Mathematical
Inequalities
& Applications
Volume 3, Number 4 (2000), 593–614

STABILITY OF SEMILINEAR STOCHASTIC EVOLUTION EQUATIONS WITH MONOTONE NONLINEARITY

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(*communicated by G. Peskir*)

Abstract. In this paper, we consider the exponentially asymptotic stability of the mild solutions of semilinear stochastic evolution equations of monotone type. An Itô-type inequality is our main tool to study the stability in the p-th moment and almost sure sample-path stability of the mild solutions. We also give some examples to illustrate the applications of the theorems.

1. Introduction

Extending the stability theory of Itô equations [11, 19] to stochastic evolution equations (or SEE's for short) has been studied by many authors. We may point out Haussmann [7], Zabczyk [26], Ichikawa [8, 9, 10], Caraballo and Real [2], Taniguchi [24], R. Liu and Mandrekar [17], K. Liu and Mao [16] and Leha, Maslowski and Ritter [14], among the others. In particular, Haussmann [7] studied pathwise exponential stability with probability one for linear SEE's, in the sense of strong and mild solutions. In [9, 10], Ichikawa considered semilinear SEE's of Lipschitz type and proved the stability of the moments, pathwise continuity and stability, and also the existence of invariant measures for the mild solutions, by the so-called Liapunov's second method. Mandrekar [17] and R. Liu and Mandrekar [18] proposed another Liapunov function, different from the one in [7, 9], to handle the asymptotic stability and ultimate boundedness of the strong and mild solutions, and K. Liu and Mao [16] studied the almost sure stability of strong solutions to non-linear SEE's, by virtue of exponential martingale formula.

It is well-known that the mild solutions do not necessarily have stochastic differentials, so one can not apply the Itô's formula to them. But this is needed when the Liapunov's direct method is used to prove the stability theorems. For this reason, the proofs in [9], [10], [14], [17] and [18] are based on an extended version of the Itô's formula for the mild solutions deduced by first introducing approximating systems with strong solutions and then using a limiting argument. This approach followed also in [15] to prove the almost sure stability, with a general decay function, of the mild solutions of semilinear SEE's with variable delay.

Mathematics subject classification (1991): 60H15, 34G20.

Key words and phrases: Stochastic evolution equations, Itô-type inequality, Stability in the p-th moment, Borel-Cantelli's lemma, Sample-path stability.

This work is partially supported by the Research Council of Sharif University of Technology.

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Paper MIA-03-53

Taniguchi [24] and recently Caraballo and K. Liu [3] studied the same stability problems as in [9, 10], by using the properties of stochastic convolution integral [6] and giving an L^p-estimate for the difference of two mild solutions with different initial values. In particular, the results in [3] allow one to ensure the stability of moments and paths in some cases where Ichikawa's ones do not give any answer.

In this paper, we will investigate the asymptotic stability of the semilinear SEE's (4.1) in which not only the operator $A(t)$ depends explicitly on t but also while the diffusion coefficient g is assumed to be Lipschitz (Hypothesis $3_\lambda(d)$ in Section 4), the non-linear drift f satisfies the semimonotone condition (Hypothesis $3_\lambda(c)$) which is weaker than the Lipschitz one. Concerning such equations, however, neither the technique employed in [9, 10] nor the one developed in [3] and [24] can be applied, since the Lipschitz condition has an essential role in both of them. To overcome the difficulty, we will give in Section 3 an Itô-type inequality which is our main tool to prove the exponential stability of the mild solutions. This way, we can derive the results in [7], [8], [9], [10], [17] and [24] as corollaries to our main theorems (Theorems 4 and 5).

The paper is organized as follows. In Section 2, we give some definitions and preliminaries including a number of notions from semigroup theory, Brownian motion and stochastic integration on Hilbert spaces. In Section 3, we prove an Itô-type inequality which extends the results obtained in [28] and [29] to the powers $p \geqslant 2$. In Section 4, we prove the exponential stability in the p-th moment of the mild solutions. In Section 5, we consider the sample-path asymptotic behaviour of the mild solutions and finally in Section 6, we will apply the results established in the previous sections to several stochastic evolution equations of monotone type.

2. Preliminaries

Let H and K be two real separable Hilbert spaces. We use the same notations $\|\,.\,\|$ and $<.\,,\,.>$ for the norm and inner product in H as well as in K. We denote by $L(K,H)$ the space of all bounded linear operators from K to H with the usual operator norm $\|\,.\,\|_L$. Moreover, $L(K) = L(K,K)$. Let T be a positive real number.

DEFINITION 1. A family $\{\, U(t,s) \,:\, 0 \leqslant s \leqslant t \leqslant T \,\}$ of bounded linear operators on H is said to be an *evolution operator* if

(a) $U(t,t) = I, \quad U(t,s)\,U(s,r) = U(t,r) \quad$ for $\quad 0 \leqslant r \leqslant s \leqslant t \leqslant T,$
where I is the identity operator;

(b) $(t,s) \to U(t,s)$ is strongly continuous for $0 \leqslant s \leqslant t \leqslant T$.

DEFINITION 2. [4] Let $\{A(t) : 0 \leqslant t \leqslant T\}$ be a family of closed densely defined linear operators on H with domain D independent of $t \in S := [0,T]$. We say the evolution operator $U(t,s)$ is an *almost strong evolution operator* with *generator* $A(t)$, if it satisfies the following:

(a) For almost every $s \leqslant t$ and for each $x \in D$

$$U(t,s)\,x - x = \int_s^t U(t,r)A(r)\,x\,dr;$$

(b) Let $x \in D$ and $s \in S$. For almost every $t > s$

$$U(t,s)D \subseteq D,$$

and

$$\int_s^t A(r)\,U(r,s)\,x\,dr = (U(t,s) - I)\,x.$$

Let $B(t,s) := A(t)[\,\mu I - A(s)\,]^{-1}$. The following are the relevant hypotheses concerning A and U:

HYPOTHESIS 1_λ. (a) *The domain* $\mathscr{D}(A(t)) = D$ *of* $A(t)$ *is independent of* t *for* $t \in S$ *and is dense in* H*;*

(b) $\{A(t) : t \in S\}$ *generates a unique almost strong evolution operator* $U(t,s)$*;*

(c) $U(t,s)$ *is exponentially bounded with parameter* λ *on* S*, i.e.,* $\|U(t,s)\|_L \leqslant e^{\lambda(t-s)}$ *for almost every* $0 \leqslant s \leqslant t \leqslant T$*;*

(d) $B(t,s)$ *is uniformly bounded in* (t,s)*; that is, for* $\mu \geqslant \lambda$ *there exists a constant* $K(\mu) > 0$ *such that* $\|B(t,s)\|_L \leqslant K(\mu)$ *for every* s,t *(this is the case if* $B(t,s)$ *is continuous in* t *in the sense of the norm* $\|\,.\,\|_L$*, at least for some* s*).*

Remark 1. Note that if an almost strong evolution operator $U(t,s)$ is exponentially bounded with parameter λ on S, then

$$< A(t)x\,,\, x > \leqslant \lambda\|x\|^2, \qquad \forall x \in D,$$

for a.e. $t \in S$.

We refer to [22] and [23] for sufficient conditions for the existence of an evolution operator with the properties $1_\lambda(a)$–(d). These conditions apply to a large class of delay equations, and to parabolic and hyperbolic equations (see for example [5]).

Assume that $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is a complete stochastic basis with a right continuous filtration. We follow [25] and define cylindrical Brownian motion on K as

DEFINITION 3. A family of random linear functionals $\{W_t : t \geqslant 0\}$ on K is called a *cylindrical Brownian motion* on K if it satisfies the following conditions:

(i) $W_0 = 0$ and $W_t(x)$ is $\mathscr{F}_t$-adapted for every $x \in K$;

(ii) For every $x \in K$ such that $x \neq 0$, $W_t(x)/\|x\|$ is a one-dimensional Brownian motion.

Note that cylindrical Brownian motion is not K-valued because its covariance is not nuclear. For the properties of cylindrical Brownian motion and its relation to other definitions of Brownian motion, see [25].

DEFINITION 4. Let $\phi : [0,\infty) \to K$ be an $\mathscr{F}_t$-adapted, K-valued, predictable process such that $E[\int_0^t \|\phi(s)\|^2 ds] < \infty$ for all $t \geqslant 0$. The stochastic integral of $\phi(t)$ with respect to $\{W_t : t \geqslant 0\}$ is a real-valued martingale given by

$$\int_0^t < \phi(s)\,,\, dW_s > = \sum_{n=1}^{\infty} \int_0^t < \phi(s)\,,\, e_n > dW_s(e_n),$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis of K.

Let $L_2(K,H)$ be the space of Hilbert-Schmidt operators from K to H with the Hilbert-Schmidt norm $\|\,.\,\|_2$. Now, we define the H-valued stochastic integral with respect to a cylindrical Brownian motion.

DEFINITION 5. Let $\Phi : [0,\infty) \to L_2(K,H)$ be an $\mathscr{F}_t$-adapted, $L_2(K,H)$-valued, predictable process such that $E[\int_0^t \|\Phi(s)\|_2^2\, ds] < \infty$ for all $t \geqslant 0$. The stochastic integral of $\Phi(t)$ is an H-valued continuous martingale given by

$$< h\,, \int_0^t \Phi(s) dW_s > = \int_0^t < \Phi^*(s)h\,,\, dW_s >, \qquad \forall h \in H,$$

where $\Phi^*(s)$ is the adjoint operator of $\Phi(s)$.

If $\Phi(t)$ satisfies the conditions of Definition 5 and if $M(t) = \int_0^t \Phi(s) dW_s$, then we have

$$E\,[\,\sup_{0 \leqslant s \leqslant t} |M(s)|\,] \leqslant 3E \left(\int_0^t \|\Phi(s)\|_2^2\, ds \right)^{\frac{1}{2}}, \qquad \forall t \geqslant 0. \tag{2.1}$$

This is a consequence of a general inequality for square-integrable martingales [21]. For other properties of stochastic integral with respect to cylindrical Brownian motion, see [6].

3. An Itô-type Inequality

Let $M(t)$ be an H-valued local martingale and consider the *stochastic convolution integral* $X(t) = \int_0^t U(t,s) dM_s$. Notice that because the integrand depends on t as well as on s, $X(t)$ is not necessarily a local martingale. However, it is possible to prove some results analogous to the well-known martingale inequalities. Kotelenez [12, 13] proved submartingale type and stopped Doob inequalities for $X(t)$. Zangeneh [28] proved an Itô-type inequality and a Burkhölder-type inequality for this object.

Let us state the main result of [28] on Itô-type inequality.

THEOREM 1. *Let $Z(t)$, $0 \leqslant t \leqslant T$, be an H-valued, cadlag, locally square-integrable semimartingale. Suppose U and A satisfy Hypothesis $1_\lambda(a)$–(c). If*

$$X(t) \;=\; U(t,0)X_0 + \int_0^t U(t,s) dZ_s,$$

then

$$\begin{aligned} \|X(t)\|^2 \;&\leqslant\; e^{2\lambda t}\|X_0\|^2 + 2\int_0^t e^{2\lambda(t-s)} < X(s^-)\,,\, dZ_s > \\ &\quad + e^{2\lambda t}\left[\int_0^{\cdot} e^{-\lambda s} dZ_s\right]_t, \qquad \forall t \in [0,T], \end{aligned}$$

where $[\quad]_t$ stands for the quadratic variation process.

In this section, we will adopt the same approach as in [28] to prove an Itô-type inequality which extends Theorem 1 (for a special semimartingale $Z(t)$) to the powers $p > 2$. Throughout the section, we assume that $p \geqslant 2$ and $f(t)$ and $g(t)$ are two processes defined on $[0,T]$ with values in H and $L_2(K,H)$, respectively, and satisfy

$$\int_0^T E\|f(t)\|^p dt < \infty, \quad \int_0^T E\|g(t)\|_2^p dt < \infty. \tag{3.1}$$

Our main result in this section is

THEOREM 2. *Let X_0 be an H-valued, $\mathscr{F}_0$-measurable random variable. Suppose U and A satisfy Hypothesis 1_λ. If*

$$X(t) = U(t,0)X_0 + \int_0^t U(t,s)f(s)ds + \int_0^t U(t,s)g(s)dW_s, \tag{3.2}$$

then

$$\begin{aligned}\|X(t)\|^p \;\leqslant\;& e^{p\lambda t}\|X_0\|^p + p\int_0^t e^{p\lambda(t-s)}\|X(s)\|^{p-2} < X(s)\,, f(s) > ds \\ &+ p\int_0^t e^{p\lambda(t-s)}\|X(s)\|^{p-2} < X(s)\,,\, g(s)dW_s > \\ &+ \frac{p(p-1)}{2}\int_0^t e^{p\lambda(t-s)}\|X(s)\|^{p-2}\|g(s)\|_2^2 ds,\end{aligned} \tag{3.3}$$

for all $t \in S$.

Remark 2. We can assume that X_0 is bounded in norm; otherwise, we replace X_0 by $X_0 1_{\{\|X_0\| \leqslant n\}}$ and pass to limit as $n \to \infty$.

In order to prove the theorem, we need several lemmas.

Suppose that $U(t,s)$ satisfies Hypothesis $1_\lambda(c)$ for some $\lambda \in \mathbb{R}$. Define

$$U_1(t,s) = e^{-\lambda(t-s)}U(t,s), \quad A_1(t) = A(t) - \lambda I, \quad Z^1(t) = \int_0^t e^{-\lambda s}dZ_s,$$

and $X^1(t) = e^{-\lambda t}X(t)$, where $Z(t) := \int_0^t f(s)ds + \int_0^t g(s)dW_s$ for all $t \in S$.

LEMMA 1. *If U and A satisfy Hypothesis 1_λ, then U_1 and A_1 satisfy Hypothesis 1_λ with $\lambda = 0$. Moreover, $X(t)$ satisfies (3.2) if and only if $X^1(t)$ satisfies*

$$X^1(t) = U_1(t,0)X_0 + \int_0^t U_1(t,s)dZ_s^1.$$

Proof. See [28]. □

In view of Lemma 1, it suffices for the theorem to be proved when $U(t,s)$ is exponentially bounded with parameter $\lambda = 0$ on S. Then, for all $x \in D$, $< A(t)x, x > \leqslant 0$ for a.e. $t \in S$.

We proceed as in [9, 10] and approximate $X(t)$ by Yosida method. Define the map $R_n(t) : H \to D$ by $R_n(t) = n(nI - A(t))^{-1}$. Then, $\|R_n(t)\|_L \leqslant 1$ for a.e.

$t \in S$. Replace $f(s)$ and $g(s)$ in (3.2) by $f_n(s) = R_n(s)f(s)$ and $g_n(s) = R_n(s)g(s)$, respectively, and let $\{X_0^n\}_{n=1}^{\infty}$ be a sequence in D which converges almost surely to X_0 such that $\|X_0^n\| \leqslant \|X_0\|$ for all n.

LEMMA 2. *If*

$$X_n(t) = U(t,0)X_0^n + \int_0^t U(t,s)f_n(s)ds + \int_0^t U(t,s)g_n(s)dW_s, \tag{3.4}$$

then $\sup_{0\leqslant t\leqslant T} \|X_n(t) - X(t)\| = \|X_n - X\|_\infty \to 0$ *in* L^p, *as* $n \to \infty$.

Proof. According to Burkhölder-type inequality for stochastic convolution integrals [28]

$$E\left\{\left\|\int_0^{\cdot} U(.,s)(g_n(s) - g(s))dW_s\right\|_\infty^p\right\} \leqslant k_p\, E\int_0^T \|g_n(s) - g(s)\|_2^p\, ds,$$

where k_p is a positive constant which depends only on p. Since $R_n(s) \to I$ strongly and $\|R_n(s) - I\| \leqslant 2$ a.e., by the dominated convergence theorem

$$E\int_0^T \|g_n(s) - g(s)\|_2^p\, ds \to 0. \tag{3.5}$$

Hence,

$$\left\|\int_0^{\cdot} U(.,s)(g_n(s) - g(s))dW_s\right\|_\infty \to 0 \qquad \text{in } L^p.$$

Note also that

$$\|U(t,0)(X_0^n - X_0)\|^p \leqslant \|X_0^n - X_0\|^p \to 0 \qquad \text{boundedly,}$$

and by Hölder inequality

$$E\left\|\int_0^{\cdot} U(.,s)(f_n(s) - f(s))ds\right\|_\infty^p \leqslant T^{p-1}E\int_0^T \|f_n(s) - f(s)\|^p ds.$$

But,

$$E\int_0^T \|f_n(s) - f(s)\|^p ds \to 0,$$

by the dominated convergence theorem. Hence, $\|X_n - X\|_\infty \to 0$ in L^p. □

LEMMA 3. *Let* $(\Omega, \mathscr{F}, \mu)$ *be a measure space and* f_n, g_n *be integrable functions on* Ω *with* $|f_n| \leqslant g_n$. *Suppose that as* $n \to \infty$, $f_n(x) \to f(x)$ *and* $g_n(x) \to g(x)$, *for almost every* $x \in \Omega$. *If* $\int_\Omega g_n d\mu \to \int_\Omega g d\mu$, *then* $\int_\Omega f_n d\mu \to \int_\Omega f\, d\mu$.

Proof. An easy consequence of Fatou's lemma. □

LEMMA 4. *Let* $X(t)$ *and* $X_n(t)$ *be as in (3.2) and (3.4), respectively. Then, there exists a subsequence, again denoted by* $\{X_n\}$, *such that*

$$E\int_0^T |\, \|X_n(t)\|^p - \|X(t)\|^p| dt \to 0.$$

Proof. By Lemma 2, one can find a set $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$ and a subsequence, again denoted by $\{X_n\}$, such that for each $\omega \in \Omega_0$

$$\|X_n(t,\omega)\|^p \to \|X(t,\omega)\|^p, \qquad \forall t \in S,$$

and $\|X_n(\omega) - X(\omega)\|_\infty^p \to 0$, as $n \to \infty$. Since

$$|\,\|X_n(t,\omega)\|^p - \|X(t,\omega)\|^p| \leqslant 2^{p-1}\|X_n(t,\omega) - X(t,\omega)\|^p + (2^{p-1}+1)\|X(t,\omega)\|^p,$$

we can apply Lemma 3 to obtain the desired result. □

LEMMA 5. *Almost all sample paths of the H-valued process $A(t)X_n(t)$ are integrable over $[0,T]$ and X_n satisfies*

$$X_n(t) = X_0^n + \int_0^t A(s)X_n(s)ds + \int_0^t f_n(s)ds + \int_0^t g_n(s)dW_s, \tag{3.6}$$

for all $t \in S$.

Proof. We first remark that by the definitions of f_n, g_n and X_0^n, $X_n(t)$ is D-valued, so $A(s)X_n(s)$ in (3.6) is meaningful. Next, the well-known Fubini theorem yields

$$\begin{aligned}\int_0^t A(r)\left(\int_0^r U(r,s)f_n(s)ds\right)dr &= \int_0^t \left(\int_s^t A(r)U(r,s)f_n(s)dr\right)ds \\ &= \int_0^t U(t,s)f_n(s)ds - \int_0^t f_n(s)ds.\end{aligned}$$

By Fubini theorem for stochastic integrals (see [6], page 109), we also have

$$\begin{aligned}\int_0^t A(r)\left(\int_0^r U(r,s)g_n(s)dW_s\right)dr &= \int_0^t \left(\int_s^t A(r)U(r,s)g_n(s)dr\right)dW_s \\ &= \int_0^t U(t,s)g_n(s)dW_s - \int_0^t g_n(s)dW_s.\end{aligned}$$

Finally, since $U(t,0)$ is an almost strong evolution operator

$$\int_0^t A(r)U(r,0)X_0^n dr = U(t,0)X_0^n - X_0^n.$$

By adding the above equations, the desired result follows. □

Proof of Theorem 2. For $p = 2$, the result is a direct consequence of Theorem 1. Therefore we address ourselves to the case $p > 2$. By applying Itô's formula (see [6], page 105) to $f(X_n(.))$ where $f(x) = \|x\|^p$, we have

$$\begin{aligned} \|X_n(t)\|^p &= \|X_0^n\|^p + p\int_0^t \|X_n(s)\|^{p-2} < X_n(s)\,,\, A(s)X_n(s) > ds \\ &\quad + p\int_0^t \|X_n(s)\|^{p-2} < X_n(s)\,,\, f_n(s) > ds \\ &\quad + p\int_0^t \|X_n(s)\|^{p-2} < X_n(s)\,,\; g_n(s)dW_s > \\ &\quad + \frac{1}{2}\int_0^t trace\{g_n^*(s)\,(p\,\|X_n(s)\|^{p-2}I\, + \\ &\quad + p(p-2)\,\|X_n(s)\|^{p-4}\, X_n(s)\otimes X_n(s)\,)g_n(s)\}ds, \end{aligned} \tag{3.7}$$

where $g_n^*(s)$ is the adjoint operator of $g_n(s)$ and the operator $a\otimes b : H \to H$ is given by

$$(a\otimes b)(h) = a < h\,,\, b >, \qquad \forall h \in H,$$

for $a, b \in H$. But,

$$|\, trace\{\, g_n^*(s) f_{xx}(X_n(s)) g_n(s)\,\}\,| \leqslant\, p(p-1)\|X_n(s)\|^{p-2}\|g_n(s)\|_2^2.$$

Moreover, $\|X_0^n\|^p \leqslant \|X_0\|^p$, $< X_n(s), A(s)X_n(s) > \leqslant 0$ and $\|g_n(s)\|_2 \leqslant \|g(s)\|_2$ for a.e. $s \in S$. So, from (3.7) we obtain that

$$\begin{aligned} \|X_n(t)\|^p &\leqslant \|X_0\|^p + p\int_0^t \|X_n(s)\|^{p-2} < X_n(s)\,,\, f_n(s) > ds \\ &\quad + p\int_0^t \|X_n(s)\|^{p-2} < X_n(s)\,,\; g_n(s)dW_s > \\ &\quad + \frac{p(p-1)}{2}\int_0^t \|X_n(s)\|^{p-2}\,\|g(s)\|_2^2\, ds. \end{aligned} \tag{3.8}$$

Therefore, to complete the proof, we only need to show that as $n \to \infty$, the right hand side of (3.8) tends to that of (3.3) in probability, along some subsequence. To this end, we proceed in three steps and to simplify notations, we suppress the variable s in integrands.

Step 1. we write

$$\begin{aligned} &\left|\int_0^t \|X_n\|^{p-2} < X_n, f_n > ds \,-\, \int_0^t \|X\|^{p-2} < X, f\; > ds\right| \\ &\leqslant \left|\int_0^t (\|X_n\|^{p-2} - \|X\|^{p-2}) < X_n, f_n > ds\right| + \left|\int_0^t \|X\|^{p-2} < X_n - X, f_n > ds\right| \\ &+ \left|\int_0^t \|X\|^{p-2} < X, f_n - f\; > ds\right| \,=\, I_n^1(t) \,+\, I_n^2(t) \,+\, I_n^3(t), \end{aligned}$$

and show that there exists a sequence $\{n_k\} \subseteq \mathbb{N}$ such that $\sup_{0\leqslant t\leqslant T} I_{n_k}^j(t) \to 0$ in probability for $j = 1, 2, 3$. For the third term on the right, one can obtain by Hölder inequality that

$$\begin{aligned}
\sup_{0\leqslant t\leqslant T} I_n^3(t) &\leqslant \int_0^T \|X\|^{p-2}|<X,f_n-f>|\,ds \\
&\leqslant \int_0^T \|X\|^{p-1}\|f_n-f\|\,ds \\
&\leqslant (T\,\|X\|_\infty^p)^{1-\frac{1}{p}}\left(\int_0^T \|f_n-f\|^p\,ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\|X\|_\infty^p<\infty$ w.p.1 and by the dominated convergence theorem, $\int_0^T \|f_n-f\|^p ds \to 0$ w.p.1, we conclude that $\sup_{0\leqslant t\leqslant T} I_n^3(t)\to 0$ w.p.1. Next, by Hölder inequality and the fact that $\|f_n(s)\|\leqslant\|f(s)\|$ for a.e. $s\in S$, we obtain

$$\begin{aligned}
\sup_{0\leqslant t\leqslant T} I_n^2(t) &\leqslant \int_0^T \|X\|^{p-2}\|X_n-X\|\,\|f_n\|ds \\
&\leqslant (T\|X\|_\infty^p)^{1-\frac{2}{p}}\left(\int_0^T \|X_n-X\|^p ds\right)^{\frac{1}{p}}\left(\int_0^T \|f\|^p ds\right)^{\frac{1}{p}} \\
&\leqslant T^{1-\frac{1}{p}}(\|X\|_\infty^p)^{1-\frac{2}{p}}(\|X_n-X\|_\infty)\left(\int_0^T \|f\|^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Now, (3.1) and Lemma 2 imply that $\sup_{0\leqslant t\leqslant T} I_n^2(t)\to 0$ in probability. Moreover, from Hölder inequality

$$\begin{aligned}
&E\,[\sup_{0\leqslant t\leqslant T} I_n^1(t)] \\
\leqslant\; &(TE\|X_n\|_\infty^p)^{\frac{1}{p}}\left(E\int_0^T \|f_n\|^p\,ds\right)^{\frac{1}{p}}\left(E\int_0^T |\,\|X_n\|^{p-2}-\|X\|^{p-2}|^{\frac{p}{p-2}}\,ds\right)^{1-\frac{2}{p}} \\
\leqslant\; &(TE\|X_n\|_\infty^p)^{\frac{1}{p}}\left(E\int_0^T \|f\|^p\,ds\right)^{\frac{1}{p}}\left(E\int_0^T |\,\|X_n\|^{p}-\|X\|^{p}|\,ds\right)^{1-\frac{2}{p}}.
\end{aligned}$$

Here, we have used the elementary inequality $|a-b|^r\leqslant|a^r-b^r|$ for all non-negative numbers a,b and all $r\geqslant 1$. By Lemma 4, there exists a sequence $\{n_k\}$ of positive integers such that

$$E\int_0^T |\,\|X_{n_k}\|^p-\|X\|^p|ds\to 0.$$

Therefore, $\sup_{0\leqslant t\leqslant T} I_{n_k}^1(t)\to 0$ in L^1 hence in probability. This finishes the proof of Step 1.

Step 2. Let us now consider the convergence of stochastic integrals. We write

$$\begin{aligned}
&\left|\int_0^t \|X_n\|^{p-2} < X_n\,,\, g_n dW_s > \;-\; \int_0^t \|X\|^{p-2} < X\,,\, g\,dW_s >\right| \\
&\leqslant \left|\int_0^t (\|X_n\|^{p-2} - \|X\|^{p-2}) < X_n\,,\, g_n dW_s >\right| \\
&+ \left|\int_0^t \|X\|^{p-2} < X_n - X\,,\, g_n dW_s >\right| \\
&+ \left|\int_0^t \|X\|^{p-2} < X\,,\, (g_n - g) dW_s >\right| \;=\; J_n^1(t) \;+\; J_n^2(t) \;+\; J_n^3(t),
\end{aligned}$$

and show that there exists a sequence $\{n_m\} \subseteq \mathbb{N}$ such that $\sup_{0\leqslant t\leqslant T} J^j_{n_m}(t) \to 0$ in L^1 for $j = 1,2,3$. First, by (2.1) and Hölder inequality

$$\begin{aligned}
E\,[\sup_{0\leqslant t\leqslant T} J_n^3(t)] \;&\leqslant\; 3\,E\left(\int_0^T \|X\|^{2p-2}\|g_n - g\|_2^2\,ds\right)^{\frac{1}{2}} \\
&\leqslant\; 3\,(E\|X\|_\infty^p)^{1-\frac{1}{p}} \left\{E\left(\int_0^T \|g_n - g\|_2^2\,ds\right)^{\frac{p}{2}}\right\}^{\frac{1}{p}} \\
&\leqslant\; 3\,T^{\frac{1}{2}-\frac{1}{p}}\,(\,E\|X\|_\infty^p)^{1-\frac{1}{p}} \left(E\int_0^T \|g_n - g\|_2^p\,ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Now, from (3.5) we conclude that $\sup_{0\leqslant t\leqslant T} J_n^3(t) \to 0$ in L^1. Next, from (2.1), Hölder inequality and taking into account that $\|g_n(s)\|_2 \leqslant \|g(s)\|_2$ for a.e. $s \in S$, one can see that

$$\begin{aligned}
E[\,\sup_{0\leqslant t\leqslant T} J_n^2(t)\,] \;&\leqslant\; 3\,E\left(\int_0^T \|X\|^{2p-4}\|X - X_n\|^2\,\|g_n\|_2^2\,ds\right)^{\frac{1}{2}} \\
&\leqslant\; 3\,(E\|X\|_\infty^p)^{1-\frac{2}{p}} \left\{E\left[\,\|X - X_n\|_\infty^{\frac{p}{2}} \left(\int_0^T \|g\|_2^2 ds\right)^{\frac{p}{4}}\right]\right\}^{\frac{2}{p}} \\
&\leqslant\; 3\,T^{\frac{1}{2}-\frac{1}{p}}\,(\,E\|X\|_\infty^p)^{1-\frac{2}{p}}\,(\,E\|X - X_n\|_\infty^p)^{\frac{1}{p}} \left(E\int_0^T \|g\|_2^p\,ds\right)^{\frac{1}{p}},
\end{aligned}$$

which by (3.1) and Lemma 2, approaches zero. Therefore, $\sup_{0\leqslant t\leqslant T} J_n^2(t) \to 0$ in L^1. Finally, since by (3.1) and the definitions of f_n and g_n, $E\|X\|_\infty^p$ and $E\|X_n\|_\infty^p$ are bounded to a constant independent of n, (2.1) and Hölder inequality imply

$$\begin{aligned} E[\sup_{0\leqslant t\leqslant T} J_n^1(t)] &\leqslant 3\,E\left(\int_0^T |\,\|X_n\|^{p-2} - \|X\|^{p-2}|^2 \|X_n\|^2 \|g_n\|_2^2\,ds\right)^{\frac{1}{2}} \\ &\leqslant 3\left\{E[\,\|X_n\|_\infty^2(\|X_n\|_\infty^{p-2} + \|X\|_\infty^{p-2})]\right\}^{\frac{1}{2}} \\ &\quad \times\left(E\int_0^T |\,\|X_n\|^{p-2} - \|X\|^{p-2}|\,\|g\|_2^2\,ds\right)^{\frac{1}{2}} \\ &\leqslant 3\left\{E\|X_n\|_\infty^p + (E\|X_n\|_\infty^p)^{\frac{2}{p}}(E\|X\|_\infty^p)^{1-\frac{2}{p}}\right\}^{\frac{1}{2}} \\ &\quad \times\left(E\int_0^T |\,\|X_n\|^p - \|X\|^p|ds\right)^{\frac{1}{2}-\frac{1}{p}}\left(E\int_0^T \|g\|_2^p\,ds\right)^{\frac{1}{p}} \\ &\leqslant \tilde{C}\left(E\int_0^T |\,\|X_n\|^p - \|X\|^p|ds\right)^{\frac{1}{2}-\frac{1}{p}}\left(E\int_0^T \|g\|_2^p\,ds\right)^{\frac{1}{p}}, \end{aligned}$$

for some positive constant $\tilde{C}$ independent of n. So, by Lemma 4 there exists a sequence $\{n_m\}$ of positive integers such that $\sup_{0\leqslant t\leqslant T} J_{n_m}^1(t) \to 0$ in L^1. This finishes the proof of Step 2.

Step 3. At last, we conclude by Hölder inequality that

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T}\left|\int_0^t (\|X_n\|^{p-2} - \|X\|^{p-2})\|g\|_2^2\,ds\right| \\ &\leqslant \left(\int_0^T |\,\|X_n\|^{p-2} - \|X\|^{p-2}|^{\frac{p}{p-2}}\,ds\right)^{1-\frac{2}{p}}\left(\int_0^T \|g\|_2^p\,ds\right)^{\frac{2}{p}} \\ &\leqslant \left(\int_0^T |\,\|X_n\|^p - \|X\|^p|\,ds\right)^{1-\frac{2}{p}}\left(\int_0^T \|g\|_2^p\,ds\right)^{\frac{2}{p}}, \end{aligned}$$

which by (3.1) and the same argument as in the proof of Lemma 4, approaches zero w.p.1 along some subsequence. The proof of the theorem is now complete. □

4. Stability in the p-th Moment

Consider on H a semilinear SEE of the form

$$dX_t = [A(t)X_t + f(t, X_t)]dt + g(t, X_t)dW_t, \tag{4.1}$$

with initial value X_0, in which $\{A(t) : t \in [0, \infty)\}$ is a family of linear operators on H satisfying the following hypotheses.

HYPOTHESIS 2_λ. (a) *There exists $\lambda \in \mathbb{R}$ such that $\forall s > 0$, $(A(s) - \lambda I)$ is the generator of a contraction semigroup;*

(b) *The operator-valued function $(-A(t) + \mu I)^{-1}$ is strongly continuously differentiable with respect to t for $\mu > \lambda$;*

(c) *There exists a fundamental solution* $U(t,s)$ *of the linear equation* $\dot{u}(t) = A(t)u(t)$ *on* $[0,T]$. *Moreover, if* $u_0 \in H$ *and* $f \in C(0,T;H)$, *then the strong solution of*

$$\begin{cases} \dot{u}(t) &= A(t)u(t) + f(t) \\ u(0) &= u_0 \end{cases} \tag{4.2}$$

is given by

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds. \tag{4.3}$$

Conversely, if $u_0 \in \mathscr{D}(A(0))$ *and* $f \in C^1(0,T;H)$, *then (4.3) is a strong solution of (4.2).*

Remark 3. It is turned out (see e.g. [1]) that Hypothesis $2_\lambda(c)$ holds, for example, if $\{A(t) : t \in [0,\infty)\}$ is a family of linear operators on H with domain D independent of t, satisfying Hypothesis $2_\lambda(a)$ and (b). Moreover, if an evolution operator $U(t,s)$ satisfies Hypothesis 2_λ, then it is an almost strong evolution operator satisfying Hypothesis $1_\lambda(b)$ and (c).

DEFINITION 6. A stochastic process X_t, $t \in [0,T]$, is called a *mild solution* of (4.1) if

(i) X_t is an H-valued, $\mathscr{F}_t$-adapted, predictable process;
(ii) $\int_0^T \|X_t\|^2 dt < \infty$, w.p.1;
(iii) X_t satisfies the integral equation

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)f(s,X_s)ds + \int_0^t U(t,s)g(s,X_s)dW_s, \tag{4.4}$$

for all $t \in [0,T]$, w.p.1.

The following are the relevant hypotheses concerning X_0, f, g, A and U:

HYPOTHESIS 3_λ. *There exist a set* $G \subseteq \Omega$ *of probability one and a constant* $C > 0$ *with the following properties:*

(a) *For each* $t \in [0,\infty)$ *and* $\omega \in G$, $x \to f(t,\omega,x)$ *is a demicontinuous function on* H, *i.e., whenever* $\{x_n\}$ *is sequence in* H *which converges strongly to* $x \in H$, *then* $f(t,\omega,x_n)$ *converges weakly to* $f(t,\omega,x)$;

(b) *For each* $t \in [0,\infty)$ *and* $\omega \in G$, $x \to f(t,\omega,x)$ *satisfies a linear growth condition:*

$$\|f(t,\omega,x)\| \leqslant C(1 + \|x\|), \qquad \forall x \in H;$$

(c) *There exists a non-negative number* M *such that for each* $t \in [0,\infty)$ *and* $\omega \in G$, $x \to -f(t,\omega,x)$ *is semimonotone with parameter* M, *i.e.*

$$< f(t,\omega,x) - f(t,\omega,y), x - y > \leqslant M\|x-y\|^2,$$

for all x, $y \in H$;

(d) $g : [0,\infty) \times \Omega \times H \to L_2(K,H)$ *is a predictable process on* H *such that*

$$\|g(t,\omega,x) - g(t,\omega,y)\|_2 \leqslant C\|x-y\|,$$

for all $t \geqslant 0$, $\omega \in G$ *and* x, $y \in H$;

(*e*) A *and* U *satisfy Hypotheses* 1_λ *and* 2_λ;

(*f*) X_0 *is an* H*-valued,* $\mathscr{F}_0$*-measurable random variable.*

Now, we give the existence and uniqueness result for mild solutions of (4.1).

THEOREM 3. [28] *Let* $p \geqslant 2$. *If* $E\|X_0\|^p$ *and* $E[\sup_{0\leqslant s\leqslant t}\|g(s,0)\|_2^p]$ *are finite for all* $t \in [0,T]$ *and Hypothesis* 3_λ *is satisfied, then the equation* (4.4) *has a unique continuous adapted strong solution* X *with*

$$E(X_t^\star)^p < \infty, \qquad \forall t \in [0,T],$$

in which $X_t^\star = \sup_{0\leqslant s\leqslant t}\|X_s\|$.

In this section, we prove the exponentially asymptotic stability in the p-th moment, $p \geqslant 2$, of the mild solutions of (4.1).

DEFINITION 7. A mild solution X_t of (4.1) with initial value X_0 is said to be *exponentially asymptotically stable in the* p*-th moment*, if there exist α, $\beta > 0$ such that for any mild solution Y_t of (4.1) with initial value Y_0,

$$E\,\|X_t - Y_t\|^p \leqslant \beta e^{-\alpha t} E\,\|X_0 - Y_0\|^p, \qquad \forall t \geqslant 0.$$

Now, we state and prove the main result of this section. First, we recall a lemma.

LEMMA 6. *Let* X_t, $t \in [0,\infty)$, *be an* H*-valued continuous process. If* M_t *is an* H*-valued continuous martingale, then for any constant* $K > 0$ *we have*

$$E\left[\sup\nolimits_{0\leqslant\theta\leqslant t}\left|\int_0^\theta < X_s\,,\,dM_s >\right|\right] \leqslant \frac{3}{2K}E(X_t^\star)^2 + \frac{3K}{2}E([M]_t),$$

where $[\quad]_t$ *stands for the quadratic variation process.*

Proof. The proof can be found in [28] but, for the sake of completeness, we give it here. By Burkhölder-Davis-Gundy inequality [21], we have

$$E\left[\sup\nolimits_{0\leqslant\theta\leqslant t}\left|\int_0^\theta < X_s\,,\,dM_s >\right|\right] \leqslant 3\,E\left\{\left[\int_0^\cdot < X_s\,,\,dM_s >\right]_t^{\frac{1}{2}}\right\}.$$

But, $[\int_0^\cdot < X_s\,,\,dM_s >]_t \leqslant (X_t^\star)^2\,[M]_t$. Therefore,

$$E\left[\sup\nolimits_{0\leqslant\theta\leqslant t}\left|\int_0^\theta < X_s\,,\,dM_s >\right|\right] \leqslant 3E\{(X_t^\star)\,[M]_t^{\frac{1}{2}}\}.$$

Since $ab \leqslant \frac{1}{2}(\frac{1}{K}a^2 + Kb^2)$ for any a, $b \in \mathbb{R}$ and any $K > 0$, the desired result follows. □

THEOREM 4. *Let* $p \geqslant 2$. *Assume that* $\lambda > 0$ *and* f, g, A *and* U *satisfy Hypothesis* $3_{-\lambda}$. *Let* X_t *and* Y_t *be two mild solutions of* (4.1) *with initial values* X_0 *and* Y_0, *respectively. Then,*

$$E[\sup\nolimits_{0\leqslant\theta\leqslant t}e^{p\lambda\theta}\|X_\theta - Y_\theta\|^p] \leqslant 2\,e^{\alpha t}E\|X_0 - Y_0\|^p, \tag{4.5}$$

for all $t \in [0,\infty)$, *where* $\alpha = 2pM + p(p-1)C^2 + (3pC)^2$.

Proof. By assumption, we have

$$X_t = U(t,0)X_0 + \int_0^t U(t,s)f(s,X_s)\,ds + \int_0^t U(t,s)g(s,X_s)dW_s, \tag{4.6}$$

and

$$Y_t = U(t,0)Y_0 + \int_0^t U(t,s)f(s,Y_s)\,ds + \int_0^t U(t,s)g(s,Y_s)dW_s. \tag{4.7}$$

Subtracting (4.7) from (4.6) and using Itô-type inequality (Theorem 2), we obtain

$$\begin{aligned}
&\|X_t - Y_t\|^p \leqslant e^{-p\lambda t}\|X_0 - Y_0\|^p\\
&+p\int_0^t e^{-p\lambda(t-s)}\|X_s - Y_s\|^{p-2} < X_s - Y_s\,, f(s,X_s) - f(s,Y_s) > ds\\
&+p\int_0^t e^{-p\lambda(t-s)}\|X_s - Y_s\|^{p-2} < X_s - Y_s\,, (g(s,X_s) - g(s,Y_s))dW_s >\\
&+\frac{p(p-1)}{2}\int_0^t e^{-p\lambda(t-s)}\|X_s - Y_s\|^{p-2}\|g(s,X_s) - g(s,Y_s)\|_2^2\,ds.
\end{aligned} \tag{4.8}$$

Multiplying both sides of (4.8) by $e^{p\lambda t}$ and using Hypothesis $3_{-\lambda}(c)$ and (d), yield

$$\begin{aligned}
&\sup_{0\leqslant\theta\leqslant t} e^{p\lambda\theta}\|X_\theta - Y_\theta\|^p \leqslant \|X_0 - Y_0\|^p\\
&+[pM + \frac{p(p-1)}{2}C^2]\int_0^t e^{p\lambda s}\|X_s - Y_s\|^p\,ds\\
&+p\sup_{0\leqslant\theta\leqslant t}\left|\int_0^\theta e^{p\lambda s}\|X_s - Y_s\|^{p-2} < X_s - Y_s\,, (g(s,X_s) - g(s,Y_s))dW_s >\right|
\end{aligned} \tag{4.9}$$

Now, we take the mathematical expectation and apply Lemma 6 to the third term on the right hand side of (4.9). Then, for any $K > 0$

$$\begin{aligned}
E(\Lambda_t) \leqslant\ & E(\Lambda_0) + \frac{3p}{2K}E(\Lambda_t)\\
&+[pM + \frac{p(p-1)}{2}C^2 + \frac{3pKC^2}{2}]\int_0^t E(\Lambda_s)ds,
\end{aligned} \tag{4.10}$$

in which

$$\Lambda_t = \sup_{0\leqslant\theta\leqslant t} e^{p\lambda\theta}\|X_\theta - Y_\theta\|^p, \qquad \forall t \geqslant 0.$$

Choose $K = 3p$ in (4.10) and note that by Theorem 3, $E(\Lambda_t) < \infty$. Then, we have

$$\frac{1}{2}E(\Lambda_t) \leqslant E(\Lambda_0) + [pM + \frac{p(p-1)}{2}C^2 + \frac{(3pC)^2}{2}]\int_0^t E(\Lambda_s)ds.$$

Therefore, by Gronwall's inequality

$$E(\Lambda_t) \leqslant 2E(\Lambda_0)\,e^{\alpha t}, \qquad \forall t \geqslant 0,$$

where $\alpha = 2pM + p(p-1)C^2 + (3pC)^2$. This completes the proof of the theorem. □

COROLLARY 1. *Suppose that all conditions of Theorem* 4 *hold except that* f *satisfies in a Lipschitz condition with constant* M, g *is* $L(K,H)$*-valued and* W_t, $t \geqslant 0$, *is a* Q*-Wiener process on* K *with a positive, self-adjoint, nuclear covariance operator* $Q \in L(K)$. *If* X_t *and* Y_t *are two mild solutions of* (4.1) *with initial values* X_0 *and* Y_0, *respectively, then*

$$E[\sup_{0 \leqslant \theta \leqslant t} e^{p\lambda\theta} \|X_\theta - Y_\theta\|^p] \leqslant 2e^{\bar{\alpha}t} E\|X_0 - Y_0\|^p, \qquad \forall t \geqslant 0,$$

where $\bar{\alpha} = 2pM + trace(Q)[p(p-1)C^2 + (3pC)^2]$.

Proof. This follows from the fact that W_t can be written as $W_t = Q^{\frac{1}{2}} \tilde{W}_t$, where $\tilde{W}_t$ is a cylindrical Brownian motion on K (see [20]). Moreover, $Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on K, so if $\Phi(t)$ is an $L(K,H)$-valued predictable process, then $\Phi(t)Q^{\frac{1}{2}}$ is an $L_2(K,H)$-valued predictable process. Therefore, we can apply Theorem 4 to obtain the desired result. □

COROLLARY 2. *If all hypotheses of Theorem* 4 *are satisfied and* X_t *is a mild solution of* (4.1) *with initial value* X_0, *then for any mild solution* Y_t *of* (4.1) *with initial value* Y_0, *we have*

$$E\|X_t - Y_t\|^p \leqslant 2\, e^{-(p\lambda-\alpha)t} E\|X_0 - Y_0\|^p,$$

for all $t \in [0,\infty)$, *where* $\alpha = 2pM + p(p-1)C^2 + (3pC)^2$. *Consequently, if* $p\lambda > \alpha$, *then* X_t *is exponentially asymptotically stable in the* p*-th moment.*

COROLLARY 3. *Assume that all conditions of Theorem* 4 *hold and* $f(t,0) = g(t,0) = 0$ *for all* $t \in [0,\infty)$. *If* $p\lambda > 2pM + p(p-1)C^2 + (3pC)^2$, *then the trivial solution of* (4.4) *is exponentially asymptotically stabe in the* p*-th moment.*

5. Asymptotic Behaviour of Sample Paths

Since by Theorem 3, the mild solution of (4.1) has continuous sample paths, it is reasonable to consider the asymptotic stability of its sample paths. In this section, we prove the pathwise exponentially asymptotic stability of the mild solutions of (4.1).

THEOREM 5. *Suppose that all conditions of Theorem* 4 *hold. Let* α *be the same as in Theorem* 4. *If* $p\lambda > \alpha$, *then for any two mild solutions* X_t *and* Y_t *of* (4.1) *with initial values* X_0 *and* Y_0, *respectively,*

$$\limsup_{t\to\infty} (1/t) \log \|X_t - Y_t\| \leqslant -(p\lambda - \alpha)/2p, \qquad w.p.1.$$

Proof. Let n be an arbitrary natural number. By assumption, we have

$$X_t = U(t,n)X_n + \int_n^t U(t,s)f(s,X_s)ds + \int_n^t U(t,s)g(s,X_s)dW_s,$$

and

$$Y_t = U(t,n)Y_n + \int_n^t U(t,s)f(s,Y_s)ds + \int_n^t U(t,s)g(s,Y_s)dW_s,$$

for $n \leqslant t \leqslant n+1$. Assume for the simplicity of the notations that $I_k(t)$, $k=1,2,3$, denotes the k-th integral on the right hand side of (4.8) with lower limit 0 replaced by n. Then, by Itô-type inequality (Theorem 2), we obtain for any fixed $\varepsilon_n > 0$ that

$$\begin{aligned}
&P\left\{\sup_{n\leqslant t\leqslant n+1} \|X_t - Y_t\|^p > \varepsilon_n\right\} \leqslant P\left\{\sup_{n\leqslant t\leqslant n+1} e^{-p\lambda(t-n)}\|X_n - Y_n\|^p > \frac{\varepsilon_n}{4}\right\}\\
&+P\left\{\sup_{n\leqslant t\leqslant n+1} p\,|I_1(t)| > \frac{\varepsilon_n}{4}\right\} + P\left\{\sup_{n\leqslant t\leqslant n+1} p\,|I_2(t)| > \frac{\varepsilon_n}{4}\right\}\\
&+P\left\{\sup_{n\leqslant t\leqslant n+1} \frac{p(p-1)}{2} I_3(t) > \frac{\varepsilon_n}{4}\right\}\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Now, by (4.5) and Markov inequality

$$\begin{aligned}
J_1 &\leqslant (4/\varepsilon_n)E\|X_n - Y_n\|^p\\
&\leqslant (8/\varepsilon_n)\,e^{-(p\lambda-\alpha)n}E\|X_0 - Y_0\|^p.
\end{aligned} \tag{5.1}$$

Furthermore, since $-f$ is semimonotone with parameter M, again by (4.5)

$$\begin{aligned}
J_2 &\leqslant (4pM/\varepsilon_n)\,E\left[\sup_{n\leqslant t\leqslant n+1}\int_n^t e^{-p\lambda(t-s)}\|X_s - Y_s\|^p ds\right]\\
&\leqslant (4pM/\varepsilon_n)\int_n^{n+1} E\|X_s - Y_s\|^p ds\\
&\leqslant (8pM/\varepsilon_n)\,e^{-(p\lambda-\alpha)n}E\|X_0 - Y_0\|^p.
\end{aligned} \tag{5.2}$$

Also, Lemma 1 implies that for any $K>0$

$$\begin{aligned}
J_3 \leqslant (4p/\varepsilon_n)\Big\{&\frac{3}{2K}e^{-p\lambda n}E\left[\sup\nolimits_{n\leqslant\theta\leqslant n+1} e^{p\lambda\theta}\|X_\theta - Y_\theta\|^p\right]\\
&+\frac{3K}{2}e^{-p\lambda n}\int_n^{n+1} e^{p\lambda s}E\|X_s - Y_s\|^{p-2}\|g(s,X_s) - g(s,Y_s)\|_2^2\,ds\Big\}.
\end{aligned} \tag{5.3}$$

Choose $K = 3e^{\alpha}$ in (5.3). Then, from Hypothesis $3_{-\lambda}(d)$ and (4.5), we conclude that

$$\begin{aligned}
J_3 &\leqslant (4p/\varepsilon_n)\{2e^{-(p\lambda-\alpha)n}E\|X_0 - Y_0\|^p\\
&\qquad + (3Ce^{\alpha})^2e^{-(p\lambda-\alpha)n}E\|X_0 - Y_0\|^p\}\\
&= (4pL/\varepsilon_n)\,e^{-(p\lambda-\alpha)n}E\|X_0 - Y_0\|^p,
\end{aligned} \tag{5.4}$$

where $L = 2 + (3Ce^{\alpha})^2$ and C is the Lipschitz constant of g. Finally, from (4.5)

$$J_4 \leqslant (4C^2p(p-1)/\varepsilon_n)\,e^{-(p\lambda-\alpha)n}E\|X_0 - Y_0\|^p. \tag{5.5}$$

Combining (5.1), (5.2), (5.4) and (5.5), we can conclude that there exists a positive constant γ such that for any $\varepsilon_n > 0$

$$P\{ \sup_{n \leqslant t \leqslant n+1} \|X_t - Y_t\|^p > \varepsilon_n \} \leqslant (\gamma/\varepsilon_n)\, e^{-(p\lambda - \alpha)n} E\|X_0 - Y_0\|^p. \tag{5.6}$$

Now, take

$$\varepsilon_n = (E\|X_0 - Y_0\|^p)\, e^{-(p\lambda - \alpha)n/2}.$$

Then, by (5.6)

$$P\left\{ \sup_{n \leqslant t \leqslant n+1} \|X_t - Y_t\|^p > (E\|X_0 - Y_0\|^p)\, e^{-(p\lambda - \alpha)n/2} \right\}$$
$$\leqslant \gamma\, e^{-(p\lambda - \alpha)n/2}.$$

Thus, the Borel-Cantelli's lemma implies that there exist a random variable $0 < T(\omega) < \infty$ and a set $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$, such that for every $\omega \in \Omega_0$ and $t > T(\omega)$

$$\|X_t - Y_t\|^p \leqslant \beta\, E\|X_0 - Y_0\|^p\, e^{-(p\lambda - \alpha)t/2},$$

where $\beta = e^{(p\lambda - \alpha)/2}$. The proof of the theorem is now complete. □

COROLLARY 4. *Suppose that all conditions of Corollary* 1 *hold. Let* $\bar{\alpha}$ *be the same as in Corollary* 1 *and* $p\lambda > \bar{\alpha}$. *Then, for any two mild solutions* X_t *and* Y_t *of* (*4.1*) *with initial values* X_0 *and* Y_0, *respectively, we have*

$$\limsup_{t\to\infty} (1/t) \log \|X_t - Y_t\| \leqslant -(p\lambda - \bar{\alpha})/2p, \qquad \text{w.p.1.}$$

COROLLARY 5. *Assume that all conditions of Corollary* 3 *hold. If* $p\lambda > \alpha$, *then for any mild solution* X_t *of* (*4.1*) *with initial value* X_0, *we have*

$$\limsup_{t\to\infty} (1/t) \log \|X_t\| \leqslant -(p\lambda - \alpha)/2p, \qquad \text{w.p.1.}$$

6. Examples

Let us apply the results established in the previous sections to several stochastic partial differential equations of monotone type.

Example 1. In the first example, we consider a semilinear stochastic heat equation with Dirichlet boundary condition. The stability theory of such equations when the nonlinear term satisfies a Lipschitz condition, has been well-studied by several authors (see e.g. [9] and [24]). Our main results in this paper ensure the exponential stability of the mild solutions, in the case of semimonotone nonlinearity.

DEFINITION 8. Let $D \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. A function $f(x,y) : D \times \mathbb{R}^n \to \mathbb{R}^m$ is said to satisfy the Carathéodory condition, if it is continuous with respect to y for almost all $x \in D$ and measurable with respect to x for all $y \in \mathbb{R}^n$.

Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x, u(t,x)) + g(x, u(t,x))\dot{W}_t \quad \text{on} \quad [0,\infty) \times [0,1], \tag{6.1}$$

$$u(t,0) = u(t,1) = 0 \quad \text{for } t \in [0,\infty), \quad u(0,x) = u_0(x) \quad \text{for } x \in [0,1], \tag{6.2}$$

where W_t is the real standard Brownian motion, $u_0 \in L^2(0,1)$ and f, g satisfy the following hypotheses.

HYPOTHESIS 4. (a) *f, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory condition;*
(b) *There exist a function $a \in L^2(0,1)$ and a constant $C > 0$ such that*

$$|f(x,y)| \leqslant a(x) + C|y|,$$

$$|g(x,y)| \leqslant a(x) + C|y|,$$

for all $x \in [0,1]$ and $y \in \mathbb{R}$;
(c) *$g(x,.)$ is uniformly Lipschitz with Lipschitz constant $C > 0$, i.e.*

$$|g(x,y_2) - g(x,y_1)| \leqslant C|y_2 - y_1|, \qquad \forall x \in [0,1], \quad y_2, y_1 \in \mathbb{R};$$

(d) *$-f(x,.)$ is semimonotone with parameter M, uniformly with respect to $x \in [0,1]$, i.e.*

$$(f(x,y_2) - f(x,y_1))(y_2 - y_1) \leqslant M(y_2 - y_1)^2.$$

Define $H = L^2(0,1)$ with the norm $\|\,.\,\|$, $K = \mathbb{R}$ and $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$\mathscr{D}(A) = \{ u \in L^2(0,1) : u', u'' \in L^2(0,1) \quad \text{and} \quad u(0) = u(1) = 0 \}.$$

Then, the operator $A : \mathscr{D}(A) \subseteq H \to H$ generates a strongly continuous semigroup $S(t)$ satisfying $\|S(t)\|_L \leqslant e^{-\pi^2 t}$ for all $t \geqslant 0$.

Now, denote by $\bar{f}$ and $\bar{g}$, the functions defined on H by $(\bar{f}(u))(x) = f(x,u(x))$ and $(\bar{g}(u))(x) = g(x,u(x))$, respectively. Then, the initial-boundary value problem (6.1) and (6.2) can be written as the SEE

$$du(t) = [Au(t) + \bar{f}(u(t))]dt + \bar{g}(u(t))dW_t, \qquad u(0) = u_0. \tag{6.3}$$

Since f and g satisfy Hypothesis $4(a)$ and (b), $\bar{f}$ and $\bar{g}$ are continuous and there exists a constant $C > 0$ such that

$$\|\bar{f}(u)\| \leqslant C(1 + \|u\|), \qquad \|\bar{g}(u)\| \leqslant C(1 + \|u\|).$$

It is easy to see that Hypothesis $4\,(c)$ and (d) imply that $\bar{f} : L^2(0,1) \to L^2(0,1)$ and $\bar{g} : L^2(0,1) \to L^2(0,1) \simeq L(\mathbb{R}, L^2(0,1))$, satisfy Hypothesis $3_{-\pi^2}(c)$ and (d). Corollary 3 now implies that if $f(x,0) = g(x,0) = 0$ for all $x \in [0,1]$, then for any mild solution $u(t)$ of (6.3)

$$E\|u(t)\|^p \leqslant 2e^{-(p\pi^2 - \alpha)}E\|u_0\|^p.$$

Thus, if $p\pi^2 > \alpha$, then the trivial solution of (6.1) is exponentially asymptotically stable in the p-th moment. Moreover, if $p\pi^2 > \alpha$, then by Corollary 5

$$\limsup_{t\to\infty} (1/t) \log \|u(t)\| \leqslant -(p\pi^2 - \alpha)/2p, \qquad \text{w.p.1.}$$

Example 2. As the second example, let us consider semilinear equations of the type introduced by Kotelenez in [12]:

$$\begin{cases} dX_t &= [(A + C(t))X_t + f(t, X_t)]dt + g(t, X_t)dW_t \\ X(0) &= X_0 \end{cases} \tag{6.4}$$

where X_0 is an H-valued, $\mathscr{F}_0$-measurable random variable, A is the generator of a C_0-semigroup $S(t)$, $t \geqslant 0$, and $C(t) \in L(H)$ is measurable in t. Assume that the family $A(t) = A + C(t)$ generates an almost strong evolution operator $U(t, s)$ satisfying

$$\|U(t, s)\|_L \leqslant e^{-\lambda(t-s)},$$

for some real $\lambda > 0$ and almost every $0 \leqslant s \leqslant t$. This holds, for example, if there exist positive costants β_1 and β_2 such that $\beta_1 > \beta_2$, $\|S(t)\|_L \leqslant e^{-\beta_1 t}$ and $\sup_{t\geqslant 0} \|C(t)\|_L \leqslant \beta_2$. Let W_t be the cylindrical Brownian motion on K and f, g satisfy Hypothesis $3_{-\lambda}(a)$–(d). For $p \geqslant 2$, if $E\|X_0\|^p$ and $E[\sup_{0\leqslant s\leqslant t} \|g(s, 0)\|_2^p]$ are finite, then by Theorem 3, the equation (6.4) has a unique, L^p-bounded, continuous mild solution X_t. Now, by Corollary 2, if $p\lambda > \alpha$ then the mild solution of (6.4) is exponentially asymptotically stable in the p-th moment. Furthermore, Corollary 5 implies that if $p\lambda > \alpha$ and $f(t, 0) = g(t, 0) = 0$ for all $t \geqslant 0$, then for any mild solution X_t with initial value X_0 of (6.4), we have

$$\limsup_{t\to\infty} (1/t) \log \|X_t\| \leqslant -(p\lambda - \alpha)/2p, \qquad \text{w.p.1.}$$

Example 3. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the stochastic initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} &= A(t, x)u(t, x) + f(t, x, u(t, x)) + g(t, x, u(t, x))\frac{\partial W(t,x)}{\partial t} \\ u(0, x) &= u_0(x) \qquad \text{on } \Omega \\ u(t, x) &= 0 \qquad \text{on } [0, \infty) \times \partial\Omega \end{cases} \tag{6.5}$$

where $W(t, x)$ is a space-time Brownian motion with self-adjoint, positive, nuclear covariance operator Q and

$$A(t, x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t, x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(t, x)\frac{\partial}{\partial x_i} + c(t, x),$$

is a uniformly elliptic operator, i.e., there exists a constant $\delta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \geqslant \delta \sum_{i=1}^{n} \xi_i^2,$$

for each $t \in [0,\infty)$, a.e. $x \in \Omega$ and for all real vectors $\xi \in \mathbb{R}^n$. Assume that the coefficients $a_{ij} = a_{ji}$, b_i and c, $i,j = 1,2,\ldots,n$, are real-valued smooth functions on $[0,\infty)\times\bar{\Omega}$ with bounded derivatives. Assume also that there exist positive constants β_1 and β_2 such that $|b_i(t,x)| \leqslant \beta_1$ and $c(t,x) \leqslant -\beta_2$ for all $t \in [0,\infty)$ and a.e. $x \in \Omega$. We will make the following further hypotheses:

HYPOTHESIS 5. *(a) For each $t \in [0,\infty)$, the functions $f(t,.,.)$, $g(t,.,.)$: $\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy the Carathéodory condition;*

(b) There exist a function $a \in L^2(\Omega)$ and a constant $C > 0$ such that

$$|f(t,x,y)| \leqslant a(x) + C|y|,$$

$$|g(t,x,y)| \leqslant a(x) + C|y|,$$

for all $t \in [0,\infty)$, $x \in \Omega$ and $y \in \mathbb{R}$;

(c) $g(t,x,.)$ is uniformly Lipschitz with Lipschitz constant $C > 0$, i.e.

$$|g(t,x,y_2) - g(t,x,y_1)| \leqslant C|y_2 - y_1|, \qquad \forall t \in [0,\infty), \quad x \in \Omega, \quad y_2, y_1 \in \mathbb{R};$$

(d) $-f(t,x,.)$ is semimonotone with parameter M, uniformly with respect to $t \in [0,\infty)$ and $x \in \Omega$, i.e.

$$(f(t,x,y_2) - f(t,x,y_1))(y_2 - y_1) \leqslant M(y_2 - y_1)^2.$$

Let $H = L^2(\Omega)$. For each $t \in [0,\infty)$ define an operator $A(t)$ on H by

$$\mathscr{D}(A(t)) = H^2(\Omega)\cap H_0^1(\Omega),$$

and

$$A(t)u = A(t,x)u(t,x) \qquad \text{for } u \in \mathscr{D}(A(t)).$$

Denote by $\bar{f}$ and $\bar{g}$, the functions defined by

$$(\bar{f}(t,u))(x) = f(t,x,u(x)), \qquad u \in H,$$

and

$$(\bar{g}(t,u)v)(x) = g(t,x,u(x))v(x), \qquad u \in H,\ v \in K,$$

respectively; here K is a Hilbert space continuously embedded into H. If $K_0 = Q^{\frac{1}{2}}K$, then the initial-boundary value problem (6.5) can be written as the SEE

$$dX_t = [A(t)X_t + \bar{f}(t,X_t)]dt + \bar{g}(t,X_t)d\tilde{W}_t, \qquad X(0) = X_0, \tag{6.6}$$

where $\bar{g} : [0,\infty)\times H \to L_2(K_0,H)$ and $\tilde{W}_t$ is a cylindrical Brownian motion on K_0. It is well-known (see e.g. [22] or [23]) that if $\beta_2 > \frac{n\beta_1}{2\delta} + \frac{\delta}{2}$, then there exists $\lambda > 0$ such that $A(t)$ satisfies Hypothesis $2_{-\lambda}(a)$ and (b). Therefore, by Remark 3, the family $A(t)$, $t \geqslant 0$, generates a unique almost strong evolution operator $U(t,s)$ which is exponentially bounded with parameter $-\lambda$ on $[0,\infty)$. It is easy to see that $\bar{f}$ and $\bar{g}$ satisfy Hypothesis $3_{-\lambda}(a)$–(d). Thus, by Corollary 2, if $p\lambda > \alpha$ then any mild solution X_t of (6.6) (which is called a *generalized solution* of (6.5)), with $\mathscr{F}_0$-measurable initial value X_0, is exponentially asymptotically stable in the p-th moment.

Moreover, Corollary 5 implies that if $f(t,x,0) = g(t,x,0) = 0$ for all $t \in [0,\infty)$ and $x \in \Omega$, and if $p\lambda > \alpha$, then for any mild solution X_t of (6.6), we have

$$\limsup_{t\to\infty} (1/t)\log\|X_t\| \leqslant -(p\lambda-\alpha)/2p, \qquad \text{w.p.1.}$$

Acknowledgement. The authors wish to express their thanks to anonymous referees for their valuable suggestions and to the editor for his careful remarks which made the paper more complete.

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(Received May 18, 1999)
(Revised February 10, 2000)

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