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SUNDMAN'S INEQUALITIES IN N–BODY PROBLEMS AND THEIR APPLICATIONS

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Abstract. In this paper, we study the necessary and sufficient conditions which make Sundman's inequalities become equalities and the relationships between the period and energy and angular momentum and the momentum of inertia for uniformly rotating planar circular solutions and the straight line solutions of N-body problems with homogeneous potentials.

1. The main results

THEOREM 1.1. *For the uniformly rotating planar circular T-periodic solutions (for example, those orbits produced by rotating planar central configurations) of N-body problems with homogeneous potentials:*

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, i = 1, \cdots, N \tag{1.1}$$

$$U(q) = U(q_1, \cdots, q_N) = \frac{1}{2} \sum_{1 \leqslant i \neq j \leqslant N} \frac{m_i m_j}{|q_i - q_j|^\alpha} \tag{1.2}$$

Where $q_i \in R^k, \alpha > 0, \alpha \neq 2$, *we have the following relationships:*

$$|C| = \frac{\alpha}{2-\alpha} \frac{1}{\pi} \times (-h)T = \frac{2\pi}{T} I \tag{1.3}$$

$$C^2 = \frac{4\alpha}{2-\alpha}(-h)I \tag{1.4}$$

where

$$C = \sum_{i=1}^{N} m_i q_i(t) \times \dot{q}_i(t) \tag{1.5}$$

denotes the angular momentum of systems (1.1)-(1.2) and

$$h = K(q) - U(q) \tag{1.6}$$

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denotes the total energy for (1.1)-(1.2),
where

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i|\dot{q}_i|^2 \tag{1.7}$$

denotes the kinetic energy for (1.1)-(1.2),
where

$$2I = \sum_{i=1}^{N} m_i|q_i|^2 \tag{1.8}$$

denotes the momentum of inertia for (1.1)-(1.2).

THERme 1.2. *For N-body problems (1.1)-(1.2),if N bodies always move on the straight lines toward the center of masses,then we have*

$$\ddot{I} - (2-\alpha)\left|\frac{d(I^{1/2})}{dt}\right|^2 - \alpha h = 0 \tag{1.9}$$

or

$$2r\cdot\ddot{r} + \alpha\dot{r}^2 - \alpha h = 0 \tag{1.10}$$

where

$$r = I^{\frac{1}{2}} \tag{1.11}$$

2. Some basic lemmas

LEMMA 2.1. *(Lagrange-Jacobi identity [2,3,4,5])*

$$\ddot{I} = (2-\alpha)U(q) + 2h = (2-\alpha)K + \alpha h \tag{2.1}$$

LEMMA 2.2. *(Sundman's inequality [2,3,4,5,6])*

$$C^2 \leqslant 4IK \tag{2.2}$$

LEMMA 2.3. *Sundman's inequality (2.2) takes the equality if and only if* $q_i(t)(i = 1,2,\ldots,N)$ *locates on the same plane and* $q_i(t)$ *rotates on the circles with the same instaneous angular velocity* ω *around the center of masses or all bodies locate at the center of masses forever.*

Proof.

$$C^2 = \left(\sum_{i=1}^{N} m_i q_i \times \dot{q}_i\right)^2 \tag{2.3}$$

$$\leqslant \left(\sum_{i=1}^{N} |m_i q_i \times \dot{q}_i|\right)^2 \tag{2.4}$$

$$\leqslant \left[\sum_{i=1}^{N} |\sqrt{m_i}q_i|\cdot|\sqrt{m_i}\dot{q}_i|\right] \tag{2.5}$$

$$\leqslant \left(\sum_{i=1}^{N} m_i|q_i|^2\right)\left(\sum_{i=1}^{N} m_i|\dot{q}_i|^2\right) \tag{2.6}$$

$$= 4IK. \tag{2.7}$$

The inequality (2.2) takes the equality if and only if inequalities (2.4)-(2.7) take equalities simultaneously, so we have

(1) $q_1 \times \dot{q}_1, q_2 \times \dot{q}_2, \cdots, q_N \times \dot{q}_N$ are parallel to each other and have the same direction.

(2) $q_i \cdot \dot{q}_i = 0, i = 1, 2, \cdots, N.$

(3) $\dfrac{|\dot{q}_1|}{|q_1|} = \cdots = \dfrac{|\dot{q}_N|}{|q_N|}$ is a positive number (which may depend on t) or

$$q_i(t) \equiv 0, (i = 1, \cdots, N)$$

That is, the Lemma 2.3 is true.

LEMMA 2.4. *(Sundman's inequality [2,3,4,5,6])*

$$2\left(\frac{dI^{1/2}}{dt}\right)^2 \leqslant \sum_{i=1}^{N} m_i\dot{q}_i^2 = \sum_{i=1}^{N} m_i\frac{(q_i\cdot\dot{q}_i)^2}{|q_i|^2} \tag{2.8}$$

LEMMA 2.5. *Sundman's inequality (2.8) takes the equality if and only if*

$$\dot{q}_i(t) = \lambda_i(t)q_i(t), \quad \lambda_i(t) \leqslant 0 \text{ or } \lambda_i(t) \geqslant 0,\ i = 1, \cdots, N \tag{2.9}$$

Proof. By the definition of I we have

$$2I^{1/2}\frac{dI^{1/2}}{dt} = \sum_{i=1}^{N} m_i q_i\cdot\dot{q}_i \tag{2.10}$$

By Cauchy-Schwartz's inequality we have

$$\left(\sum_{i=1}^{N} m_i q_i\cdot\dot{q}_i\right)^2 \leqslant \sum_{i=1}^{N} m_i q_i^2 \sum_{i=1}^{N} m_i\dot{q}_i^2 = 2I\sum_{i=1}^{N} m_i\dot{q}_i^2 \tag{2.11}$$

Hence by (2.10) and (2.11) we have

$$2\left(\frac{dI^{1/2}}{dt}\right)^2 \leqslant \sum_{i=1}^{N} m_i\dot{q}_i^2 \tag{2.12}$$

$$= \sum_{i=1}^{N} m_i\frac{(q_i\cdot\dot{q}_i)^2}{|q_i|^2} \tag{2.13}$$

The inequality (2.8) or (2.13) takes the equality if and only if inequality (2.11) takes the equality, hence we have (2.9)

LEMMA 2.6. *(Sundman's inequality [2,3,4,5,6])*

$$\frac{2}{2-\alpha}\ddot{I}+\frac{2\alpha}{2-\alpha}(-h)-2\left(\frac{dI^{1/2}}{dt}\right)^2 \geqslant \frac{C^2}{2I} \tag{2.14}$$

LEMMA 2.7. *Let* $\lambda_1,\cdots,\lambda_i,\cdots,\lambda_N \in R;\alpha_1,\cdots,\alpha_i,\cdots,\alpha_N \in R^K$*, then we have*
(1) $|\lambda_1\alpha_1+\cdots+\lambda_N\alpha_N|^2 \leqslant (\lambda_1^2+\cdots+\lambda_N^2)\cdot(\alpha_1^2+\cdots+\alpha_N^2)$,
(2) the above inequality takes the equality if and only if

$$\lambda_i\alpha_j=\lambda_j\alpha_i,\ 1\leqslant i\neq j\leqslant N.$$

LEMMA 2.8. *Sundman's inequality (2.14) takes the equality if and only if there holds (2.9).*

Proof. By Lemma 2.7 we have

$$C^2=\left(\sum_{i=1}^N m_i(q_i\times\dot{q}_i)\right)^2 \leqslant \left(\sum_{i=1}^N m_i|q_i|^2\right)\left(\sum_{i=1}^N \frac{m_i|q_i\times\dot{q}_i|^2}{|q_i|^2}\right) \tag{2.15}$$

Hence we have

$$\frac{C^2}{2I}\leqslant\sum_{i=1}^N m_i\frac{|q_i\times\dot{q}_i|^2}{|q_i|^2} \tag{2.16}$$

By (2.16) and (2.8) we have

$$2\left(\frac{dI^{1/2}}{dt}\right)^2+\frac{C^2}{2I}\leqslant\sum_{i=1}^N\frac{m_i}{|q_i|^2}[(q_i\cdot\dot{q}_i)^2+|q_i\times\dot{q}_i|^2] \tag{2.17}$$

$$=\sum_{i=1}^N\frac{m_i}{|q_i|^2}[q_i^2\cdot\dot{q}_i^2] \tag{2.18}$$

$$=\sum_{i=1}^N m_i\dot{q}_i^2=2K \tag{2.19}$$

By Lemma 2.1 we have

$$2\left(\frac{dI^{1/2}}{dt}\right)^2+\frac{C^2}{2I}\leqslant\frac{2}{2-\alpha}\ddot{I}-\frac{2\alpha h}{2-\alpha} \tag{2.20}$$

The inequality (2.20) or (2.14) takes the equality if and only if inequalities (2.15) and (2.8) take the equalities simultaneously, hence the result is true.

3. Jacobi's coordinates and Sundman's inequalities

We introduce the Jacobi's coordinates (For $N = 3$ see [1]):

$$x_1 = q_2 - q_1$$

$$x_2 = q_3 - \frac{\sum_{j=1}^{2} m_j q_j}{M_2}$$

$$\cdots$$

$$x_i = q_{i+1} - \frac{\sum_{j=1}^{i} m_j q_j}{M_i} \tag{3.1}$$

$$\cdots$$

$$x_{N-1} = q_N - \frac{\sum_{j=1}^{N-1} m_j q_j}{M_{N-1}}$$

where

$$M_i = \sum_{j=1}^{i} m_j \tag{3.2}$$

We assume the center of masses $m_1, \cdots, m_N$ is at the origin:

$$\sum_{i=1}^{N} m_i q_i = 0 \tag{3.3}$$

Then we have

THEOREM 3.1. *The inverse transformations of Jacobi's coordinates are:*

$$q_N = \frac{M_{N-1}}{M_N} x_{N-1}$$

$$q_{N-1} = \frac{M_{N-2}}{M_{N-1}} x_{N-2} - \frac{m_N}{M_N} x_{N-1}$$

$$q_{N-2} = \frac{M_{N-3}}{M_{N-2}} x_{N-3} - \frac{m_{N-1}}{M_{N-1}} x_{N-2} - \frac{m_N}{M_N} x_{N-1}$$

$$\cdots$$

$$q_{N-i} = \frac{M_{N-i-1}}{M_{N-i}} x_{N-i-1} - \sum_{j=N-i+1}^{N} \frac{m_j}{M_j} x_{j-1}, i \neq N-1, \tag{3.4}$$

When $i = N - 1$,

$$q_1 = -\sum_{j=2}^{N} \frac{m_j}{M_j} x_{j-1} \tag{3.5}$$

THEOREM 3.2.. *Under the Jacobi's Coordinates, I, K and C have the following representations:*

$$I = \frac{1}{2}\sum_{i=1}^{N} m_i |q_i|^2 = \frac{1}{2}\sum_{j=1}^{N-1} \frac{m_{j+1}M_j}{M_{j+1}} x_j^2 \equiv \tilde{I}\ (x) \tag{3.6}$$

$$K = \frac{1}{2}\sum_{j=1}^{N} m_i |\dot{q}_i|^2 = \frac{1}{2}\sum_{j=1}^{N-1} \frac{m_{j+1}M_j}{M_{j+1}} \dot{x}_j^2 \equiv \tilde{K}\ (\dot{x}) \tag{3.7}$$

$$C = \sum_{i=1}^{N} m_i \dot{q}_i \times q_i = \sum_{j=1}^{N-1} \frac{m_{j+1}M_j}{M_{j+1}} \dot{x}_j \times x_j \equiv \tilde{C} \tag{3.8}$$

By Theorem 3.2 and similar proof of Lemma 2.2 we have

THEOREM 3.3. *The Sundman's inequality*

$$\tilde{C}^2 \leqslant 4\ \tilde{I} \cdot \tilde{K} \tag{3.9}$$

takes the equality if and only if there hold the following conditions:

(1°) $x_1 \times \dot{x}_1, \cdots, x_{N-1} \times \dot{x}_{N-1}$ *are parallel to each other and have the same direction.*

(2°) $x_i \cdot \dot{x}_i = 0, i = 1, \cdots, N-1.$

(3°) $\dot{x}_i = \lambda_i(t) \cdot x_i(t), \quad \lambda_i(t) \geqslant 0$ *or* $\lambda_i \leqslant 0,\ i = 1, \cdots, N-1.$

Using the Theorem 2.2 and Lemma 2.1 and the similar proof of Lemma 2.8, we have

THEOREM 3.4. *Sundman's inequalities:*

$$\left(\frac{d\ \tilde{I}^{1/2}}{dt}\right)^2 \leqslant \tilde{K}\ (x) \tag{3.10}$$

$$\frac{2}{2-\alpha}\ddot{\tilde{I}} + \frac{2\alpha}{2-\alpha}(-h) - 2\left(\frac{d\ \tilde{I}^{1/2}}{dt}\right)^2 \geqslant \frac{\tilde{C}^2}{2\ \tilde{I}} \tag{3.11}$$

take equality,respectively,if and only if there hold

$$\dot{x}_i(t) = \lambda_i(t) x_i(t), \quad \lambda_i \geqslant 0 \text{ or } \lambda_i \leqslant 0\ i = 1, \cdots, .N-1 \tag{3.12}$$

Similar to the Thorem 1.2 ,we have

THEOREM 3.5. *If the relative motion $x_i(t)$ of $q_{i+1}(i = 1, \cdots, N-1)$ with respect to the mass center of the former i bodies satisfies (3.12),that is, in the Jacobi's relative coordinates, the N-1 bodies with masses $m_{j+1}M_j/M_{j+1}(j = 1, \cdots, N-1)$ always move on the straight lines toward the center of masses, then we bave the relationship between $\tilde{I}$ and h:*

$$\ddot{\tilde{I}} - (2-\alpha)\left(\frac{d\ \tilde{I}^{1/2}}{dt}\right)^2 - \alpha h = 0 \tag{3.13}$$

or

$$2\,\tilde{r}\cdot\ddot{\tilde{r}}+\alpha\dot{\tilde{r}}^{2}-\alpha h=0 \tag{3.14}$$

where

$$\tilde{r}=\tilde{I}^{1/2} \tag{3.15}$$

4. The proofs of Theorem 1.1 and Theorem 1.2

Firstly we prove Theorem 1.1: By Lemma 2.1 and Lemma 2.2 we have

$$C^2 \leqslant 4I\left(\frac{1}{2-\alpha}\ddot{I}-\frac{\alpha}{2-\alpha}h\right) \tag{4.1}$$

By the assumptions for the orbit $q(t)=(q_1(t),\cdots,q_N(t))$ of the N-body problems, we have that (2.2) and (4.1) take equalities and $\ddot{I}=0$, hence we have

$$C^2=\frac{4\alpha}{2-\alpha}(-h)I \tag{4.2}$$

For $q(t)$ we use (1.5) to get

$$|C|=\sum_{i=1}^{N}m_i|q_i||\dot{q}_i|=\left(\sum_{i=1}^{N}m_i|q_i|^2\right)\omega=2I\cdot\omega=2I\cdot\frac{2\pi}{T} \tag{4.3}$$

Hence by (4.2) and (4.3) we have

$$|C|=\frac{2\alpha}{2-\alpha}(-h)\omega^{-1}=\frac{\alpha}{2-\alpha}\frac{1}{\pi}(-h)T \tag{4.4}$$

For the proof of Theorem 1.2, we notice that for the motions on the straight lines toward the center of masses,we have $\dot{q}_i(t)=\lambda_i(t)q_i(t),\lambda_i(t)\leqslant 0$ and the angular momentum $C=0$, hence Theorem 1.2 follows from Lemma 2.8.

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