is convex, then

$$\Phi(f(A)) - g(\Phi(A)) \geqslant \max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \min_{m\leqslant t\leqslant M} \{k(t) - g(t)\} I \geqslant \min_{m\leqslant t\leqslant M} \{h_r(t)\} I$$

where $h_r(t) = f(r) + f'(r)(t-r) - g(t)$, $r \in [m,M]$. We choose $r = f'^{-1}(\mu_g)$ when $f'(m) \leqslant \mu_g \leqslant f'(M)$, $r = m$ when $\mu_g < f'(m)$ or $r = M$ when $f'(M) < \mu_g$. In the case of convexity of g the function h_r is concave and so its minimum is attained at m or M. (Specially, we have $h_r(m) = h_r(M)$ when $r = f'^{-1}(\mu_g)$). In the case of concavity of g the function h_r is convex and so its minimum is attained at $t_0 \in [m,M]$. In the same way we can obtain the opposite inequality (4.9).

Indeed, in the case of convexity

REMARK 4.11. If we put $g = f$ in Corollary 4.9 we have [9, Corollary 2.4] with remark that t^* was not properly determined.

Also, when we replace $\Phi(A)$ with $\sum_{i=1}^{n} \omega_i \phi_i(A_i)$ and for $g = f$ a convex function we obtain a matrix analogous to an operator case [13, Theorem 3]. Further under the hypothesis of Corollary 4.9 we have

$$f(m) - g(m) \leqslant \max_{m\leqslant t\leqslant M} \{\mu_f \cdot t + \nu_f - g(t)\} \leqslant f(m) - g(m) + \left[\mu_f - g'(m)\right](M-m)$$

(Proof is given in [19, Theorem 7]). Hence for $f = g$ we have estimate in [13, Theorem 3]:

$$0 < \max_{m \leqslant t \leqslant M} \{\mu_f \cdot t + \nu_f - f(t)\} < \left[\mu_f - f'(m)\right](M - m).$$

COROLLARY 4.12. *Let the hypothesis of Theorem 4.1 be satisfied. Suppose in addition that either of the following conditions holds* (i) $g(t) > 0$ *for all* $t \in [m, M]$ *or* (ii) $g(t) < 0$ *for all* $t \in [m, M]$. *If* f *is a convex function (resp. a concave function) on* $[m, M]$, *then the following inequalities hold*

$$\Phi(f(A)) \leqslant \left[\max_{m \leqslant t \leqslant M} \left\{\frac{\mu_f \cdot t + \nu_f}{g(t)}\right\}\right] g(\Phi(A)) \tag{4.11}$$

$$\left(\text{resp. } \Phi(f(A)) \geqslant \left[\min_{m \leqslant t \leqslant M} \left\{\frac{\mu_f \cdot t + \nu_f}{g(t)}\right\}\right] g(\Phi(A))\right) \tag{4.12}$$

in case (i), *or*

$$\Phi(f(A)) \leqslant \left[\min_{m \leqslant t \leqslant M} \left\{\frac{\mu_f \cdot t + \nu_f}{g(t)}\right\}\right] g(\Phi(A)) \tag{4.13}$$

$$\left(\text{resp. } \Phi(f(A)) \geqslant \left[\max_{m \leqslant t \leqslant M} \left\{\frac{\mu_f \cdot t + \nu_f}{g(t)}\right\}\right] g(\Phi(A))\right) \tag{4.14}$$

in case (ii). *Suppose in addition that* $f(m) > 0$, $f(M) > 0$ *in case* (i) *or* $f(m) < 0$, $f(M) < 0$ *in case* (ii) *and* g *is a strictly convex (resp. strictly concave) twice differentiable function on* $[m, M]$ *then the inequalities* (4.11), (4.13) *(resp.* (4.12), (4.14)*) are sharp and the equality is attained for a real value normalized positive linear map* Φ *defined by* (4.7) *and*

$$t_0 = \begin{cases} \text{the solution of } \mu_f g(t) = (\mu_f \cdot t + \nu_f)\, g'(t) & \text{if} \quad f(m)\frac{g'(m)}{g(m)} < \mu_f < f(M)\frac{g'(M)}{g(M)} \\ M & \text{if} \quad \mu_f \geqslant f(M)\frac{g'(M)}{g(M)} \\ m & \text{if} \quad \mu_f \leqslant f(m)\frac{g'(m)}{g(m)} \end{cases} \tag{4.15}$$

(resp.

$$t_0 = \begin{cases} \text{the solution of } \mu_f g(t) = (\mu_f \cdot t + \nu_f)\, g'(t) & \text{if} \quad f(M)\frac{g'(M)}{g(M)} < \mu_f < f(m)\frac{g'(m)}{g(m)} \\ M & \text{if} \quad \mu_f \leqslant f(M)\frac{g'(M)}{g(M)} \\ m & \text{if} \quad \mu_f \geqslant f(m)\frac{g'(m)}{g(m)} \end{cases}$$

)

Proof. The inequalities (4.11), (4.12) for case (i) and (4.13), (4.14) for case (ii) follow from Theorem 4.7 if we choose α such that $\beta = 0$. Next, to show the inequality (4.11) is sharp for a convex function f and a strictly convex function g, we proceed only with case (i) since the proof in case (ii) is essentially the same. Since $f(m) > 0$, $f(M) > 0$ and $g(t) > 0$, we have $(\mu_f \cdot t + \nu_f)/g(t) > 0$ and according to Corollary 4.4 we have $\max_{m \leqslant t \leqslant M} \left\{\frac{\mu_f \cdot t + \nu_f}{g(t)}\right\} = \frac{\mu_f \cdot t_0 + \nu_f}{g(t_0)}$ for $t_0 \in [m, M]$

determined by (4.15). Using Lemma 4.6 for $t^* = t_0$ and map Φ defined by (4.7) we have $\Phi(f(A)) = \frac{\mu_f \cdot t_0 + \nu_f}{g(t_0)} g(\Phi(A))$. Hence,

$$\Phi(f(A)) = \left[\max_{m \leqslant t \leqslant M} \left\{\frac{\mu_f \cdot t + \nu_f}{g(t)}\right\}\right] g(\Phi(A)).$$

REMARK 4.13. Similarly to Remark 4.10, we obtain the opposite inequality of (4.11) in Corollary 4.12. Let f and g be two twice differentiable positive valued functions on $[m,M]$ and let f be convex. If $g(m)\frac{f'(m)}{f(m)} \leqslant \mu_g \leqslant g(M)\frac{f'(M)}{f(M)}$ then

$$\frac{f(r)}{\mu_g r + \nu_g} g(\Phi(A)) \leqslant \Phi(f(A)),$$

when g is a strictly convex function and

$$\frac{f(r)}{\mu_g r + \nu_g} \frac{\mu_g t_0 + \nu_g}{f(t_0)} g(\Phi(A)) \leqslant \Phi(f(A)),$$

when g is a strictly concave, where r is the unique solution in $[m,M]$ of $\frac{f'(r)}{f(r)} = \frac{\mu_g}{\mu_g r + \nu_g}$ and t_0 is the unique solution in $[m,M]$ of $\frac{g'(t)}{g(t)} = \frac{\mu_g}{\mu_g t + \nu_g}$. Otherwise, if $\frac{\mu_g}{g(m)} < \frac{f'(m)}{f(m)}$ or $\frac{f'(M)}{f(M)} < \frac{\mu_g}{g(M)}$ then

$$\max_{s \in \{m,M\}} \min_{m \leqslant t \leqslant M} \left\{\frac{f(s) + f'(s)(t-s)}{g(t)}\right\} g(\Phi(A)) \leqslant \Phi(f(A)).$$

Indeed, because f is convex, then

$$\Phi(f(A)) \geqslant \max_{\substack{k \in \{conx.\} \\ k \leqslant f}} \min_{m \leqslant t \leqslant M} \left\{\frac{k(t)}{g(t)}\right\} g(\Phi(A)) \geqslant \min_{m \leqslant t \leqslant M} \{h_r(t)\} g(\Phi(A))$$

where $h_r(t) = \frac{f(r) + f'(r)(t-r)}{g(t)}$, $r \in [m,M]$. We choose r which is the unique solution in $[m,M]$ of $\frac{f'(r)}{f(r)} = \frac{\mu_g}{\mu_g r + \nu_g}$ when $g(m)\frac{f'(m)}{f(m)} \leqslant \mu_g \leqslant g(M)\frac{f'(M)}{f(M)}$, $r = m$ when $\frac{\mu_g}{g(m)} < \frac{f'(m)}{f(m)}$ or $r = M$ when $\frac{f'(M)}{f(M)} < \frac{\mu_g}{g(M)}$. In the case of convexity of g, the function h_r is concave and so its minimum is attained at m or M. (Specially, we have $h_r(m) = h_r(M)$ when r is the unique solution of $\frac{f'(r)}{f(r)} = \frac{\mu_g}{\mu_g r + \nu_g}$). In the case of concavity of g the function h_r is convex and so its minimum is attained at $t_0 \in [m,M]$. In the same way we can obtain the opposite inequality (4.12) and in case $f, g < 0$ the opposite inequalities of (4.13), (4.14).

REMARK 4.14. If we put $g = f > 0$ in Corollary 4.12 we have [9, Corollary 2.4] for the ratio case.

Also, when we replace $\Phi(A)$ with $\sum_{i=1}^{n}\omega_i\phi_i(A_i)$ and for $g=f$ a convex function we obtain a matrix analogous to an operator case [13, Theorem 4]. Further let the hypothesis of Corollary 4.12 be satisfied and f, g are convex functions. Then

$$\max\left\{\frac{f(m)}{g(m)},\frac{f(M)}{g(M)}\right\}\leqslant\max_{m\leqslant t\leqslant M}\left\{\frac{\mu_f\cdot t+\nu_f}{g(t)}\right\}$$

in case (i) or

$$0\leqslant\min_{m\leqslant t\leqslant M}\left\{\frac{\mu_f\cdot t+\nu_f}{g(t)}\right\}\leqslant\min\left\{\frac{f(m)}{g(m)},\frac{f(M)}{g(M)}\right\}$$

in case (ii). For $f=g$ an extreme of $h(t)=(\mu_f\cdot t+\nu_f)/g(t)$ on $[m,M]$ is at $t_0\in(m,M)$ and we have the estimate in [13, Theorem 3]: $1<\max_{m\leqslant t\leqslant M}\left\{\frac{\mu_f\cdot t+\nu_f}{f(t)}\right\}$ in case (i) or $0<\min_{m\leqslant t\leqslant M}\left\{\frac{\mu_f\cdot t+\nu_f}{f(t)}\right\}<1$ in case (ii).

5. Applications I

In this section, as application of our general theorem, we shall consider the upper and lower bounds of the difference and the ratio in inequalities of the power functions and the means. We use our approach to obtain results for the cases that were not covered in [9].

5.1. Application to power functions

THEOREM 5.1. *Let A be a positive definite Hermitian matrix with spectrum contained in $[m,M]$, where $0<m<M$. Let Φ be a normalized positive linear map from H_n to $H_{\tilde{n}}$. Put $q\in\mathbb{R}$. If f is a real valued continuous convex function on $[m,M]$, then for a given real number α*

$$\Phi(f(A))\leqslant\alpha\Phi(A)^q+\beta I \tag{5.1}$$

holds for

$$\beta=\begin{cases}\alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}}+\nu_f, & \text{if}\quad m<\left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}}<M \text{ and } \alpha q(q-1)>0\\ \max\{f(M)-\alpha M^q, f(m)-\alpha m^q\}, & \text{otherwise.}\end{cases}$$

But if f is a real valued continuous concave function on $[m,M]$, then for a given real number α

$$\Phi(f(A))\geqslant\alpha\Phi(A)^q+\beta I \tag{5.2}$$

holds for

$$\beta=\begin{cases}\alpha(q-1)\left(\frac{\mu_f}{\alpha q}\right)^{\frac{q}{q-1}}+\nu_f & \text{if}\quad m<\left(\frac{\mu_f}{\alpha q}\right)^{\frac{1}{q-1}}<M \text{ and } \alpha q(q-1)<0\\ \min\{f(M)-\alpha M^q, f(m)-\alpha m^q\} & \text{otherwise.}\end{cases}$$

Inequalities (5.1) and (5.2) are sharp in the sense that there is a normalized positive linear map Φ for which the equality is attained.

Proof. In the case of $\alpha q(q-1)>0$, we obtain inequality (5.1) if we put $g(t)=\alpha t^q$ in Corollary 4.9. Similarly we prove the concave case.

We have the following Corollary by applying $f(t)=t^p$ to Theorem 5.1.

COROLLARY 5.2. *Let the hypothesis of Theorem 5.1 be satisfied. If* $p\in\mathbb{R}\backslash[0,1)$ *(resp.* $p\in(0,1]$ *) and* $q\in\mathbb{R}$*, then for a given real number* α

$$\Phi(A^p)\leqslant\alpha\Phi(A)^q+\beta_1 I \quad (\textit{resp.}\ \ \Phi(A^p)\geqslant\alpha\Phi(A)^q+\beta_2 I) \tag{5.3}$$

holds for

$$\beta_1=\begin{cases}\alpha(q-1)\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{q}{q-1}}+\nu_{t^p} & \text{if}\quad m<\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{1}{q-1}}<M \text{ and } \alpha q(q-1)>0\\ \max\{m^p-\alpha m^q,M^p-\alpha M^q\} & \text{otherwise,}\end{cases}$$

(resp.

$$\beta_2=\begin{cases}\alpha(q-1)\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{q}{q-1}}+\nu_{t^p} & \text{if}\quad m<\left(\frac{1}{\alpha q}\mu_{t^p}\right)^{\frac{1}{q-1}}<M \text{ and } \alpha q(q-1)<0\\ \min\{m^p-\alpha m^q,M^p-\alpha M^q\} & \text{otherwise.}\end{cases}$$

)
The inequality (5.3) *is sharp.*

Next we shall show the following two theorems, which are extensions of [9, Theorem 3.1 and Theorem 3.2].

THEOREM 5.3. *Let the hypothesis of Theorem 5.1 be satisfied. If* $p\in\mathbb{R}\backslash\{0\}$ *and* $q\in\mathbb{R}$*, then*

$$\beta_2 I\leqslant\Phi(A^p)-\Phi(A)^q\leqslant\beta_1 I \tag{5.4}$$

with

$$\beta_1=\begin{cases}(q-1)\left(\frac{1}{q}\mu_{t^p}\right)^{\frac{q}{q-1}}+\nu_{t^p} & \text{if}\quad m<\left(\frac{1}{q}\mu_{t^p}\right)^{\frac{1}{q-1}}<M \text{ and } q(q-1)>0\\ \max\{m^p-m^q,M^p-M^q\} & \text{otherwise}\end{cases}$$

if $p\in\mathbb{R}\backslash[0,1]$,

$$\beta_1=\begin{cases}\left(\frac{q}{p}\right)^{\frac{p}{p-q}}-\left(\frac{q}{p}\right)^{\frac{q}{p-q}} & \text{if}\quad m<\left(\frac{q}{p}\right)^{\frac{1}{p-q}}<M \text{ and } 0<p<q\\ \max\{m^p-m^q,M^p-M^q\} & \text{otherwise}\end{cases}$$

if $p\in(0,1]$

and

$$\beta_2 = \begin{cases} (q-1)\left(\frac{1}{q}\mu_{t^p}\right)^{\frac{q}{q-1}} + \nu_{t^p} & \textit{if} \quad m < \left(\frac{1}{q}\mu_{t^p}\right)^{\frac{1}{q-1}} < M \textit{ and } q(q-1) < 0 \\ \min\{m^p - m^q, M^p - M^q\} & \textit{otherwise} \end{cases}$$

if $p \in (0,1)$,

$$\beta_2 = \begin{cases} \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} & \textit{if} \quad m < \left(\frac{q}{p}\right)^{\frac{1}{p-q}} < M \textit{ and } q(p-q) > 0 \\ \min\{m^p - m^q, M^p - M^q\} & \textit{otherwise} \end{cases}$$

if $p \in [-1,0)$ *or* $p \in [1,2]$,

$$\beta_2 = \begin{cases} (1-p)\left(\frac{1}{p}\mu_{t^q}\right)^{\frac{p}{p-1}} - \nu_{t^q} \quad \textit{if } m \leqslant \left(\frac{1}{p}\mu_{t^q}\right)^{\frac{1}{p-1}} \leqslant M \textit{ and } q(q-1) > 0 \\ (1-p)\left(\frac{1}{p}\mu_{t^q}\right)^{\frac{p}{p-1}} + (q-1)\left(\frac{1}{q}\mu_{t^q}\right)^{\frac{q}{q-1}} \\ \qquad\qquad \textit{if } m \leqslant \left(\frac{1}{p}\mu_{t^q}\right)^{\frac{1}{p-1}} \leqslant M \textit{ and } q(q-1) < 0 \\ \max\limits_{s\in\{m,M\}} \min\limits_{m\leqslant t\leqslant M} \left\{(1-p)s^p + ps^{p-1}t - t^q\right\} \quad \textit{otherwise} \end{cases}$$

if $p < -1$ *or* $p > 2$.

The right hand inequality (5.4) *is sharp for all values of* p. *The left hand of inequality* (5.4) *is sharp when* $p \in [-1,2]$.

Proof. We first consider β_1.

Case 1. Suppose $p > 1$ or $p < 0$. Put $\alpha = 1$ in Corollary 5.2. Then we have β_1 and the right hand inequality is sharp.

Case 2. Suppose $0 < p \leqslant 1$. Then the function $f(t) = t^p$ is matrix concave. We use $k = f$ in inequality (4.2) of Corollary 4.2 to determinate β_1. This inequality is sharp and the equality is attained for map (4.7), where t_0 is determined in usual way.

Next, we consider β_2.

Case 1. Suppose $0 < p < 1$. Put $\alpha = 1$ in Corollary 5.2. Then we have β_2 and the left hand inequality is sharp.

Case 2. Suppose $-1 \leqslant p < 0$ or $1 \leqslant p \leqslant 2$. Then the function $f(t) = t^p$ is matrix convex. We use $k = f$ in inequality (4.2) of Corollary 4.2 to determine β_2. This inequality is sharp and the inequality is attained for map defined by (4.7), where t_0 is determined in usual way.

Case 3. Suppose $p > 2$ or $p < -1$. Then $f(t) = t^p$ is a convex function and we use Remark 4.10 to determine β_2.

REMARK 5.4. If we put $q = p$ in Theorem 5.3 we obtain [9, Theorem 3.1]:

$$\beta_2 I \leqslant \Phi(A^p) - \Phi(A)^p \leqslant \beta_1 I$$

with

$$\beta_2 = \begin{cases} -\Delta & \text{if } p > 2 \ \ or \ \ p < -1 \\ 0 & \text{if } \ -1 \leqslant p < 0 \ \ or \ \ 1 \leqslant p \leqslant 2 \\ \Delta & \text{if } 0 < p < 1, \end{cases}$$

$$\beta_1 = \begin{cases} \Delta & \text{if } p > 1 \ \ or \ \ p < 0 \\ 0 & \text{if } 0 < p \leqslant 1, \end{cases}$$

where

$$\Delta = M^p \frac{1-\gamma^{1-p}}{1-\gamma} + m^p(p-1)\left\{\frac{p(\gamma-1)}{\gamma^p - 1}\right\}^{\frac{p}{1-p}}$$

and $\gamma = M/m$. The right hand inequality is sharp for all values of p and the left hand inequality when $p \in [-1,2]$.

THEOREM 5.5. *Let the hypothesis of Theorem 5.1 be satisfied. If $p \in \mathbb{R}\backslash\{0\}$ and $q \in \mathbb{R}$, then*

$$\alpha_2 \Phi(A)^q \leqslant \Phi(A^p) \leqslant \alpha_1 \Phi(A)^q \tag{5.5}$$

with

$$\alpha_1 = \begin{cases} C_{t^p}(m,M;q) & if \quad m < \frac{q}{1-q}\nu_{t^p}/\mu_{t^p} < M\ , \ \ q(q-1) > 0 \ \ and \ \ pq > 0 \\ \max\{\frac{m^p}{m^q}, \frac{M^p}{M^q}\} & otherwise \end{cases}$$

if $p \in \mathbb{R}\backslash[0,1]$,

$$\alpha_1 = \begin{cases} m^{p-q} & if \ \ p < q \\ M^{p-q} & if \ \ p \geqslant q \end{cases}$$

if $p \in (0,1]$, and

$$\alpha_2 = \begin{cases} C_{t^p}(m,M;q) & if \quad m < \frac{q}{1-q}\nu_{t^p}/\mu_{t^p} < M \ and \ \ q(q-1) < 0 \\ \min\{\frac{m^p}{m^q}, \frac{M^p}{M^q}\} & otherwise \end{cases}$$

if $p \in (0,1)$,

$$\alpha_2 = \begin{cases} m^{p-q} & if \ \ p > q \\ M^{p-q} & if \ \ p \leqslant q \end{cases}$$

if $p \in [-1,0)$ or $p \in [1,2]$,

$$\alpha_2 = \begin{cases} C_{t^q}(m,M;p)^{-1} & if \quad pm^{q-1} \leqslant \mu_{t^q} \leqslant pM^{q-1} \ and \ \ q(q-1) > 0 \\ \frac{1-p}{1-q} C_{t^q}(m,M;p)^{-1} C_{t^q}(m,M;q) & \\ \qquad\qquad if \quad pm^{q-1} \leqslant \mu_{t^q} \leqslant pM^{q-1} \ and \ \ q(q-1) < 0 \\ \max\limits_{s\in\{m,M\}} \min\limits_{m\leqslant t\leqslant M} \left\{\frac{(1-p)s^p + ps^{p-1}t}{t^q}\right\} & otherwise \end{cases}$$

if $p < -1$ or $p > 2$.

Here $C_{t^p}(m,M;q)$ is Furuta's constant for $f(t) = t^p$ and also $C_{t^q}(m,M;p)$ is Furuta's constant for $f(t) = t^q$. The right hand inequality of (5.5) is sharp for all values of p. The left hand inequality of (5.5) is sharp when $p \in [-1,2]$.

Proof. We prove this theorem by a similar method as in Corollary 5.3.

We first consider α_1.

Case 1. Suppose $p > 1$ or $p < 0$. Then we have α_1 as a unique constant which satisfies $\beta_1 = 0$ in Corollary 5.2. The right hand inequality is sharp.

Case 2. Suppose $0 < p \leqslant 1$. Then the function $f(t) = t^p$ is matrix concave. We use $k = f$ in inequality (4.3) to determinate α_1. The sharp is attained for map (4.7), where t_0 may be determined in usual way.

Next, we consider α_2.

Case 1. Suppose $0 < p < 1$. Then we have α_2 as a unique constant which satisfies $\beta_2 = 0$ in Corollary 5.2. The left hand inequality is sharp.

Case 2. Suppose $-1 \leqslant p < 0$ or $1 \leqslant p \leqslant 2$. Then the function $f(t) = t^p$ is matrix convex. We use $k = f$ in inequality 4.3 to determinate α_2. The sharp is attained for map (4.7), where t_0 may be determined in usual way.

Case 3. Suppose $p > 2$ or $p < -1$. Then $f(t) = t^p$ is a convex function and we use Remark 4.13 to determine α_2.

REMARK 5.6. If we put $q = p$ in Theorem 5.5 we obtain [9, Theorem 3.2]:

$$\alpha_2 \Phi(A)^p \leqslant \Phi(A^p) \leqslant \alpha_1 \Phi(A)^p$$

with

$$\alpha_2 = \begin{cases} \Delta^{-1} & \text{if } p > 2 \ \ or \ \ p < -1 \\ 1 & \text{if } -1 \leqslant p < 0 \ \ or \ \ 1 \leqslant p \leqslant 2 \\ \Delta & \text{if } 0 < p < 1, \end{cases}$$

$$\alpha_1 = \begin{cases} \Delta & \text{if } p > 1 \ \ or \ \ p < 0 \\ 1 & \text{if } 0 < p \leqslant 1, \end{cases}$$

where

$$\Delta = \frac{\gamma^p - \gamma}{(1-p)(\gamma - 1)} \left(\frac{(p-1)(\gamma^p - 1)}{p(\gamma^p - \gamma)} \right)^p$$

and $\gamma = M/m$. The right hand inequality is sharp for all values of p and the left hand inequality when $p \in [-1, 2]$.

5.2. Application to means

We recall Jensen's type inequalities of power means on a positive linear map: If A is a positive definite Hermitian matrix, then

$$\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s}$$

holds for either $r \leqslant s$ with $r, s \neq (-1, 1)$, or $1/2 \leqslant r \leqslant 1 \leqslant s$ or $r \leqslant -1 \leqslant s \leqslant -1/2$.

In this section, we shall investigate the lower and upper estimates of the difference and the ratio in power means on a positive linear map. We prepare the following intervals:

$$\begin{array}{ll} (i) & s \geqslant r, s \notin (-1,1), r \notin (-1,1) \\ (ii) & s \geqslant 1 \geqslant r \geqslant 1/2 \\ (iii) & r \leqslant -1 \leqslant s \leqslant -1/2 \\ (iv) & s \geqslant 1, -1 < r < 1/2, r \neq 0 \\ (v) & r \leqslant -1, -1/2 < s < 1, s \neq 0 \\ (vi) & s > r, s \notin (-1,1) \text{ or } r \notin (-1,1) \end{array} \tag{5.6}$$

If we put $p = s/r$ in Remark 5.6 and replace A by A^r or $p = r/s$, A by A^s, then we have the following theorem:

THEOREM 5.7. *Let A be a Hermitian matrix with spectrum contained in $[m,M]$, where $0 < m < M$. Let Φ be a normalized positive linear map from H_n to $H_{\tilde{m}}$. Also let r,s be nonzero real numbers such that $r \leqslant s$. Then*

$$\alpha_2 \Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant \alpha_1 \Phi(A^r)^{1/r} \tag{5.7}$$

with

$$\alpha_2 = \begin{cases} 1 & \text{if either } (i) \text{ or } (ii) \text{ or } (iii), \\ \tilde{\Delta}^{-1} & \text{if either } (iv) \text{ or } (v), \end{cases} \qquad \alpha_1 = \tilde{\Delta} \text{ if } (vi),$$

where

$$\tilde{\Delta} = \left\{ \frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)} \right\}^{\frac{1}{s}} \left\{ \frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)} \right\}^{-\frac{1}{r}}.$$

and $\gamma = M/m$. The right hand inequality is sharp when r,s satisfy (vi) and the left hand inequality when r,s satisfy (i), or (ii), or (iii).

Proof. We prove this Theorem by a similar method as in [20, Theorem (Mond-Pečarić)1]. We shall consider only the case when $s \neq r$.

Suppose that $s \geqslant 1$ and $r < 1$. In this case we put $p = \frac{s}{r}$. If $r > 0$ then $m^r \leqslant A^r \leqslant M^r$ and Remark 5.6 (for $1 < p \leqslant 2$ or $p > 2$) gives

$$\begin{array}{ll} \Phi(A)^{s/r} \leqslant \Phi(A^{s/r}) \leqslant C_{t^{s/r}}(m,M;\tfrac{s}{r})\Phi(A)^{s/r} & \text{if } s/2 \leqslant r < 1, \\ \text{or } \; C_{t^{s/r}}(m,M;\tfrac{s}{r})^{-1}\Phi(A)^{s/r} \leqslant \Phi(A^{s/r}) \leqslant C_{t^{s/r}}(m,M;\tfrac{s}{r})\Phi(A)^{s/r} & \text{if } 0 < r < s/2, \end{array}$$

where $C_{t^{s/r}}(m,M;\frac{s}{r})$ is Furuta's constant for $f(t) = t^{s/r}$. Then replacing A by A^r we have

$$\begin{array}{ll} \Phi(A^r)^{s/r} \leqslant \Phi(A^s) \leqslant C_{t^{s/r}}(m^r,M^r;\tfrac{s}{r})\Phi(A^r)^{s/r} & \text{if } s/2 \leqslant r < 1, \\ \text{or } \; C_{t^{s/r}}(m^r,M^r;\tfrac{s}{r})^{-1}\Phi(A^r)^{s/r} \leqslant \Phi(A^s) \leqslant C_{t^{s/r}}(m^r,M^r;\tfrac{s}{r})\Phi(A^r)^{s/r} & \text{if } 0 < r < s/2, \end{array}$$

where (see [20, proof of Theorem (Mond-Pečarić) 1])

$$C_{t^{s/r}}(m^r, M^r; \frac{s}{r}) = \frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)} \left(\frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)}\right)^{-\frac{s}{r}}.$$

The function $f(t) = t^{\frac{1}{s}}$ is matrix increasing if $s \geqslant 1$ and it follows that

$$\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant C_{t^{s/r}}(m^r, M^r; \tfrac{s}{r})^{1/s}\Phi(A^r)^{1/r} \quad \text{if} \quad s/2 \leqslant r < 1, \text{ or}$$
$$C_{t^{s/r}}(m^r, M^r; \tfrac{s}{r})^{-1/s}\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant C_{t^{s/r}}(m^r, M^r; \tfrac{s}{r})^{1/s}\Phi(A^r)^{1/r} \quad \text{if} \quad 0 < r < s/2,$$

where $C_{t^{s/r}}(m^r, M^r; \frac{s}{r})^{1/s} = \left\{\frac{r(\gamma^s-\gamma^r)}{(s-r)(\gamma^r-1)}\right\}^{\frac{1}{s}} \left\{\frac{s(\gamma^r-\gamma^s)}{(r-s)(\gamma^s-1)}\right\}^{-\frac{1}{r}} = \tilde{\Delta}$. Furthermore, consider the case of $s = 1$. Then for $1/2 \leqslant r \leqslant 1$

$$\Phi(A^r)^{1/r} \leqslant \Phi(A),$$

so for arbitrary $s > 1$, we have

$$\Phi(A^r)^{1/r} \leqslant \Phi(A) \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r} \quad \text{if} \quad 1/2 \leqslant r < 1,$$
$$\text{or} \quad \tilde{\Delta}^{-1}\Phi(A^r)^{1/r} \leqslant \Phi(A) \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r} \quad \text{if} \quad 0 < r < 1/2.$$

If $r < 0$ then $M^r \leqslant A^r \leqslant m^r$ and Remark 5.6 (for $-1 \leqslant p < 0$ or $p < -1$) with the fact that the function $f(t) = t^{\frac{1}{s}}$ is matrix increasing gives

$$\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant C_{t^{s/r}}(M^r, m^r; \tfrac{s}{r})^{1/s}\Phi(A^r)^{1/r} \quad \text{if} \quad r \leqslant -s, \quad \text{or}$$
$$C_{t^{s/r}}(M^r, m^r; \tfrac{s}{r})^{-1/s}\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant C_{t^{s/r}}(M^r, m^r; \tfrac{s}{r})^{1/s}\Phi(A^r)^{1/r} \quad \text{if} \quad -s < r < 0,$$

where (see [20, proof of Theorem (Mond-Pečarić) 1]) $C_{t^{s/r}}(M^r, m^r; \frac{s}{r})^{\frac{1}{s}} = \tilde{\Delta}$. Therefore, similarly to above we have

$$\Phi(A^r)^{1/r} \leqslant \Phi(A) \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r} \quad \text{if} \quad r \leqslant -1,$$
$$\text{or} \quad \tilde{\Delta}^{-1}\Phi(A^r)^{1/r} \leqslant \Phi(A) \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r} \quad \text{if} \quad -1 < r < 0.$$

Now, suppose that $1 \leqslant r < s$. In this case we put $p = \frac{r}{s}$ and Remark 5.6 (for $0 < p \leqslant 1$) with the fact that the function $f(t) = t^{\frac{1}{r}}$ is matrix increasing gives

$$C_{t^{r/s}}(m^s, M^s; \frac{r}{s})^{1/r}\Phi(A^s)^{1/s} \leqslant \Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s}$$

where (see [20, proof of Theorem (Mond-Pečarić) 1]) $C_{t^{r/s}}(m^s, M^s; \frac{r}{s})^{1/r} = \tilde{\Delta}^{-1}$, so that we obtain

$$\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r} \qquad \text{if} \ 1 \leqslant r \leqslant s.$$

Therefore, we obtain the desired results in the cases of (i), (ii) and (iii) for $s \geqslant 1$ and $r \leqslant s$.

We prove first the case when $-1 < s < 1$. In this case we put $p = \frac{r}{s}$. If $0 < s < 1$ then $m^s \leqslant A^s \leqslant M^s$ and Remark 5.6 (for $p < -1$) gives

$$C_{t^{r/s}}(m^s, M^s; \frac{r}{s})^{-1}\Phi(A^r) \leqslant \Phi(A^s)^{r/s} \leqslant C_{t^{r/s}}(m^s, M^s; \frac{r}{s})\Phi(A^r).$$

Therefore, the function $f(t) = t^{\frac{1}{r}}$ is matrix decreasing if $r \leqslant -1$ and it follows that

$$C_{t^{r/s}}(m^s, M^s; \frac{r}{s})^{-1/r}\Phi(A^r)^{1/r} \geqslant \Phi(A^s)^{1/s}$$
$$\geqslant C_{t^{r/s}}(m^s, M^s; \frac{r}{s})^{1/r}\Phi(A^r)^{1/r} \quad \text{if } 0 < s < 1,$$

so that we obtain the desired inequality.

Similarly, if $-1 < s < 0$ then $M^s \leqslant A^s \leqslant m^s$ and Remark 5.6 (for $1 \leqslant p \leqslant 2$ or $p > 2$) with the fact that the function $f(t) = t^{\frac{1}{r}}$ is matrix decreasing gives

$$C_{t^{r/s}}(M^s, m^s; \frac{r}{s})^{-1/r}\Phi(A^r)^{1/r} \geqslant \Phi(A^s)^{1/s} \geqslant \Phi(A^r)^{1/r}, \text{ if } -1 < s \leqslant r/2,$$

or

$$C_{t^{s/r}}(M^r, m^r; \frac{s}{r})^{-1/s}\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s}$$
$$\leqslant C_{t^{s/r}}(M^r, m^r; \frac{s}{r})^{1/s}\Phi(A^r)^{1/r} \text{ if } r/2 \leqslant r < 0,$$

where $C_{t^{r/s}}(M^s, m^s; \frac{r}{s})^{1/r} = \tilde{\Delta}^{-1}$.

Furthermore, consider the case of $r = -1$. Then for $-1 \leqslant s \leqslant -1/2$

$$\Phi(A^{-1})^{-1} \leqslant \Phi(A^s)^{1/s},$$

so for arbitrary $r < -1$, we have

$$\Phi(A^r)^{1/r} \leqslant \Phi(A^{-1})^{-1} \leqslant \Phi(A^s)^{1/s}.$$

Therefore we have

$$\begin{aligned} &\Phi(A^r)^{1/r} \leqslant \Phi(A^{-1})^{-1} \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r} && \text{if } -1 < s \leqslant -1/2, \\ \text{or } &\tilde{\Delta}^{-1}\Phi(A^r)^{1/r} \leqslant \Phi(A^{-1})^{-1} \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r} && \text{if } -1/2 < s < 0. \end{aligned}$$

Finally, let $r \leqslant s \leqslant -1$. In this case we put $p = \frac{s}{r}$ and then $M^r \leqslant A^r \leqslant m^r$. Remark 5.6 (for $0 < p \leqslant 1$) with the fact that the function $f(t) = t^{\frac{1}{s}}$ is matrix decreasing gives

$$\Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant \tilde{\Delta}\Phi(A^r)^{1/r},$$

so that we obtain the desired results in the cases of (i), (iii) and (v) for $s < 1$ and $r \leqslant s$.

Further, we shall give the estimate $\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r}$ for $r < s$. To establish the result, we need the following corollary.

COROLLARY 5.8. *Let the hypothesis of Theorem 5.7 be satisfied. If* $1 \leqslant r \leqslant s$ *or* $r \leqslant -1 \leqslant s$ *then*

$$[\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} \leqslant \Phi(A^r)^{1/r} \tag{5.8}$$
$$\leqslant \begin{cases} \left[\bar{\mu}\Phi(A^s) + \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\bar{\nu}\right)^{\frac{r}{r-s}} I\right]^{1/r} & \textit{if } -1/2 < s < 1, s \neq 0, \\ \Phi(A^s)^{1/s} & \textit{otherwise}, \end{cases}$$

while if $r \leqslant s \leqslant -1$ *or* $r \leqslant 1 \leqslant s$ *then*

$$\left[\frac{1}{\bar{\mu}}\Phi(A^r) - \frac{\bar{\nu}}{\bar{\mu}}I\right]^{1/s} \geqslant \Phi(A^s)^{1/s} \tag{5.9}$$
$$\geqslant \begin{cases} \left[\frac{1}{\bar{\mu}}\Phi(A^r) - \frac{1}{\bar{\mu}}\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar{\nu}\right)^{\frac{r}{r-s}} I\right]^{1/s} & \textit{if } -1 < r < 1/2, r \neq 0, \\ \Phi(A^r)^{1/r} & \textit{otherwise}, \end{cases}$$

where $\bar{\mu} = \frac{M^r - m^r}{M^s - m^s}$ *and* $\bar{\nu} = \frac{M^s m^r - M^r m^s}{M^s - m^s}$.

Proof. Let the hypothesis of Theorem 5.7 be satisfied. First we prove two inequalities similar to that in Remark 5.4. Let $p \in \mathbb{R}\backslash\{0\}$. If $0 < p \leqslant 1$ then

$$\mu_{t^p}\Phi(A) + \nu_{t^p} I \leqslant \Phi(A^p) \leqslant \Phi(A)^p. \tag{5.10}$$

This inequality is sharp. While if $p < 0$ or $p > 1$ then

$$\mu_{t^p}\Phi(A) + \nu_{t^p} I \geqslant \Phi(A^p) \geqslant \begin{cases} \Phi(A)^p & \text{if } -1 \leqslant p < 0 \ \ or\ \ 1 \leqslant p \leqslant 2 \\ \mu_{t^p}\Phi(A) + \nu^*_{t^p} I & \text{if } p < -1 \ \ or\ \ p > 2 \end{cases} \tag{5.11}$$

where $\nu^*_{t^p} = (1-p)(\mu_{t^p}/p)^{p/(p-1)}$. The left hand inequality is sharp for all values of p and the right hand inequality when $-1 \leqslant p < 0$ and $1 \leqslant p \leqslant 2$.

Indeed, the right hand inequality (5.10) for $0 < p \leqslant 1$ and left hand inequality (5.11) for $-1 \leqslant p < 0$ or $1 \leqslant p \leqslant 2$ follow from Remark 5.4. The left hand inequality (5.10) for $0 < p \leqslant 1$ and the right hand inequality (5.11) for $p < 0$ or $p > 1$ follow from Corollary 4.9 if we put $f(t) = t^p$ and $g(t) = \mu_{t^p} t$. Finally, the left hand inequality (5.11) for $p < -1$ or $p > 2$ follows from Remark 4.10 for some functions $f(t) = t^p$ and $g(t) = \mu_{t^p} t$.

Now, we prove inequalities (5.8) and (5.9) by a similar method as [20, Theorem (Mond-Pečarić)2]. We shall consider only the case when $s \neq r$. We prove first (5.8) if $r \notin (-1,1)$. In this case we replace A by A^s and put $p = \frac{r}{s}$ in both inequalities (5.10) and (5.11). Then

$$\begin{array}{ll} \Phi(A^s)^{r/s} \leqslant \Phi(A^r) \leqslant \bar{\mu}\Phi(A^s) + \bar{\nu}I & \text{if } r \leqslant -1 \text{ and } (r \leqslant s \leqslant r/2 \text{ or } -r \leqslant s), \\ \bar{\mu}\Phi(A^s) + \nu^* I \leqslant \Phi(A^r) \leqslant \bar{\mu}\Phi(A^s) + \bar{\nu}I & \text{if } r \leqslant -1,\ r/2 < s < -r,\ s \neq 0, \\ \bar{\mu}\Phi(A^s) + \bar{\nu}I \leqslant \Phi(A^r) \leqslant \Phi(A^s)^{r/s} & \text{if } 1 \leqslant r < s, \end{array}$$

where $\nu^* = \left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar{\mu}\right)^{\frac{r}{r-s}}$. Using the fact that the function $f(t) = t^{\frac{1}{r}}$ is a matrix increasing if $r \geqslant 1$ and a matrix decreasing if $r \leqslant -1$ we have

$$\begin{array}{ll} \Phi(A^s)^{1/s} \geqslant \Phi(A^r)^{1/r} \geqslant [\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} & \text{if } r \leqslant -1 \text{ and } (r \leqslant s \leqslant r/2 \text{ or } -r \leqslant s), \\ [\bar{\mu}\Phi(A^s) + \nu^* I]^{1/r} \geqslant \Phi(A^r)^{1/r} \geqslant [\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} & \text{if } r \leqslant -1,\ r/2 < s < -r,\ s \neq 0, \\ [\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} \leqslant \Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} & \text{if } 1 \leqslant r < s. \end{array} \tag{5.12}$$

If we put $r=-1$, then we have

$$\Phi(A^r)^{1/r} \leqslant \Phi(A^{-1})^{-1} \leqslant \Phi(A^s)^{1/s}$$

for $r \leqslant -1$ and $-1 \leqslant s \leqslant -1/2$. Therefore, it follows from a similar way to Theorem 5.7 that

$$\begin{array}{ll} [\bar{\mu}\Phi(A^s)+\bar{\nu}I]^{1/r} \leqslant \Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} & \text{if either } (iii) \text{ or } (r \leqslant -1,\ 1 \leqslant s), \\ [\bar{\mu}\Phi(A^s)+\bar{\nu}I]^{1/r} \leqslant \Phi(A^r)^{1/r} \leqslant [\bar{\mu}\Phi(A^s)+\nu^* I]^{1/r} & \text{if } (v), \\ [\bar{\mu}\Phi(A^s)+\bar{\nu}I]^{1/r} \leqslant \Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} & \text{if } 1 \leqslant r < s. \end{array}$$

Next we prove (5.9) if $s \notin (-1,1)$. In this case we replace A by A^r and put $p=\frac{s}{r}$ in both inequalities (5.10) and (5.11). Then

$$\begin{array}{ll} \Phi(A^r)^{s/r} \leqslant \Phi(A^s) \leqslant \tilde{\mu}\Phi(A^r)+\tilde{\nu}I & \text{if } s \geqslant 1 \text{ and } (s/2 \leqslant r \leqslant s \text{ or } r \leqslant -s), \\ \tilde{\mu}\Phi(A^r)+\tilde{\nu}^* I \leqslant \Phi(A^s) \leqslant \tilde{\mu}\Phi(A^r)+\tilde{\nu}I & \text{if } s \geqslant 1,\ -s < r < s/2,\ r \neq 0, \\ \tilde{\mu}\Phi(A^r)+\tilde{\nu}I \leqslant \Phi(A^s) \leqslant \Phi(A^r)^{s/r} & \text{if } r < s \leqslant -1, \end{array}$$

where

$$\tilde{\mu} = \frac{M^s-m^s}{M^r-m^r} = \frac{1}{\bar{\mu}}, \quad \tilde{\nu} = \frac{M^r m^s - M^s m^r}{M^r-m^r} = -\frac{\bar{\nu}}{\bar{\mu}},$$
$$\tilde{\nu}^* = \left(1-\frac{s}{r}\right)\left(\frac{r}{s}\tilde{\mu}\right)^{\frac{s}{s-r}} = -\frac{\nu^*}{\bar{\mu}}.$$

Using the fact that the function $f(t)=t^{\frac{1}{s}}$ is matrix increasing if $s \geqslant 1$ and matrix decreasing if $s \leqslant -1$ we have

$$\begin{array}{ll} \Phi(A^r)^{1/r} \leqslant \Phi(A^s)^{1/s} \leqslant \left[\frac{1}{\bar{\mu}}\Phi(A^r)-\frac{\bar{\nu}}{\bar{\mu}}I\right]^{1/s} & \text{if } s \geqslant 1 \text{ and } (s/2 \leqslant r \leqslant s \text{ or } r \leqslant -s), \\ \left[\frac{1}{\bar{\mu}}\Phi(A^r)+\tilde{\nu}^* I\right]^{1/s} \leqslant \Phi(A^s)^{1/s} \leqslant \left[\frac{1}{\bar{\mu}}\Phi(A^r)-\frac{\bar{\nu}}{\bar{\mu}}I\right]^{1/s} & \text{if } s \geqslant 1,\ -s < r < s/2,\ r \neq 0, \\ \left[\frac{1}{\bar{\mu}}\Phi(A^r)-\frac{\bar{\nu}}{\bar{\mu}}I\right]^{1/s} \geqslant \Phi(A^s)^{1/s} \geqslant \Phi(A^r)^{1/r} & \text{if } r < s \leqslant -1. \end{array} \tag{5.13}$$

Therefore we have the desired inequality similarly to above.

We shall show the bound of the difference in power means on a positive linear map.

THEOREM 5.9. *Let the hypothesis of Theorem 5.7 be satisfied. Then*

$$\beta_2 I \leqslant \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leqslant \beta_1 I \tag{5.14}$$

with

$$\beta_2 = \begin{cases} 0 & \text{if either } (i) \text{ or } (ii) \text{ or } (iii), \\ \Delta^* & \text{if either } (v) \text{ or } (iv), \end{cases} \qquad \beta_1 = \Delta \ \text{ if } \ (vi),$$

where

$$\Delta = \max_{\theta\in[0,1]} \left\{[\theta M^s+(1-\theta)m^s]^{\frac{1}{s}} - [\theta M^r+(1-\theta)m^r]^{\frac{1}{r}}\right\},$$

$$\Delta^* = \min_{\theta\in[0,1]\cup[\frac{d}{M^r-m^r},\frac{d}{M^r-m^r}+1]} \left\{[\theta M^s+(1-\theta)m^s]^{\frac{1}{s}} - [\theta M^r+(1-\theta)m^r-d]^{\frac{1}{r}}\right\},$$

and

$$d = \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\frac{M^r - m^r}{M^s - m^s}\right)^{\frac{r}{r-s}}.$$

The right hand inequality is sharp when r,s satisfy (vi) and the left hand inequality when r,s satisfy (i), or (ii), or (iii).

Proof. We shall consider only the case when $s \neq r$. We prove first (5.14) if $r \notin (-1,1)$. Using (5.12) and the matrix calculus we have

$$\begin{aligned}
&0 = \Phi(A^s)^{1/s} - \Phi(A^s)^{1/s} \leqslant \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \\
&\qquad \leqslant \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} \qquad \text{if either } (iii) \text{ or } (r \leqslant -1, 1 \leqslant s); \\
&\Phi(A^s)^{1/s} - \left[\bar{\mu}\Phi(A^s) + \left(1 - \tfrac{r}{s}\right)\left(\tfrac{s}{r}\bar{\nu}\right)^{\frac{r}{r-s}} I\right]^{1/r} \leqslant \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \\
&\qquad \leqslant \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} \qquad \text{if } (v); \\
&0 = \Phi(A^s)^{1/s} - \Phi(A^s)^{1/s} \leqslant \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \\
&\qquad \leqslant \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} \qquad \text{if } 1 \leqslant r < s.
\end{aligned} \tag{5.15}$$

Therefore, the right hand inequalities (5.15) become

$$\begin{aligned}
\Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} &\leqslant \Phi(A^s)^{1/s} - [\bar{\mu}\Phi(A^s) + \bar{\nu}I]^{1/r} \\
&\leqslant \max_{t \in \bar{T}} \left\{ t^{\frac{1}{s}} - (\bar{\mu}\, t + \bar{\nu})^{\frac{1}{r}} \right\} I,
\end{aligned}$$

where T denotes the open interval joining m^s to M^s, and $\bar{T}$ is the closure of T. We set $\theta = (t - m^s)/(M^s - m^s)$. Then a simple calculation implies $\bar{\mu} \cdot t + \bar{\nu} = \theta M^r + (1-\theta)m^r$, and hence $\max_{t \in \bar{T}} \left\{ t^{\frac{1}{s}} - (\bar{\mu}\, t + \bar{\nu})^{\frac{1}{r}} \right\} = \Delta$. Therefore, we obtain $\beta_1 = \delta$ for $1 \leqslant r \leqslant s$ and $r \leqslant -1 \leqslant s$.

In the case (v) the left hand second inequality (5.15) becomes

$$\begin{aligned}
\Phi(A^s)^{1/s} \;-\; \Phi(A^r)^{1/r} &\geqslant \Phi(A^s)^{1/s} - \left[\bar{\mu}\Phi(A^s) + \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\bar{\nu}\right)^{\frac{r}{r-s}} I\right]^{1/r} \\
&\geqslant \min_{t \in \bar{T}} \left\{ t^{\frac{1}{s}} - \left(\bar{\mu}\, t + \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\bar{\nu}\right)^{\frac{r}{r-s}}\right)^{\frac{1}{r}} \right\} I \\
&= \min_{t \in \bar{T}} \left\{ t^{\frac{1}{s}} - (\bar{\mu}\, t + \bar{\nu} - d)^{\frac{1}{r}} \right\} I \\
&= \min_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta)m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta)m^r - d]^{\frac{1}{r}} \right\} I \geqslant \Delta^* I
\end{aligned}$$

Next, we prove (5.14) if $s \notin (-1,1)$. Using (5.13) and the matrix calculus we have

$$\begin{aligned}
&0=\Phi(A^r)^{1/r}-\Phi(A^r)^{1/r}\leqslant \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r}\\
&\qquad\leqslant \left[\tfrac{1}{\bar\mu}\Phi(A^r)-\tfrac{\bar\nu}{\bar\mu}I\right]^{1/s}-\Phi(A^r)^{1/r} \qquad \text{if either } (ii) \text{ or } (r\leqslant -1, 1\leqslant s);\\
&\left[\tfrac{1}{\bar\mu}\Phi(A^r)-\tfrac{1}{\bar\mu}\left(1-\tfrac{r}{s}\right)\left(\tfrac{s}{r}\bar\nu\right)^{\frac{r}{r-s}}I\right]^{1/s}-\Phi(A^r)^{1/r}\leqslant \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r}\\
&\qquad\leqslant \left[\tfrac{1}{\bar\mu}\Phi(A^r)-\tfrac{\bar\nu}{\bar\mu}I\right]^{1/s}-\Phi(A^r)^{1/r} \qquad \text{if } (iv);\\
&0=\Phi(A^r)^{1/r}-\Phi(A^r)^{1/r}\leqslant \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r}\\
&\qquad\leqslant \left[\tfrac{1}{\bar\mu}\Phi(A^r)-\tfrac{\bar\nu}{\bar\mu}I\right]^{1/s}-\Phi(A^r)^{1/r} \qquad \text{if } r<s\leqslant -1.
\end{aligned} \tag{5.16}$$

Therefore, the right hand inequalities (5.16) become

$$\begin{aligned}
\Phi(A^s)^{1/s}-\Phi(A^r)^{1/r} &\leqslant \left[\frac{1}{\bar\mu}\Phi(A^r)-\frac{\bar\nu}{\bar\mu}I\right]^{1/s}-\Phi(A^r)^{1/r}\\
&\leqslant \max_{t\in\bar T_1}\left\{\left(\frac{1}{\bar\mu}\,t-\frac{\bar\nu}{\bar\mu}\right)^{\frac{1}{s}}-t^{\frac{1}{r}}\right\}I,
\end{aligned}$$

where T_1 denotes the open interval joining m^r to M^r, and $\bar T_1$ is the closure of T_1. We set $\theta=(t-m^r)/(M^r-m^r)$. Then simple calculation implies $\frac{1}{\bar\mu}\cdot t-\frac{\bar\nu}{\bar\mu}=\theta M^s+(1-\theta)m^s$, and hence $\max_{t\in\bar T_1}\left\{\left(\frac{1}{\bar\mu}\,t-\frac{\bar\nu}{\bar\mu}\right)^{\frac{1}{s}}-t^{\frac{1}{r}}\right\}=\Delta$. Therefore, we obtain $\beta_1=\delta$ for $r\leqslant s\leqslant -1$ and $-1\leqslant r\leqslant 1\leqslant s$.

In the case (iv) the left hand of second inequality (5.16) becomes

$$\begin{aligned}
\Phi(A^s)^{1/s}-\Phi(A^r)^{1/r} &\geqslant \left[\frac{1}{\bar\mu}\Phi(A^r)-\frac{1}{\bar\mu}\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar\nu\right)^{\frac{r}{r-s}}I\right]^{1/s}-\Phi(A^r)^{1/r}\\
&\geqslant \min_{t\in\bar T_1}\left\{\left[\frac{1}{\bar\mu}t-\frac{1}{\bar\mu}\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar\nu\right)^{\frac{r}{r-s}}\right]^{1/s}-t^{\frac{1}{r}}\right\}I\\
&= \min_{\theta\in[0,1]}\left\{[\theta M^s+(1-\theta)m^s+\frac{d}{\bar\mu}]^{\frac{1}{s}}-[\theta M^r+(1-\theta)m^r]^{\frac{1}{r}}\right\}I\\
&= \min_{\theta\in[\frac{d}{M^r-m^r},\frac{d}{M^r-m^r}+1]}\left\{[\theta M^s+(1-\theta)m^s]^{\frac{1}{s}}-[\theta M^r+(1-\theta)m^r-d]^{\frac{1}{r}}\right\}I\\
&\geqslant \Delta^* I.
\end{aligned}$$

As applications, we have the following corollaries [9, Corollaries 3.3 and 3.4] which are given both inequalities but with errors in the bounds estimate: In fact, we obtain Corollary 5.10 (resp. Corollary 5.11) if we put $s=1$ and $r=p$ in Theorem 5.7 (resp. Theorem 5.9) respectively.

COROLLARY 5.10. *Let Φ be a positive linear map and A a positive definite Hermitian matrix with spectrum contained in $[m,M]$. Let p be a nonzero real number. If $\gamma=M/m$, then*

$$\alpha_2\Phi(A^p)^{1/p}\leqslant \Phi(A)\leqslant \alpha_1\Phi(A^p)^{1/p}$$

with

$$\alpha_2 = \begin{cases} \bar{\Delta}^{-1} & \text{if } -1 < p < 0 \text{ or } 0 < p < 1/2 \\ 1 & \text{if } p \leqslant -1 \text{ or } 1/2 \leqslant p \leqslant 1 \\ \bar{\Delta} & \text{if } p > 1, \end{cases}$$

$$\alpha_1 = \begin{cases} \bar{\Delta} & \text{if } p < 0 \text{ or } 0 < p < 1, \\ 1 & \text{if } p \geqslant 1, \end{cases}$$

where

$$\bar{\Delta} = (\gamma - 1)^{\frac{1}{p}} \frac{p}{\gamma^p - 1} \left\{ \frac{p-1}{\gamma^p - \gamma} \right\}^{\frac{1-p}{p}}.$$

COROLLARY 5.11. *Let Φ be a positive linear map and A a positive definite Hermitian matrix with spectrum contained in $[m,M]$. Let p be a nonzero real number. If $\gamma = M/m$, then*

$$\beta_2 I \leqslant \Phi(A) - \Phi(A^p)^{1/p} \leqslant \beta_1 I$$

with

$$\beta_2 = \begin{cases} -\Delta & \text{if } -1 < p < 0 \text{ or } 0 < p < 1/2 \\ 0 & \text{if } p \leqslant -1 \text{ or } 1/2 \leqslant p \leqslant 1 \\ \Delta & \text{if } p > 1, \end{cases}$$

$$\beta_1 = \begin{cases} \Delta & \text{if } p < 0 \text{ or } 0 < p < 1, \\ 0 & \text{if } p \geqslant 1, \end{cases}$$

where

$$\Delta = M \frac{1-\gamma^{p-1}}{1-\gamma^p} + m\left(\frac{1}{p} - 1\right) \left\{ \frac{p(\gamma-1)}{\gamma^p - 1} \right\}^{\frac{1}{1-p}}.$$

6. Applications II

6.1. Application to matrix version of Ky Fan inequality

By application of Theorem 3.1 on a special map $\Phi(A) = UAU^*$, we obtain the following result which is an extension of [19, Theorem 12].

COROLLARY 6.1. *Let A_j be positive definite Hermitian matrices of order n such that $0 < m \leqslant A_j \leqslant M$ $(j = 1,2,\cdots,n)$ and also let U_j $(j = 1,2,\cdots,n)$ be $r \times n$ matrices such that $\sum_{j=1}^{n} U_j U_j^* = I$. Let f and g be real valued continuous functions on $[m,M]$. Let $F(u,v)$ be a real valued function defined on $U \times V$, matrix non-decreasing in u, where U and V are intervals such that $U \supset f[m,M]$ and $V \supset g[m,M]$. Then the following inequalities hold*

$$\begin{aligned} \left\{ \max_{\substack{k \in \{conx.\} \\ k \leqslant f}} \min_{m \leqslant t \leqslant M} F[k(t), g(t)] \right\} I &\leqslant F[\sum_{j=1}^{n} U_j f(A_j) U_j^*, g(\sum_{j=1}^{n} U_j A_j U_j^*)] \\ &\leqslant \left\{ \min_{\substack{k \in \{conc.\} \\ k \geqslant f}} \max_{m \leqslant t \leqslant M} F[k(t), g(t)] \right\} I. \qquad (6.1) \end{aligned}$$

Proof. Let U be any unitary matrix of order n. We define map $\Phi : H_n \to \mathbb{C}$ with $\Phi(B) = UBU^*$ $(\forall B \in H_n)$. It is evident that this map is a normalized positive linear map from H_n to H_n. Then from Theorem 3.1 the following inequality

$$\left\{ \max_{\substack{k\in\{conx.\}\\ k\leqslant f}} \min_{m\leqslant t\leqslant M} F\left[k(t), g(t)\right] \right\} I \leqslant F[Uf(A)U^*, g(UAU^*)]$$
$$\leqslant \left\{ \min_{\substack{k\in\{conc.\}\\ k\geqslant f}} \max_{m\leqslant t\leqslant M} F\left[k(t), g(t)\right] \right\} I \qquad (6.2)$$

holds for any Hermitian matrix A with spectrum contained in $[m, M]$. Furthermore, for A_j and U_j $(j = 1, 2, \cdots, n)$ from the hypothesis of corollary we have $\sum_{j=1}^{n} U_j A_j U_j^* = UAU^*$, where $A = A_1 \dot{+} A_2 \dot{+} \cdots \dot{+} A_n$, $U = [U_1 U_2 \cdots U_k]$ and $\sum_{j=1}^{n} U_j f(A_j) U_j^* = Uf(A)U^*$. If we put this A and U in (6.2) we obtain the desired inequality (6.1).

COROLLARY 6.2. *If the conditions of Corollary 6.1 are satisfied, then for a given real number* α

$$\alpha g(\sum_{j=1}^{n} U_j A_j U_j^*) + \beta_2 I \leqslant \sum_{j=1}^{n} U_j f(A_j) U_j^* \leqslant \alpha g(\sum_{j=1}^{n} U_j A_j U_j^*) + \beta_1 I$$

holds for

$$\beta_1 = \left[\min_{\substack{k \in \{conc.\}\\ k \geqslant f}} \max_{m\leqslant t\leqslant M} \{k(t) - \alpha g(t)\} \right] I,$$

$$\beta_2 = \left[\max_{\substack{k \in \{conx.\}\\ k \leqslant f}} \min_{m\leqslant t\leqslant M} \{k(t) - \alpha g(t)\} \right] I.$$

We now consider the means

$$M_n^{[r]}(A;U) = \left(\sum_{i=1}^{n} U_i A_i^r U_i^* \right)^{1/r}, \quad r \neq 0,$$

for positive definite Hermitian matrices A_j of order n such that $0 < m \leqslant A_j \leqslant M$ $(j = 1, 2, \cdots, n)$ and matrices U_j $(j = 1, 2, \cdots, n)$ of order $r \times n$ such that $\sum_{j=1}^{n} U_j U_j^* = I$.

If we put $\Phi(B) = UBU^*$, $B \in H_n$ in Theorems 5.7 and 5.9 we have the following results which are an extension of [12, Theorems 2 and 3].

COROLLARY 6.3. *Let A_j be positive definite Hermitian matrices of order n such that $0 < m \leqslant A_j \leqslant M$ $(j = 1, 2, \cdots, n)$ and also let U_j $(j = 1, 2, \cdots, n)$ be $r \times n$*

matrices such that $\sum_{j=1}^{n} U_jU_j^* = I$. *Also let* r *and* s *be nonzero real numbers and* (i) - (vi) *as in (5.6). Then*

$$\alpha_2 M_n^{[r]}(A;U) \leqslant M_n^{[s]}(A;U) \leqslant \alpha_1 M_n^{[r]}(A;U) \tag{6.3}$$

with

$$\alpha_2 = \begin{cases} 1 & \text{if either } (i) \text{ or } (ii) \text{ or } (iii) \\ \tilde{\Delta}^{-1} & \text{if either } (iv) \text{ or } (v), \end{cases} \qquad \alpha_1 = \tilde{\Delta} \quad \text{if} \quad (vi),$$

where

$$\tilde{\Delta} = \left\{ \frac{r(\gamma^s - \gamma^r)}{(s-r)(\gamma^r - 1)} \right\}^{\frac{1}{s}} \left\{ \frac{s(\gamma^r - \gamma^s)}{(r-s)(\gamma^s - 1)} \right\}^{-\frac{1}{r}}.$$

and $\gamma = M/m$.

COROLLARY 6.4. *Let* A_j, U_j $(j = 1, 2, \cdots, n)$ *and* r, s, γ *be as in Corollary 6.3. Then*

$$\beta_2 I \leqslant M_n^{[s]}(A;U) - M_n^{[r]}(A;U) \leqslant \beta_1 I \tag{6.4}$$

with

$$\beta_2 = \begin{cases} 0 & \text{if either } (i) \text{ or } (ii) \text{ or } (iii), \\ \Delta^* & \text{if either } (v) \text{ or } (iv), \end{cases} \qquad \beta_1 = \Delta \quad \text{if} \quad (vi),$$

where

$$\Delta = \max_{\theta \in [0,1]} \left\{ [\theta M^s + (1-\theta)m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta)m^r]^{\frac{1}{r}} \right\},$$

$$\Delta^* = \min_{\theta \in [0,1] \cup [\frac{d}{M^r - m^r}, \frac{d}{M^r - m^r} + 1]} \left\{ [\theta M^s + (1-\theta)m^s]^{\frac{1}{s}} - [\theta M^r + (1-\theta)m^r - d]^{\frac{1}{r}} \right\},$$

and

$$d = \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right) \left(\frac{s}{r} \frac{M^r - m^r}{M^s - m^s} \right)^{\frac{r}{r-s}}.$$

REMARK 6.5. In [9] it was claimed that the results of Mond and Pečarić: $M_n^{[s]}(A;U) \geqslant M_n^{[r]}(A;U)$ when r, s satisfy (i), or (ii), or (iii), can be easily extended to arbitrary normalized positive maps Φ so that one replaces A by A^r and p by r/s in Corollary 5.11. This is not correct because we need Corollary 5.8 for the proof of Theorem 5.9.

6.2. Application to Hadamard product

In this section, we shall show an Hadamard product version corresponding to Theorem 3.1. The Hadamard product of matrices is expressed as the image of a normalized positive linear map. Let $E_{ij} \in H_n$ be the matrix of zeros except in the (i,j) position. Define a $n \times n^2$ matrix P^T such that

$$P^T = [E_{11} : E_{22} : \cdots : E_{nn}].$$

If $A,B \in H_n$ then

$$A \circ B = P^T (A \otimes B) P,$$

where $\circ$ and $\otimes$ denote the Hadamard and Kronecker products, respectively [21, p. 276].

For the sake of convenience, we prepare some notations. Let A and B be positive definite Hermitian matrices such that $0 < m_1 \leqslant A \leqslant M_1$ and $0 < m_2 \leqslant B \leqslant M_2$. Put $m = m_1 m_2$ and $M = M_1 M_2$. Also, let f be a real valued continuous function defined on an interval including $[m,M]$. Then f is called super-multiplicative (resp. sub-multiplicative) if $f(xy) \geqslant f(x)f(y)$ (resp. $f(xy) \leqslant f(x)f(y)$) (cf [5]). We define: $X_f = [m_1,M_1] \cup [m_2,M_2] \cup [m,M]$. Also, we denote $\{supcc.\}$ (resp. $\{subcx.\}$) the set of real valued continuous super-multiplicative matrix concave (resp. sub-multiplicative matrix convex) functions defined on X_f.

We shall show the following theorem which are an extension of [20, Theorem 2].

THEOREM 6.6. *Let A and B be Hermitian matrices such that $0 < m_1 \leqslant A \leqslant M_1$ and $0 < m_2 \leqslant B \leqslant M_2$. Put $m = m_1 m_2$ and $M = M_1 M_2$. Let Φ_1 and Φ_2 be normalized positive linear maps from H_n to $H_{\tilde{n}}$. Let f and g be two real valued continuous function, f defined on the interval X_f, g on $[m,M]$. Let J_1 an interval including $\{f(t)f(s) : t \in [m_1,M_1], s \in [m_2,M_2]\}$ and J_2 an interval including $\{g(t) : t \in [m,M]\}$. If $F(u,v)$ is a real valued function defined on $J_1 \times J_2$, matrix non-decreasing in u, then the following inequalities hold*

$$F[\Phi_1(f(A)) \circ \Phi_2(f(B)), g(\Phi_1(A) \circ \Phi_2(B))] \leqslant \left\{ \min_{\substack{k \in \{supcc.\} \\ k \geqslant f}} \max_{m \leqslant t \leqslant M} F[k(t), g(t)] \right\} I, \tag{6.5}$$

$$F[\Phi_1(f(A)) \circ \Phi_2(f(B)), g(\Phi_1(A) \circ \Phi_2(B))] \geqslant \left\{ \max_{\substack{k \in \{subcx.\} \\ k \leqslant f}} \min_{m \leqslant t \leqslant M} F[k(t), g(t)] \right\} I. \tag{6.6}$$

Proof. Since k is a continuous super-multiplicative matrix concave function such that $f(t) \leqslant k(t)$ for all $t \in X_f$, it follows from the spectral theorem that $f(A) \leqslant k(A)$. Using the positivity of Φ_1 we have $\Phi_1(f(A)) \leqslant \Phi_1(k(A))$. Applying Jensen's inequality on function $-k$ (which is a matrix convex function) we obtain inequality $\Phi_1(k(A)) \leqslant k(\Phi_1(A))$. Then it follows that $\Phi_1(f(A)) \leqslant k(\Phi_1(A))$. Also, $\Phi_2(f(B)) \leqslant k(\Phi_2(B))$ for any Hermitian matrix B with spectrum contained in $[m_2,M_2]$. Further, using the following general formula for tensor products (see [1, p. 216]): $A_1 \geqslant A_2 \geqslant 0$ and $B_1 \geqslant B_2 \geqslant 0$ imply $A_1 \otimes B_1 \geqslant A_2 \otimes B_2$, then we obtain

$$\begin{aligned} \Phi_1(f(A)) \circ \Phi_2(f(B)) &= P^T (\Phi_1(f(A)) \otimes \Phi_2(f(B))) P \\ &\leqslant P^T (k(\Phi_1(A)) \otimes k(\Phi_2(B))) P = k(\Phi_1(A)) \circ k(\Phi_2(B)), \end{aligned}$$

so that it follows from the super-multiplicative and matrix concavity of k that

$$\begin{aligned} P^T (k(\Phi_1(A)) \otimes k(\Phi_2(B))) P &\leqslant k(P^T ((\Phi_1(A)) \otimes (\Phi_2(B))) P) \\ &= k(\Phi_1(A) \circ \Phi_2(B)). \end{aligned}$$

Using the matrix non-decreasing character of $F(\cdot,v)$, we have

$$\begin{aligned}F[\Phi_1(f(A))\circ\Phi_2(f(B)),g(\Phi_1(A)\circ\Phi_2(B))]&\\ \leqslant F[k(\Phi_1(A)\circ\Phi_2(B)),g(\Phi_1(A)\circ\Phi_2(B))]&\\ \leqslant\left\{\max_{t\in\sigma(\Phi_1(A)\circ\Phi_2(B))}F[k(t),g(t)]\right\}I\leqslant\left\{\max_{m\leqslant t\leqslant M}F[k(t),g(t)]\right\}I.&\end{aligned}$$

Now we minimize this bound over all continuous super-multiplicative matrix concave functions k to obtain the inequality (6.5). The inequality (6.6) is proved in the same way.

REMARK 6.7. Notice that we can obtain similar results in case $F(u,v)$ is a real valued function matrix non-increasing in u.

REMARK 6.8. Notice that the constant function $k(t)=\max_{m\leqslant s\leqslant M}f(s)$ for all $t\in X_f$ is a continuous super-multiplicative matrix concave function that bounds f from above. Since we are optimizing over the right-hand side of (6.5) super-multiplicative matrix concave functions, we can show that there are indeed a function k that attains the extreme.

THEOREM 6.9. *Let the hypothesis of Theorem 6.6 be satisfied and $h(t)$ be defined by*

$$h(t)\equiv h(t;m,M,f)=\mu_f\cdot t+\nu_f. \tag{6.7}$$

If f is a real valued continuous convex function on X_f and $h(t)$ is a super-multiplicative function on $[m,M]$, then the following inequality holds

$$F[\Phi_1(f(A))\circ\Phi_2(f(B)),g(\Phi_1(A)\circ\Phi_2(B))]\leqslant\left\{\max_{m\leqslant t\leqslant M}F[h(t),g(t)]\right\}I, \tag{6.8}$$

but, if f is a real valued continuous concave function and $h(t)$ is a sub-multiplicative function on $[m,M]$, then the following inequality holds

$$F[\Phi_1(f(A))\circ\Phi_2(f(B)),g(\Phi_1(A)\circ\Phi_2(B))]\geqslant\left\{\min_{m\leqslant t\leqslant M}F[h(t),g(t)]\right\}I. \tag{6.9}$$

Proof. The convexity of f ensures that $f(t)\leqslant h(t)$ for all $t\in[m,M]$. Obviously $h(t)$ is a matrix concave function. If k is a matrix concave function and $f(t)\leqslant k(t)$ for all $t\in[m,M]$ then $h(m)=f(m)\leqslant k(m)$ and $h(M)=f(M)\leqslant k(M)$. Because a matrix concave function is necessarily concave, we have $h(t)\leqslant k(t)$ for all $t\in[m,M]$. Using the (matrix) non-decreasing character of $F(\cdot,v)$, we have

$$F[h(t),g(t)]I\leqslant F[k(t),g(t)]I\quad\text{for all } t\in[m,M].$$

It follows from this that the minimum in the right hand of (6.5) is attained at h. Thus we proved the inequality (6.8). The inequality (6.9) can be proved in the same way.

A positive definite matrix, with all its main diagonal entries equal to 1 and all its entries bounded in absolute value by 1, is called a correlation matrix. If we put

$\Phi(A) = A \circ B$ where B is a correlation matrix, then it follows that Φ is a normalized positive linear map. Therefore we have the following corollary by Theorem 3.1.

COROLLARY 6.10. *Let A and B be positive definite Hermitian matrices such that $0 < m_1 \leqslant A \leqslant M_1$ and $0 < m_2 \leqslant B \leqslant M_2$. Put $m = m_1 m_2$ and $M = M_1 M_2$. Let f and g be real valued continuous functions on $[m_1, M_1]$ and $[m, M]$, respectively. Let J_1 an interval including $\{f(t) \cdot s : t \in [m_1, M_1], s \in [m_2, M_2]\}$ and J_2 an interval including $\{g(t) : t \in [m, M]\}$. If $F(u, v)$ is a real valued function defined on $J_1 \times J_2$, matrix non-decreasing in u, then the following inequalities hold*

$$F(f(A) \circ B, g(A \circ B)) \leqslant \left\{ \min_{\substack{k \in \{conc.\} \\ k \geqslant f}} \max_{m \leqslant t \leqslant M} F[k(t), g(t)] \right\} I,$$

and

$$F(f(A) \circ B, g(A \circ B)) \geqslant \left\{ \max_{\substack{k \in \{conx.\} \\ k \leqslant f}} \min_{m \leqslant t \leqslant M} F[k(t), g(t)] \right\} I.$$

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