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INEQUALITIES FOR MEAN VALUES OVER QUASIBALLS FOR FUNCTIONS DEFINED ON ARBITRARY OPEN SETS

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Abstract. Sharp two-sided estimates for mean values over quasiballs are established, which allow reducing the problem of obtaining certain weighted integral inequalities for arbitrary open sets to appropriate inequalities for quasiballs.

1. Quasidistances, quasiballs, quasispheres

Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be any quasidistance on $\mathbb{R}^n$, i. e.,
i) $d(x,y) \geqslant 0$; $d(x,y) = 0 \Longleftrightarrow x = y$, $x, y \in \mathbb{R}^n$;
ii) $d(x,y) = d(y,x)$, $x,y \in \mathbb{R}^n$;
iii) *for some* $k \geqslant 1$

$$d(x,z) \leqslant k(d(x,y) + d(y,z)), \ x,y,z \in \mathbb{R}^n.$$

We note that from *ii*), *iii*) it follows that
iii') $d(x,z) \geqslant \frac{1}{k}d(x,y) - d(y,z)$, $\frac{1}{k}d(y,z) - d(x,y)$.
(If $k = 1$, then these inequalities are equivalent to $|d(x,y) - d(y,z)| \leqslant d(x,z)$.)

Moreover, let $B_d(x,r)$ and $S_d(x,r)$ be an open quasiball, a quasisphere respectively, centered at the point $x \in \mathbb{R}^n$ of radius $r > 0$, i.e., an open ball, a sphere respectively, with respect to the quasidistance d: $B_d(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\}$, $S_d(x,r) = \{y \in \mathbb{R}^n : d(x,y) = r\}$.

Also, for $x \in \mathbb{R}^n, G \subset \mathbb{R}^n$, let $\varrho_d(x,G)$ be the quasidistance from x to G, i.e., $\varrho_d(x,G) = \inf_{y \in G} d(x,y)$.

In general, a quasidistance on $\mathbb{R}^n$, as a function of $(x,y) \in \mathbb{R}^{2n}$, is not continuous with respect to the Euclidean distance or even measurable with respect to the Lebesgue measure. For example, if $d(x,y) = g(|x-y|)$, where $x,y \in \mathbb{R}^n$, $|x-y|$ is the Euclidean distance and $g : [0,\infty) \to [0,\infty)$ is an arbitrary (in particular non-measurable) function satisfying for some $0 < \alpha_1 < \alpha_2 < \infty$ the inequality $\alpha_1 u \leqslant g(u) \leqslant \alpha_2 u$, $u \geqslant 0$, then $d(\cdot,\cdot)$ is a quasidistance. Another example of a quasidistance could be obtained if $g : [0,\infty) \to [0,\infty)$ satisfies the following conditions: $g(0) = 0$, g is almost

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increasing, i.e., for some $\alpha_3 \geqslant 1$, $g(u_1) \leqslant \alpha_3 g(u_2)$, $0 \leqslant u_1 \leqslant u_2 < \infty$, and also, for some $\alpha_4 \geqslant 1$, $g(u_1+u_2) \leqslant \alpha_4(g(u_1)+g(u_2))$, $0 \leqslant u_1, u_2 < \infty$.

We shall also consider quasidistances satisfying the following condition:

$iv)$ for each $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$d(x,z) \leqslant (1+\varepsilon)d(x,y) + C(\varepsilon)d(x,y), \; x,y,z \in \mathbb{R}^n.$$

We note that from $iv)$ it follows that

$iv')$ $|d(x,y) - d(y,z)| \leqslant \varepsilon d(x,y) + C(\varepsilon)d(x,z)$.

Indeed, by $iv)$

$$\frac{d(x,y) - C(\varepsilon)d(x,z)}{1+\varepsilon} \leqslant d(y,z) \leqslant (1+\varepsilon)d(x,y) + C(\varepsilon)d(x,z). \tag{1}$$

Hence,

$$\frac{\varepsilon d(x,y) + C(\varepsilon)d(x,z)}{1+\varepsilon} \leqslant d(y,z) - d(x,y) \leqslant \varepsilon d(x,y) + C(\varepsilon)d(x,z),$$

and $iv')$ follows.

Any quasidistance d satisfying $iv)$ is a continuous function (with respect to the quasidistance d). This follows since by $iv')$, for all $x_0, y_0, x, y \in \mathbb{R}^n$ and for all $\varepsilon > 0$,

$$|d(x,y) - d(x_0,y_0)| \leqslant |d(x,y) - d(x_0,y)| + |d(x_0,y) - d(x_0,y_0)|$$
$$\leqslant \varepsilon d(x_0,y) + C(\varepsilon)d(x,x_0) + \varepsilon d(x_0,y_0) + C(\varepsilon)d(y,y_0)$$
$$\leqslant \varepsilon(2d(x_0,y_0) + C(1)d(y,y_0)) + C(\varepsilon)d(x,x_0) + \varepsilon d(x_0,y_0) + C(\varepsilon)d(y,y_0)$$
$$\leqslant 3\varepsilon d(x_0,y_0) + (C(\varepsilon) + \varepsilon C(1))(d(x,x_0) + d(y,y_0)).$$

By the continuity of d it follows, in particular, that all quasispheres are closed sets (with respect to the quasidistance d). Also all quasiballs $B_d(x,r)$ are open sets (with respect to the quasidistance d). Indeed, if $y \in B_d(x,r)$, then the quasiball $B_d(x,\varrho) \subset B_d(x,r)$, where $\varrho = \frac{1}{2}(r - d(x,y))\left(C\left(\frac{r-d(x,y)}{2d(x,y)}\right)\right)^{-1}$. This follows from $iv)$, where $\varepsilon = \frac{r-d(x,y)}{2d(x,y)}$, since, for all $z \in B_d(y,\varrho)$

$$d(z,x) \leqslant (1+\varepsilon)d(x,y) + C(\varepsilon)d(y,z) < (1+\varepsilon)d(x,y) + C(\varepsilon)\varrho = r.$$

We shall say that a quasidistance on $\mathbb{R}^n$ is *regular* if it satisfies $iv)$ and also

$v)$ *it is continuous with respect to the Euclidean distance*,

$vi)$ *for all* $x \in \mathbb{R}^n$ *and* $r > 0$ $\quad \operatorname{meas} S_d(x,r) = 0$,

$vii)$ *there exist two functions* $C_j : (0,\infty) \to (0,\infty)$, $j = 1,2$, *such that for all* $x \in \mathbb{R}^n$

$$C_1\left(\frac{r_2}{r_1}\right) \leqslant \frac{\operatorname{meas} B_d(x,r_2)}{\operatorname{meas} B_d(x,r_1)} \leqslant C_2\left(\frac{r_2}{r_1}\right).$$

If a quasidistance is regular, then all quasiballs $B_d(x,r)$ are open sets with respect to the Euclidean distance (hence, Lebesgue measurable sets of positive measure). Since, as was already proved, these quasiballs are open with respect to the quasidistance d, to prove this it is enough to verify that each quasiball $B_d(x,r)$ contains an Euclidean ball $B(x,\varrho)$ for sufficiently small ϱ, which follow from the continuity of a regular quasidistance with respect to the Euclidean distance.

A typical example of a regular quasidistance, which will be of interest for us in view of applications, is

$$d(x,y) = \left(\alpha_1|x_1-y_1|^{\beta_1} + \cdots + \alpha_n|x_n-y_n|^{\beta_n}\right)^{\gamma}, \tag{2}$$

where $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in \mathbb{R}^n$ and $\alpha_1,\beta_1,\dots,\alpha_n,\beta_n,\gamma$ are fixed positive numbers.

Property $iv)$ follows from the elementary inequality

$$(a+b)^{\sigma} \leqslant (1+\varepsilon)a^{\sigma} + c(\varepsilon)b^{\sigma}, \;\; a,b \geqslant 0, \;\varepsilon,\sigma > 0,$$

where $c(\varepsilon) = 1$ if $\sigma \leqslant 1$ and $c(\varepsilon) = \left(1-(1+\varepsilon)^{\frac{1}{1-\sigma}}\right)^{1-\sigma}$ if $\sigma > 1$.

2. Regularity properties of the mean values over quasiballs

Let $\Omega \subset \mathbb{R}^n$ be an open set,[1] $0 < \varepsilon \leqslant 1$ and let d a quasidistance on $\mathbb{R}^n$. Assume that $\varphi \in L(\Omega)$, i.e., φ is Lebesgue summable on Ω. We shall be interested in the mean values $\varphi_{d,\varepsilon}$ of the function φ over the quasiballs $B_d(x,\varepsilon\varrho_d(x))$, where $x \in \Omega$ and $\varrho_d(x) \equiv \varrho_d(x,\partial\Omega)$:

$$\varphi_{d,\varepsilon}(x) = \frac{1}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))} \int\limits_{B_d(x,\varepsilon\varrho_d(x))} \varphi(y)\,dy. \tag{3}$$

LEMMA 1. *Let d be a quasidistance on $\mathbb{R}^n$ such that all quasiballs $B_d(x,r)$ are Lebesgue measurable sets of positive finite measure, and let $0 < \varepsilon \leqslant 1$. Moreover, let $\Omega \subset \mathbb{R}^n$ be an open set, $\Omega \neq \mathbb{R}^n$ and $\varphi \in L(\Omega)$.*

Suppose that the set

$$\mathscr{A}_{d,\varepsilon} = \{(x,y) \in \Omega\times\Omega : \; d(x,y) < \varepsilon\varrho_d(x)\} \subset \mathbb{R}^{2n} \tag{4}$$

is Lebesgue measurable.

Then the mean value $\varphi_{d,\varepsilon}$ is a function measurable on Ω.

Proof. Since the set $\mathscr{A}_{d,\varepsilon}$ is measurable, by the Fubini theorem the function $\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))$ is measurable on Ω. The characteristic function $\chi_{\mathscr{A}_{d,\varepsilon}}$ of the set $\mathscr{A}_{d,\varepsilon}$ is measurable on $\mathbb{R}^{2n}$. Hence, for all $k \in \mathbb{N}$, $\chi_{\mathscr{A}_{d,\varepsilon}}\varphi \in L((\Omega\cap B(0,k))\times\Omega)$,

[1] Here and in the sequel we mean an open set with respect to the Euclidean distance.

where $B(0,k)$ is the Euclidean ball. Again by the Fubini theorem it follows that the function

$$\int\limits_{\Omega} \chi_{\mathscr{A}_{d,\varepsilon}}(x,y)\,\varphi(y)\,dy = \int\limits_{\Omega} \chi_{B_d(x,\varepsilon\varrho_d(x))}(y)\varphi(y)\,dy = \int\limits_{B_d(x,\varepsilon\varrho_d(x))} \varphi(y)\,dy$$

is measurable on $\Omega \cap B(0,k)$ for all $k \in \mathbb{N}$ and, hence, on Ω.

Since $0 < \operatorname{meas} B_d(x,\varepsilon\,\varrho_d(x)) < \infty$ for all $x \in \Omega$, the function $\varphi_{d,\varepsilon}$ is also measurable on Ω. $\square$

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and that for each $x \in \Omega$ a set $G(x) \subset \mathbb{R}^n$ is measurable. We shall say that the set-valued function $G(\cdot)$ is continious on Ω if, for each $x_0 \in \Omega$, $\lim\limits_{x \to x_0} \operatorname{meas}(G(x)\Delta G(x_0)) = 0$, where $G(x)\Delta G(x_0) = (G(x) \setminus G(x_0)) \cup (G(x_0) \setminus G(x))$ is the symmetric difference of the sets $G(x)$ and $G(x_0)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\varphi \in L^{loc}(\Omega)$, i.e., $\varphi \in L(K)$ for each compact $K \subset \Omega$. Suppose that a set valued function $G(\cdot)$ is continious on Ω and is such that $\overline{G(x)} \subset \Omega$ for each $x \in \Omega$. Then by the absolute continuity of the Lebesgue integral it follows that the function $\int\limits_{G(x)} \varphi(y)\,dy$ is continuous on Ω, since

$$\left| \int\limits_{G(x)} \varphi(y)\,dy - \int\limits_{G(x_0)} \varphi(y)\,dy \right| \leqslant \int\limits_{G(x)\Delta G(x_0)} |\varphi(y)|\,dy. \tag{5}$$

LEMMA 2. *Let d be a regular quasidistance on $\mathbb{R}^n$ and let $0 < \varepsilon < 1$. Moreover, let $\Omega \subset \mathbb{R}^n$ be an open set, $\Omega \neq \mathbb{R}^n$ and $\varphi \in L(\Omega)$.*

Then the mean value $\varphi_{d,\varepsilon}$ is a function continuous on Ω.

Proof. Let us prove that the set-valued function $B_d(x,\varepsilon\,\varrho_d(x))$ is continuous on Ω. To do this, first we note that the function ϱ_d is continuous on Ω. Indeed, let $x \in \Omega$ be fixed and let $y \in \Omega$, $z \in \partial\Omega$. By (1) it follows that for all $\gamma > 0$

$$\frac{d(x,z) - C(\gamma)d(x,y)}{1+\gamma} \leqslant d(y,z) \leqslant (1+\gamma)d(x,z) + C(\gamma)d(x,y). \tag{6}$$

By taking the infimums with respect to $z \in \partial\Omega$ we have

$$\frac{\varrho_d(x) - C(\gamma)\,d(x,y)}{1+\gamma} \leqslant \varrho_d(y) \leqslant (1+\gamma)\varrho_d(x) + C(\gamma)d(x,y).$$

Hence,

$$-\Big(\gamma\,\varrho_d(x) + C(\gamma)d(x,y)\Big) \leqslant \frac{-\gamma\,\varrho_d(x) + C(\gamma)d(x,y)}{1+\gamma}$$
$$\leqslant \varrho_d(y) - \varrho_d(x) \leqslant \gamma\,\varrho_d(x) + C(\gamma)d(x,y).$$

Thus, for all $\gamma > 0$

$$\big|\varrho_d(y) - \varrho_d(x)\big| \leqslant \gamma\,\varrho_d(x) + C(\gamma)d(x,y).$$

Consequently, for all $\gamma > 0$

$$\limsup_{y\to x} |\varrho_d(y) - \varrho_d(x)| \leqslant \gamma\varrho_d(x) + C(\gamma)\lim_{y\to x} d(x,y) = \gamma\,\varrho_d(x).$$

Hence, $\lim\limits_{y\to x} |\varrho_d(y) - \varrho_d(x)| = 0$ and $\lim\limits_{y\to x} \varrho_d(y) = \varrho_d(x)$.

We claim that for all $\sigma > 0$ there exists $\delta > 0$ such that if $|y-x| < \delta$, then

$$B(x, \varepsilon\varrho_d(x) - \sigma) \subset B_d(y, \varepsilon\varrho_d(y)) \subset B_d(x, \varepsilon\varrho_d(x) + \sigma). \tag{7}$$

Indeed, let $z \in B(x, \varepsilon\varrho_d(x) - \sigma)$, then for all $\gamma > 0$

$$d(x,y) \leqslant (1+\gamma)\,d(z,x) + C(\gamma)d(y,x) < (1+\gamma)(\varepsilon\varrho_d(x) - \sigma) + C(\gamma)d(y,x)$$

$$= \varepsilon\varrho_d(y) + \varepsilon(\varrho_d(x) - \varrho_d(y)) + \gamma\varrho_d(x) + C(\gamma)d(x,y) - \sigma.$$

Let $\gamma = \frac{\sigma}{3\varrho_d(x)}$. By the continuity of d and ϱ_d there exists $\delta > 0$ such that $\varepsilon(\varrho_d(x) - \varrho_d(y)) < \frac{\sigma}{3}$ and $C(\gamma)d(y,x) < \frac{\sigma}{3}$ if $|y-x| < \delta$. Hence, for such y, for all $z \in B(x, \varepsilon\varrho_d(x) - \sigma)$ we have $d(z,y) < \varepsilon\varrho_d(y)$ and the first inclusion (7) follows. The second one is proved similarly.

Since by (7)

$$B_d(y, \varepsilon\varrho_d(y))\Delta B_d(x, \varepsilon\varrho_d(x)) \subset B_d(x, \varepsilon\varrho_d(x) + \sigma) \setminus B_d(x, \varepsilon\varrho_d(x) - \sigma)$$

if $|y-x| < \delta$, and

$$\lim_{\sigma\to 0} \operatorname{meas}\,(B_d(x, \varepsilon\varrho_d(x) + \sigma) \setminus B_d(x, \varepsilon\varrho_d(x) - \sigma))$$

$$= \operatorname{meas}\left(\bigcap_{\sigma>0} (B_d(x, \varepsilon\varrho_d(x) + \sigma) \setminus B_d(x, \varepsilon\varrho_d(x) - \sigma))\right)$$

$$= \operatorname{meas} S_d(x, \varepsilon\varrho_d(x)) = 0,$$

it follows that

$$\lim_{y\to x} \operatorname{meas} B_d(y, \varepsilon\varrho_d(y))\Delta B_d(x, \varepsilon\varrho_d(x)) = 0.$$

Since,

$$|\operatorname{meas} B_d(y, \varepsilon\varrho_d(y)) - \operatorname{meas} B_d(x, \varepsilon\varrho_d(x))|$$

$$\leqslant \operatorname{meas}\Big(B_d(y, \varepsilon\varrho_d(y))\Delta B_d(x, \varepsilon\varrho_d(x))\Big),$$

it follows that the function $B_d(x, \varepsilon\varrho_d(x))$ is continuous on Ω.

Since $\varphi \in L(\Omega)$, by the absolute continuity of the Lebesgue integral, the function $\int\limits_{B_d(x,\varepsilon\varrho_d(x))} \varphi(y)\,dy$ is continuous on Ω, and the continuity of $\varphi_{d,\varepsilon}$ on Ω follows. □

REMARK 1. If all quasiballs $B_d(x,r)$ are bounded, then one can replace the assumption $\varphi \in L(\Omega)$ by $\varphi \in L^{loc}(\Omega)$. This follows since for sufficiently small σ the set $\overline{B_d(x, \varepsilon\varrho_d(x) + \sigma)} \subset \Omega$. Hence, for $\varphi \in L^{loc}(\Omega)$, we have $\varphi \in L(B_\delta(x, \varepsilon\varrho_d(x)))$.

3. Estimates for the mean values over quasiballs

Together with the quasiballs $B_d(x, \varepsilon\varrho_d(x))$ we shall consider the *conjugate* quasiballs

$$B_d^*(x, \varepsilon\varrho_d(x)) = \{y \in \mathbb{R}^n : d(y,x) < \varepsilon\varrho_d(y)\} \tag{8}$$

The estimates in Theorems 1 – 3 below will be based on the following statement.

LEMMA 3. *Let d be a quasidistance on $\mathbb{R}^n$ such that all quasiballs $B_d(x,r)$ are Lebesgue measurable sets of positive finite measure and let $0 < \varepsilon < 1$. Moreover, let $\Omega \subset \mathbb{R}^n$ be an open set, $\Omega \neq \mathbb{R}^n$. Suppose that the set $\mathscr{A}_{d,\varepsilon}$ defined by (4) is Lebesgue measurable. Then*

$$c_1(d,\varepsilon) = \inf_{\varphi\in L(\Omega),\varphi\not\sim 0} \frac{\| |\varphi|_{d,\varepsilon}\|_{L(\Omega)}}{\|\varphi\|_{L(\Omega)}} \leqslant \sup_{\varphi\in L(\Omega),\varphi\not\sim 0} \frac{\| |\varphi|_{d,\varepsilon}\|_{L(\Omega)}}{\|\varphi\|_{L(\Omega)}} = c_2(d,\varepsilon), \tag{9}$$

where

$$c_1(d,\varepsilon) = \operatorname*{ess\,inf}_{y\in\Omega} \int\limits_{B_d^*(y,\varepsilon\varrho_d(y))} \frac{dy}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))} \tag{10}$$

and

$$c_2(d,\varepsilon) = \operatorname*{ess\,sup}_{y\in\Omega} \int\limits_{B_d^*(y,\varepsilon\varrho_d(y))} \frac{dy}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))}. \tag{11}$$

Proof. Since the characteristic function $\chi_{\mathscr{A}_{d,\varepsilon}}(x,y)$ is measurable on $\Omega\times\Omega$, and the functions $\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))$ and $|\varphi(y)|$ are measurable on Ω, the function $\frac{\chi_{\mathscr{A}_{d,\varepsilon}}(x,y)\,|\varphi(y)|}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))}$ is nonnegative and measurable on $\Omega\times\Omega$.

Hence, by the Fubini theorem, taking into account that

$$\chi_{\mathscr{A}_{d,\varepsilon}}(x,y) = \chi_{B_d(x,\varepsilon\varrho_d(x))}(y) = \chi_{B_d^*(y,\varepsilon\varrho_d(y))}(x), \quad x,\ y\in\mathbb{R}^n,$$

we have

$$I = \int\limits_\Omega |\varphi|_{d,\varepsilon}(x)\,dx = \int\limits_\Omega \left(\frac{1}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))} \int\limits_{B_d(x,\varepsilon\varrho_d(x))} |\varphi(y)|\,dy\right) dx$$

$$= \int\limits_\Omega \left(\int\limits_\Omega \frac{\chi_{B_d(x,\varepsilon\varrho_d(x))}|\varphi(y)|}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))}\,dy\right) dx = \int\limits_{\Omega\times\Omega} \frac{\chi_{\mathscr{A}_{d,\varepsilon}}(x,y)|\varphi(y)|}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))}\,dx\,dy$$

$$= \int\limits_\Omega \left(\int\limits_\Omega \frac{\chi_{B_d^*(y,\varepsilon\varrho_d(y))}|\varphi(y)|}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))}\,dx\right) dy$$

$$= \int\limits_\Omega \left(\int\limits_{B_d^*(y,\varepsilon\varrho_d(y))} \frac{dx}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))}\right)|\varphi(y)|\,dy.$$

In particular it follows that the function $\int\limits_{B_d^*(y,\varepsilon\varrho_d(y))} \frac{dx}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))}$, which could be infinite for some $x \in \Omega$, is measurable on Ω. Consequently,

$$c_1(d,\varepsilon)\,\|\varphi\|_{L(\Omega)} \leqslant I \leqslant c_2(d,\varepsilon)\,\|\varphi\|_{L(\Omega)}.$$

If $c_2(d,\varepsilon) < \infty$, then by the definition of the essential supremum it follows that for all $k \in \mathbb{N}$ there exist measurable sets $\Omega_k \subset \Omega$ such that $0 < \operatorname{meas}\Omega_k < \infty$ and

$$\operatorname*{ess\,sup}_{y\in\Omega_k} \int\limits_{B_d^*(y,\varepsilon\varrho_d(y))} \frac{dx}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))} \geqslant c_2(d,\varepsilon) - \frac{1}{k}.$$

By considering the functions $\varphi_k = \chi_{\Omega_k}$ it follows that the constant $c_2(d,\varepsilon)$ is sharp. (If $c_2(d,\varepsilon) = \infty$, then it follows that $\sup\limits_{\varphi\in L(\Omega),\varphi\not\sim 0} \frac{I}{\|\varphi\|_{L(\Omega)}} = \infty$). The case of $c_1(d,\varepsilon)$ is similar. $\square$

THEOREM 1. *Let d be a quasidistance on $\mathbb{R}^n$ such that all quasiballs $B_d(x,r)$ are Lebesgue measurable sets of positive finite measure. Moreover, let $\Omega \subset \mathbb{R}^n$ be an open set, $\Omega \neq \mathbb{R}^n$.*

Suppose that for all $0 < \varepsilon < \frac{1}{k}$, the set $\mathscr{A}_{d,\varepsilon}$ defined by (4) is Lebesgue measurable and for all $0 < \varepsilon < \frac{1}{k}$ and $0 < \delta < \infty$

$$0 < \operatorname*{ess\,inf}_{(x,y)\in\mathscr{A}_{d,\varepsilon}} \frac{\operatorname{meas} B_d(y,\delta\varrho_d(y))}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))} \leqslant \operatorname*{ess\,sup}_{(x,y)\in\mathscr{A}_{d,\varepsilon}} \frac{\operatorname{meas} B_d(y,\delta\varrho_d(y))}{\operatorname{meas} B_d(x,\varepsilon\varrho_d(x))} < \infty. \tag{12}$$

Then for each $0 < \varepsilon < \frac{1}{k}$ there exist $c_3(d,\varepsilon), c_4(d,\varepsilon) > 0$ such that for all $f \in L(\Omega)$

$$c_3(d,\varepsilon)\int\limits_{\Omega} |f|\,dx \leqslant \int\limits_{\Omega} |f|_{d,\varepsilon}\,dx \leqslant c_4(d,\varepsilon)\int\limits_{\Omega} |f|\,dx. \tag{13}$$

Proof. Let $x, y \in \Omega$ and $z \in \partial\Omega$. Then

$$\frac{1}{k}\,d(x,z) - d(x,y) \leqslant d(y,z) \leqslant k\,(d(x,z) - d(x,y)).$$

By taking the infimums with respect to $z \in \Omega$ it follows that

$$\frac{1}{k}\,\varrho_d(x) - d(x,y) \leqslant \varrho_d(y) \leqslant k(\varrho_d(x) + d(x,y)). \tag{14}$$

Furthermore, for $0 < \varepsilon < \frac{1}{k}$,

$$B_d\Big(y, \frac{\varepsilon}{k(1+\varepsilon)}\,\varrho_d(y)\Big) \subset B_d^*(y,\varepsilon\varrho_d(y)) \subset B_d\Big(y, \frac{k\varepsilon}{1-k\varepsilon}\,\varrho_d(y)\Big). \tag{15}$$

Indeed, if $x \in B_d(y, \frac{\varepsilon\varrho_d(y)}{k(1+\varepsilon)})$, then by (14)

$$d(x,y) < \frac{\varepsilon\varrho_d(y)}{k(1+\varepsilon)} \leqslant \frac{\varepsilon}{1+\varepsilon}(\varrho_d(x) + d(x,y)).$$

Hence, $d(x,y) < \varepsilon\varrho_d(x)$ and $x \in B_d^*(y, \varrho_d(y))$. On the other hand, if $x \in B_d^*(y, \varrho_d(y))$, then $d(x,y) < \varepsilon\varrho_d(x) \leqslant k\varepsilon(\varrho_d(y) + d(x,y))$. Hence, $d(x,y) < \frac{k\varepsilon}{1-k\varepsilon}\varrho_d(y)$ and $x \in B_d\left(y, \frac{k\varepsilon}{1-k\varepsilon}\varrho_d(y)\right)$. Consequently,

$$c_2(d,\varepsilon) \leqslant \sup_{(x,y)\in\mathscr{A}_{d,\varepsilon}} \frac{\operatorname{meas} B_d(y, \frac{k\varepsilon}{1-k\varepsilon}\varrho_d(y))}{\operatorname{meas} B_d(x, \varepsilon\varrho_d(x))} \equiv c_4(d,\varepsilon) < \infty. \tag{16}$$

Similarly,

$$c_1(d,\varepsilon) \geqslant \inf_{(x,y)\in\mathscr{A}_{d,\varepsilon}} \frac{\operatorname{meas} B_d(y, \frac{\varepsilon}{k(1+\varepsilon)}\varrho_d(y))}{\operatorname{meas} B_d(x, \varepsilon\varrho_d(x))} \equiv c_3(d,\varepsilon) > 0. \tag{17}$$

Hence, (13) follows from (9). $\square$

THEOREM 2. *Let d be a regular quasidistance on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be an open set, $\Omega \neq \mathbb{R}^n$. Then for each $0 < \varepsilon < 1$ there exist $c_5(d,\varepsilon)$, $c_6(d,\varepsilon) > 0$ such that for all $f \in L(\Omega)$*

$$c_5(d,\varepsilon)\int_\Omega |f|\,dx \leqslant \int_\Omega |f|_{d,\varepsilon}\,dx \leqslant c_6(d,\varepsilon)\int_\Omega |f|\,dx. \tag{18}$$

Proof. We claim that for $0 < \varepsilon < 1$

$$B_d\left(y, \frac{\varepsilon}{k(1+\varepsilon)}\varrho_d(y)\right) \subset B_d^*(y, \varepsilon\varrho_d(y)) \subset B_d\left(y, \frac{2\varepsilon\, C(\frac{1-\varepsilon}{2\varepsilon})}{1-\varepsilon}\varrho_d(y)\right). \tag{19}$$

The first inclusion coincides with the first inclusion (15).

Let $x \in B_d^*(y, \varrho_d(y))$, then by applying (6) we have

$$d(x,y) < \varepsilon\varrho_d(x) \leqslant \varepsilon(C(\gamma)\varrho_d(y) + (1+\gamma)d(x,y)).$$

Hence,

$$d(x,y) < \frac{\varepsilon C(\gamma)}{1-\varepsilon(1+\gamma)}\varrho_d(y) \tag{20}$$

if $1-\varepsilon(1+\gamma) > 0$. Taking $\gamma = \frac{1-\varepsilon}{2\varepsilon}$ we get the second inclusion (19).

We note that for $x \in B_d^*(y, \varepsilon\varrho_d(y))$

$$B_d(x, \varepsilon\varrho_d(x)) \subset B_d\left(y, \frac{k\varepsilon}{1-\varepsilon}\left((1-\varepsilon)k + 2(1+k\varepsilon)C\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right)\varrho_d(y)\right). \tag{21}$$

Indeed, if $z \in B_d(x, \varepsilon\varrho_d(x))$, then by *iii*), (14) and (20) (with $\gamma = \frac{1-\varepsilon}{2\varepsilon}$)

$$d(z,y) \leqslant k(d(z,x) + d(x,y)) \leqslant k(\varepsilon\varrho_d(x) + d(x,y))$$

$$\leqslant k(\varepsilon\, k(\varrho_d(y) + d(x,y)) + d(x,y)) = \varepsilon\, k^2\varrho_d(y) + k(1+\varepsilon\, k)d(x,y)$$

$$\leqslant \left(\varepsilon\, k^2 + k(1+k\varepsilon)\frac{2\varepsilon\, C(\frac{1-\varepsilon}{2\varepsilon})}{1-\varepsilon}\right)\varrho_d(y)$$

$$= \frac{k\varepsilon}{1-\varepsilon}\left((1-\varepsilon)k + 2(1+k\varepsilon)C\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right)\varrho_d(y),$$

and (21) follows.

Finally, we notice that for $x \in B_d(y, \frac{\varepsilon}{2k}\varrho_d(y))$

$$B_d\left(y, \frac{\varepsilon}{2k}\varrho_d(y)\right) \subset B_d(x, \varepsilon\varrho_d(x)). \tag{22}$$

Indeed, if $z \in B_d(y, \frac{\varepsilon}{2k}\varrho_d(y))$, then

$$d(z,x) \leqslant k(d(z,y) + d(x,y)) \leqslant k\left(\frac{\varepsilon}{2k}\varrho_d(y) + \frac{\varepsilon}{2k}\varrho_d(y)\right) = \varepsilon\varrho_d(y).$$

Consequently, by (19) and (20) and *vii*)

$$c_2(d,\varepsilon) \leqslant \sup_{(x,y)\in\mathscr{A}_{d,\varepsilon}} \frac{\operatorname{meas} B_d\left(y, \frac{2\varepsilon\, C(\frac{1-\varepsilon}{2\varepsilon})}{1-\varepsilon}\varrho_d(y)\right)}{\operatorname{meas} B_d(x, \varepsilon\varrho_d(x))}$$

$$\leqslant \sup_{y\in\Omega} \frac{\operatorname{meas} B_d\left(y, \frac{2\varepsilon\, C(\frac{1-\varepsilon}{2\varepsilon})}{1-\varepsilon}\varrho_d(y)\right)}{\operatorname{meas} B_d\left(y, \frac{\varepsilon}{2k}\varrho_d(y)\right)} \equiv c_6(d,\varepsilon) < \infty. \tag{23}$$

Similarly, by (19), (21) and *vii*)

$$c_1(d,\varepsilon) \geqslant \inf_{(x,y)\in\mathscr{A}_{d,\varepsilon}} \frac{\operatorname{meas} B_d\left(y, \frac{\varepsilon}{k(1+\varepsilon)}\varrho_d(y)\right)}{\operatorname{meas} B_d(x, \varepsilon\varrho_d(x))}$$

$$\geqslant \inf_{y\in\Omega} \frac{\operatorname{meas} B_d\left(y, \frac{\varepsilon}{k(1+\varepsilon)}\varrho_d(y)\right)}{\operatorname{meas} B_d\left(y, \frac{k\varepsilon}{1-\varepsilon}\left((1-\varepsilon)k + 2(1+k\varepsilon)C(\frac{1-\varepsilon}{2\varepsilon})\right)\varrho_d(y)\right)} \equiv c_5(d,\varepsilon) > 0. \tag{24}$$

Hence, (18) follows from (9). □

If $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$ is a norm on $\mathbb{R}^n$, we shall write $B_{\|\cdot\|}(x,r)$, $\varrho_{\|\cdot\|}(x,\sigma)$ for $B_d(x,r)$, $\varrho_d(x,\sigma)$ respectively, where $d(u,v) = \|u-v\|$.

We note that

$$\operatorname{meas} B_{\|\cdot\|}(x,r) = v_{\|\cdot\|,n} r^n, \tag{25}$$

where $v_{\|\cdot\|,n} = \operatorname{meas} B_{\|\cdot\|}(0,1)$ is the volume of the unit quasiball in $\mathbb{R}^n$ with respect to the norm $\|\cdot\|$.

THEOREM 3. *Let* $\|\cdot\|$ *be any norm on* $\mathbb{R}^n$ *and let* $0 < \varepsilon < 1$. *Moreover, let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $\Omega \neq \mathbb{R}^n$ *and for* $x \in \Omega$, *let* $\varrho_{\|\cdot\|}(x) \equiv \varrho_{\|\cdot\|}(x, \partial\Omega)$.

Then for all $f \in L(\Omega)$

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n \int\limits_\Omega |f|\,dx \leqslant \int\limits_\Omega |f|_{\|\cdot\|,\varepsilon}\,dx \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n \int\limits_\Omega |f|\,dx. \tag{26}$$

COROLLARY 1. *For all*[2] $f \in L(\Omega)$

$$\lim_{\varepsilon\to 0+} \int\limits_\Omega |f|_{\|\cdot\|,\varepsilon}\,dx = \int\limits_\Omega |f|\,dx. \tag{27}$$

Proof. In the case under consideration by (25) it follows that

$$c_1(\|\cdot\|,\varepsilon) = \frac{1}{v_{\|\cdot\|,n}\varepsilon^n} \operatorname*{ess\,inf}_{y\in\Omega} \int\limits_{B^*_{\|\cdot\|}(y,\varepsilon\varrho_{\|\cdot\|}(y))} \frac{dx}{\varrho^n_{\|\cdot\|}(x)}$$

and

$$c_2(\|\cdot\|,\varepsilon) = \frac{1}{v_{\|\cdot\|,n}\varepsilon^n} \operatorname*{ess\,sup}_{y\in\Omega} \int\limits_{B^*_{\|\cdot\|}(y,\varepsilon\varrho_{\|\cdot\|}(y))} \frac{dx}{\varrho^n_{\|\cdot\|}(x)}.$$

Inclusions (15) take the form

$$B_{\|\cdot\|}\left(y, \frac{\varepsilon}{1+\varepsilon}\varrho_{\|\cdot\|}(y)\right) \subset B^*_{\|\cdot\|}((y,\varepsilon\varrho_{\|\cdot\|}(y)) \subset B_{\|\cdot\|}\left(y, \frac{\varepsilon}{1-\varepsilon}\varrho_{\|\cdot\|}(y)\right). \tag{28}$$

Since, by the triangle inequality

$$|\varrho_{\|\cdot\|}(x) - \varrho_{\|\cdot\|}(y)| \leqslant \|x-y\|,$$

it follows that for $x \in B^*_{\|\cdot\|}((y,\varepsilon\varrho_{\|\cdot\|}(y))$

$$\begin{aligned}\varrho_{\|\cdot\|}(y) - \varepsilon\varrho_{\|\cdot\|}(x) &< \varrho_{\|\cdot\|}(y) - \|x-y\| \leqslant \varrho_{\|\cdot\|}(x)\\ &\leqslant \varrho_{\|\cdot\|}(y) + \|x-y\| < \varrho_{\|\cdot\|}(y) + \varepsilon\varrho_{\|\cdot\|}(x).\end{aligned}$$

Hence

$$\frac{\varrho_{\|\cdot\|}(y)}{1+\varepsilon} < \varrho_{\|\cdot\|}(x) < \frac{\varrho_{\|\cdot\|}(y)}{1-\varepsilon}. \tag{29}$$

Consequently,

$$c_1(\|\cdot\|,\varepsilon) \geqslant \frac{(1-\varepsilon)^n}{v_{\|\cdot\|,n}\varepsilon^n} \inf_{y\in\Omega} \frac{\operatorname{meas} B_{\|\cdot\|}(y, \frac{\varepsilon}{1+\varepsilon}\varrho_{\|\cdot\|}(y))}{\varrho_{\|\cdot\|}(y)^n} = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n.$$

Similarly

$$c_2(\|\cdot\|,\varepsilon) \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^n.$$

Hence, (26) follows from (9). □

[2] Note that in this case the Dominated Convergence Theorem cannot be applied for passing to the limit under the integral sign, because the function $(Mf)(x) = \sup_{0<\varepsilon<\frac{1}{2}} |f|_{\|\cdot\|,\varepsilon}(x)$, which is a variant of the maximal function of the function f, in general, does not belong to $L(\Omega)$. (See the books [1, 2] for details.)

4. Examples

EXAMPLE 1. In this example we consider the one-dimensional case, which is much simpler that the multidimensional one.

Since in this case all norms $\|\cdot\|$ are of the form: for some $c > 0$, $\|x\| = c|x|$, $x \in \mathbb{R}$, and each open set $\Omega \subset \mathbb{R}$ is a finite or infinite union of disjoint open intervals, the problem reduces to the case $\Omega = (a,b)$ and $\|x\| = |x|$.

LEMMA 4. *Let $\Omega \subset \mathbb{R}$ be an open set, $\Omega \neq \mathbb{R}$, $\varrho(x) = dist\,(x, \partial\Omega) = \min\limits_{y\in\partial\Omega} |x-y|$ and $0 < \varepsilon < 1$.*

Then for all $f \in L(\Omega)$

$$\int\limits_{\Omega} \left(\frac{1}{2\varepsilon\varrho(x)} \int\limits_{x-\varepsilon\varrho(x)}^{x+\varepsilon\varrho(x)} |f|\,dy \right) dx = \frac{1}{2\varepsilon} \ln \frac{1+\varepsilon}{1-\varepsilon} \int\limits_{\Omega} |f|\,dx, \tag{30}$$

if all constituent intervals are infinite, and

$$\frac{1}{\varepsilon}\ln(1+\varepsilon) \int\limits_{\Omega} |f|\,dx \leqslant \int\limits_{\Omega} \left(\frac{1}{2\varepsilon\varrho(x)} \int\limits_{x-\varepsilon\varrho(x)}^{x+\varepsilon\varrho(x)} |f|\,dy \right) dx$$
$$\leqslant \frac{1}{2\varepsilon} \ln \frac{1+\varepsilon}{1-\varepsilon} \int\limits_{\Omega} |f|\,dx \tag{31}$$

otherwise.

Both of the constants in inequality (31) *are sharp.*

Proof. If all constituent intervals are infinite, then $\Omega = (a,\infty), -\infty < a < \infty$ or $\Omega = (-\infty, b), -\infty < b < \infty$ or $\Omega = (-\infty,b) \cup (a,\infty), -\infty < b \leqslant a < \infty$, and equality (30) follows since, for example, for $-\infty < a < \infty$

$$\int\limits_{a}^{\infty} \left(\frac{1}{2\varepsilon(x-a)} \int\limits_{x-\varepsilon(x-a)}^{x+\varepsilon(x-a)} |f|\,dy \right) dx = \frac{1}{2\varepsilon} \ln \frac{1+\varepsilon}{1-\varepsilon} \int\limits_{a}^{\infty} |f|\,dy. \tag{32}$$

If at least one of the constituent intervals is finite, then inequality (31) follows since for $-\infty < a < b < \infty$

$$\int\limits_{a}^{b} \left(\frac{1}{2\varepsilon\varrho(x)} \int\limits_{x-\varepsilon\varrho(x)}^{x+\varepsilon\varrho(x)} |f|\,dy \right) dx = \int\limits_{a}^{b} \mu(y)|f(y)|\,dy, \tag{33}$$

where $-\infty < a < b < \infty$, $\varrho(x) = \min\,(x-a, b-x)$ and

$$\mu(y) = \begin{cases} \frac{1}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}, & \text{if } a \leqslant y \leqslant \frac{a+b}{2} - \varepsilon\frac{b-a}{2}, \\ \frac{1}{2\varepsilon}\ln\left(\frac{b-a}{y-a}\frac{b-a}{b-y}\frac{(1+\varepsilon)^2}{4}\right), & \text{if } \frac{a+b}{2} - \varepsilon\frac{b-a}{2} \leqslant y \leqslant \frac{a+b}{2} + \varepsilon\frac{b-a}{2}, \\ \frac{1}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}, & \text{if } \frac{a+b}{2} + \varepsilon\frac{b-a}{2} \leqslant y \leqslant b. \end{cases} \tag{34}$$

(Formula (33) could be derived by changing the order of integration.) To obtain (31) it suffices to notice that $\min_{a\leqslant y\leqslant b}\mu(y)=\frac{1}{\varepsilon}\ln(1+\varepsilon)$ and $\max_{a\leqslant y\leqslant b}\mu(y)=\frac{1}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}$.

Let (a,b) be a finite constituent interval of Ω. If $f\sim 0$ on $(\frac{a+b}{2}-\varepsilon\frac{a-b}{2},\frac{a+b}{2}+\varepsilon\frac{a-b}{2})$ and on $\Omega\setminus(a,b)$, then in the second inequality (31) there is equality. The sharpness of the constant in the first inequality (31) can be verified by taking $f=\chi_{(\frac{a+b}{2}-\frac{1}{k},\frac{a+b}{2}+\frac{1}{k})}$ and passing to the limit as $k\to\infty$. □

EXAMPLE 2. Let $\Omega=\mathbb{R}^n_+=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_n>0\}$ and let $\|x\|=\max\{|x_1|,\dots,|x_n|\}$. Then $\varrho_{\|\cdot\|}(x)=x_n$ and for $y\in\mathbb{R}^n_+$ and $n\geqslant 2$

$$B^*_{\|\cdot\|}(y,\varepsilon y_n)=\{x\in\mathbb{R}^n_+ : |x_k-y_k|<\varepsilon x_n,\ k=1,\dots,n\}$$

$$=\{x\in\mathbb{R}^n_+ : |x_k-y_k|<\varepsilon x_n,\ k=1,\dots,n-1,\ |x_n-y_n^*|<b\},$$

where $b=\frac{\varepsilon y_n}{1-\varepsilon^2}$ and $y_n^*=\frac{y_n}{1-\varepsilon^2}$. Furthermore,

$$\int\limits_{B^*_{\|\cdot\|}(y,\varepsilon y_n)}\frac{dx}{x_n^n}=\int\limits_{\frac{y_n}{1+\varepsilon}}^{\frac{y_n}{1-\varepsilon}}\frac{1}{x_n^n}\Bigg(\int\limits_{y_1-\varepsilon x_n}^{y_1+\varepsilon x_n}dx_1\cdots\int\limits_{y_1-\varepsilon x_n}^{y_1+\varepsilon x_n}dx_{n-1}\Bigg)dx_n=(2\varepsilon)^{n-1}\ln\frac{1+\varepsilon}{1-\varepsilon}.$$

Since $v_{\|\cdot\|,n}=2^n$, it follows that for all $f\in L(\mathbb{R}^n_+)$

$$\int\limits_{\mathbb{R}^n_+}\Bigg(\frac{1}{2^n\varepsilon^n x_n^n}\int\limits_{B_{\|\cdot\|}(x,\varepsilon x_n)}f(y)\,dy\Bigg)dx=\frac{1}{2\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}\int\limits_{\mathbb{R}^n_+}f\,dx. \qquad (35)$$

This equality could also be easily derived by using the Fubini theorem and the one-dimensional equality (32) where $a=0$.

EXAMPLE 3. Let $\Omega=\mathbb{R}^n_+$ and let $\|x\|=|x|=\sqrt{x_1^2+\cdots+x_n^2}$. Then $\varrho_{\|\cdot\|}(x)=x_n$ and for $y\in\mathbb{R}^n_+$ and[3] $n\geqslant 2$

$$B^*(y,\varepsilon y_n)=\{x\in\mathbb{R}^n_+ : |x-y|<\varepsilon x_n\}$$

$$=\Bigg\{x\in\mathbb{R}^n_+ : \Big(\frac{x_1-y_1}{a}\Big)^2+\cdots+\Big(\frac{x_{n-1}-y_{n-1}}{a}\Big)^2+\Big(\frac{x_n-y_n^*}{b}\Big)^2<1\Bigg\},$$

where $a=\frac{\varepsilon y_n}{\sqrt{1-\varepsilon^2}}$, and b and y_n^* are defined as in Example 2. (If $n=1$, then $B^*(y_1,\varepsilon y_1)=\{x_1\in\mathbb{R}_+ : |x_1-y_1^*|<b\}$.)

By changing variables $x_1=y_1+a\xi_1,\dots,x_{n-1}=y_{n-1}+a\xi_{n-1},x_n=y_n^*+b\xi_n=b(\frac{1}{\varepsilon}+\xi_n)$ we get

$$\int\limits_{B^*_{\|\cdot\|}(y,\varepsilon y_n)}\frac{dx}{x_n^n}=(1-\varepsilon^2)^{\frac{n-1}{2}}\int\limits_{|\xi|<1}\frac{d\xi}{(\frac{1}{\varepsilon}+\xi_n)^n}.$$

[3] $B(x,\varepsilon\varrho(x))$ and $B^*(x,\varepsilon\varrho(x))$, without index $\|\cdot\|$, denote ordinary (=with respect to the Euclidean distance) ball, conjugate ball respectively.

Consequently, for all $f \in L(\mathbb{R}^n_+)$

$$\int\limits_{\mathbb{R}^n_+} \left(\frac{1}{v_n \varepsilon^n x_n^n} \int\limits_{B(x,\varepsilon x_n)} f(y)\, dy \right) dx = A_n(\varepsilon) \int\limits_{\mathbb{R}^n_+} f\, dx, \tag{36}$$

where

$$A_n(\varepsilon) = \frac{(1-\varepsilon^2)^{\frac{n-1}{2}}}{v_n \varepsilon^n} \int\limits_{|\xi|<1} \frac{d\xi}{(\frac{1}{\varepsilon} + \xi_n)^n}.$$

If $n = 1$, then $A_1(\varepsilon) = \frac{1}{2\varepsilon} \ln \frac{1+\varepsilon}{1-\varepsilon}$, hence (35) coincides with (32) where $a = 0$. If $n \geqslant 2$, then one can verify that

$$A_2(\varepsilon) = \frac{2(1-\sqrt{1-\varepsilon^2})}{\varepsilon^2}.$$

In particular, in contrast to the case $n = 1$, $\lim\limits_{\varepsilon \to 1-} A_2(1) = 2 < \infty$. Hence, equality (35) for $n = 2$ holds also for $\varepsilon = 1$. The same is true for all $n \geqslant 2$. Indeed, if $n \geqslant 2$ and $\varepsilon = 1$, then for $y \in \mathbb{R}^n_+$

$$B^*_{\|\cdot\|}(y, y_n) = \{x \in \mathbb{R}^n_+ : |x-y| < x_n\}$$

$$= \left\{ x \in \mathbb{R}^n_+ : x_n > \frac{1}{2}\left(y_n + \frac{(x_1-y_1)^2 + \cdots + (x_{n-1}-y_{n-1})^2}{y_n} \right) \right\}.$$

Let $\bar{x} = (x_1, \ldots, x_{n-1})$. By integrating with respect to x_n, passing to the spherical coordinates and setting $\varrho = y_n t$, we get

$$\int\limits_{B^*_{\|\cdot\|}(y,y_n)} \frac{dx}{x_n^n} = \int\limits_{\mathbb{R}^{n-1}} \left(\int\limits_{x_n > \frac{1}{2}(y_n + \frac{|\bar{x}-\bar{y}|^2}{y_n})} \frac{dx_n}{x_n^n} \right) d\bar{x}$$

$$= \frac{2^{n-1}\sigma_{n-1}}{(n-1)} \int\limits_0^\infty \frac{\varrho^{n-2}\, d\varrho}{(y_n + \frac{\varrho^2}{y_n})^{n-1}} = 2^{n-1} v_{n-1} \int\limits_0^\infty \frac{t^{n-2}\, dt}{(1+t^2)^{n-1}},$$

where σ_{n-1} is the surface area of the unit sphere in $\mathbb{R}^{n-1}$. Consequently,[4]

$$A_n(1) = \frac{2^{n-1} v_{n-1}}{v_n} B\left(\frac{n-1}{2}, \frac{n-1}{2} \right) = \frac{2^{n-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}+1)\, \Gamma(\frac{n-1}{2})^2}{\Gamma(\frac{n+1}{2})\, \Gamma(n-1)} = \frac{n}{n-1}.$$

[4] We apply the formulas:

$$\int\limits_0^\infty \frac{x^m\, dx}{(1+x^k)^l} = \frac{1}{k} B\left(\frac{m+1}{k}, l - \frac{m+1}{k} \right),$$

where $m > -1$, $k > 0$, $l > \frac{m+1}{k}$, $v_n = \pi^{\frac{n}{2}} (\Gamma(\frac{n}{2}+1))^{-1}$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \alpha > 0$ and the doubling formula

$$\Gamma(n-1) = \frac{2^{n-2}}{\sqrt{\pi}} \Gamma\left(\frac{n-1}{2} \right) \Gamma\left(\frac{n}{2} \right).$$

Here Γ and B are the gamma and the beta functions. (See, for example, [3].)

Thus, for $n \geqslant 2$

$$\int\limits_{\mathbb{R}^n_+} \left(\frac{1}{v_n x_n^n} \int\limits_{B_{\|\cdot\|}(x,x_n)} f(y)\,dy \right) dx = \frac{n}{n-1} \int\limits_{\mathbb{R}^n_+} f\,dx. \tag{37}$$

EXAMPLE 4. Let $\Omega = \mathbb{R}^n_+$ and let $\|x\| = |x_1| + \cdots + |x_n|$. In this case also $\varrho_{\|\cdot\|}(x) = x_n$. Furthermore

$$B^*_{\|\cdot\|}(y, \varepsilon y_n) = \{x \in \mathbb{R}^n_+ : \|\bar{x} - \bar{y}\| + |x_n - y_n| < \varepsilon x_n\}$$

$$= \left\{ x \in \mathbb{R}^n_+ : \|\bar{x} - \bar{y}\| < \varepsilon y_n,\ \frac{y_n + \|\bar{x} - \bar{y}\|}{1+\varepsilon} < x_n < \frac{y_n - \|\bar{x} - \bar{y}\|}{1-\varepsilon} \right\}.$$

By changing variables $x_1 = y_1 + \xi_1 y_n, \dots, x_{n-1} = y_{n-1} + \xi_{n-1} y_n, x_n = \xi_n y_n$ and appying symmetry, we get

$$\int\limits_{B^*_{\|\cdot\|}(y,\varepsilon y_n)} \frac{dx}{x_n^n} = \int\limits_{\|\bar{x}-\bar{y}\|<\varepsilon y_n} d\bar{x} \int\limits_{\frac{y_n+\|\bar{x}-\bar{y}\|}{1+\varepsilon}}^{\frac{y_n-\|\bar{x}-\bar{y}\|}{1-\varepsilon}} \frac{dx_n}{x_n^n} = 2^{n-1} \int\limits_{0<\xi_1+\cdots+\xi_{n-1}<\varepsilon} d\bar{\xi} \int\limits_{\frac{1+\xi_1+\cdots+\xi_{n-1}}{1+\varepsilon}}^{\frac{1-\xi_1-\cdots-\xi_{n-1}}{1-\varepsilon}} \frac{d\xi_n}{\xi_n^n}.$$

Since $v_{\|\cdot\|,n} = \frac{2^n}{n!}$, it follows that for all $f \in L(\mathbb{R}^n_+)$

$$\int\limits_{\mathbb{R}^n_+} \left(\frac{n!}{2^n \varepsilon^n x_n^n} \int\limits_{B_{\|\cdot\|}(x,\varepsilon x_n)} f(y)\,dy \right) dx = B_n(\varepsilon) \int\limits_{\mathbb{R}^n_+} f\,dx, \tag{38}$$

where

$$B_n(\varepsilon) = \frac{(n-2)!\,n}{2\varepsilon^n}$$

$$\times \int\limits_{0<\xi_1+\cdots+\xi_{n-1}<\varepsilon} \left[\left(\frac{1+\varepsilon}{1+\xi_1+\cdots+\xi_{n-1}} \right)^{n-1} - \left(\frac{1-\varepsilon}{1-\xi_1-\cdots-\xi_{n-1}} \right)^{n-1} \right] d\bar{\xi}.$$

If $n = 2$, then

$$B_2(\varepsilon) = \frac{(1+\varepsilon)\ln(1+\varepsilon) + (1-\varepsilon)\ln(1-\varepsilon)}{\varepsilon^2}.$$

As in Example 3 $\lim\limits_{\varepsilon \to 1-} B_2(\varepsilon) = 2\ln 2 < \infty$. Hence, equality (38) is also valid for $\varepsilon = 1$. The same is true for all $n \geqslant 2$.

REMARK 2. From the formulas for conjugate balls in Examples $2-4$ it follows, in particular, that inclusions (28) are sharp, i.e., one cannot replace $\frac{\varepsilon}{1+\varepsilon}$ in the first inclusion by a larger quantity, $\frac{\varepsilon}{1-\varepsilon}$ in the second inclusion by a smaller quantity respectively.

REMARK 3. From Theorems $1-3$ it follows, in particular, that under appropriate assumptions $\|\,(|f|^p)_{d,\varepsilon}\|^{\frac{1}{p}}_{L_1(\Omega)}$ is equivalent to $\|f\|_{L_p(\Omega)}$, where $0 < p < \infty$. This equivalence could be used to derive some integral inequalities for arbitrary open sets starting with appropriate inequalities for quasiballs.

REFERENCES

[1] E. M. STEIN, *Singular integrals and differentiabilty properties of functions*, Princeton University Press, 1970.
[2] M. DE GUZMÁN, *Differentiation of integrals in* $\mathbb{R}^n$, Springer-Verlag, 1975.
[3] I. S. GRADSHTEIN, I. M. RYZHIK, *Table of integrals, series, and products*, Academic Press, 1980.

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