

ON THE HYERS–ULAM–RASSIAS STABILITY OF A PEXIDERIZED QUADRATIC INEQUALITY

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Abstract. In this paper we prove the stability of the Pexiderized quadratic inequality $\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \varphi(x,y)$ in the spirit of D. H. Hyers, S. M. Ulam, Th. M. Rassias and P. Găvruta.

1. Introduction

In 1940, S. M. Ulam [31] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, Hyers [6] proved that if $f : V \rightarrow X$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in V$, where V and X are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T : V \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in V$.

Throughout the paper, let V and X be a normed space and a Banach space, respectively. Th. M. Rassias [21] gave a generalization of the Hyers' result in the following way:

THEOREM 1.1. *Let $f : V \rightarrow X$ be a mapping such that $f(tx)$ is continuous in t for each fixed x . Assume that there exist $\theta \geq 0$ and $p < 1$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$ (for all $x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique linear mapping $T : V \rightarrow X$ such that

$$\|T(x) - f(x)\| \leq \frac{2\theta}{|2 - 2^p|} \|x\|^p$$

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for all $x \in V$ (for all $x \in V \setminus \{0\}$ if $p < 0$).

Th. M. Rassias [27] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [3] following the same approach as in Th. M. Rassias [21], gave an affirmative solution to Rassias' question for $p > 1$.

However, it was showed that a similar result for the case $p = 1$ does not hold (see [3,28]). Recently, P. Găvruta[5] also obtained a further generalization of the Hyers-Rassias theorem (see also [7-12,15,16,19,20,22-24]).

Lee and Jun [17,18] also obtained the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y) = g(x) + h(y)$ (see also [13]):

THEOREM 1.2. *Let $f, g, h : V \rightarrow X$ be mappings. Assume that there exist $\theta \geq 0$ and $p \in [0, \infty) \setminus \{1\}$ such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \rightarrow X$ such that

$$\begin{aligned} \|T(x) - f(x) + f(0)\| &\leq \frac{4\theta}{|2^p - 2|} \|x\|^p + M \\ \|T(x) - g(x) + g(0)\| &\leq \frac{(4 + 2^p)\theta}{|2^p - 2|} \|x\|^p + M \\ \|T(x) - h(x) + h(0)\| &\leq \frac{(4 + 2^p)\theta}{|2^p - 2|} \|x\|^p + M \end{aligned}$$

where $M = \|f(0) - g(0) - h(0)\|$ (if $1 < p$ then $M = 0$).

In 1983, the stability theorem for the quadratic functional equation

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0$$

was proved F. Skof [30] for the function $f : V \rightarrow X$. In 1984, P. W. Cholewa [1] extended the Skof's result to the case where V is an Abelian group G .

In 1992, S. Czerwik [2] gave a generalization of the Skof-Cholewa's result in the following way:

THEOREM 1.3. *Let $p \neq 2$, $\theta > 0$ be real numbers. Suppose that the function $f : V \rightarrow X$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p).$$

Then there exists exactly one quadratic function $g : V \rightarrow X$ such that

$$\|f(x) - g(x)\| \leq c + k\theta\|x\|^p$$

for all x in V if $p \geq 0$ and for all $x \in V \setminus \{0\}$ if $p < 0$, where: when $p < 2$, $c = \frac{\|f(0)\|}{3}$, $k = \frac{2}{4-2^p}$ and $g = \lim_{n \rightarrow \infty} 4^{-n} f(2^n x)$. When $p > 2$, $c = 0$, $k = \frac{2}{2^p-4}$

and $g(x) = \lim_{n \rightarrow \infty} 4^n f(2^{-n}x)$ for all x in V . Also, if the mapping $t \rightarrow f(tx)$ from R to X is continuous for each fixed x in V , then $g(tx) = t^2g(x)$ for all t in R .

Since then, the stability problem of the quadratic equation have been extensively investigated by a number of mathematician([25,26,29]) In this paper, we prove the stability of the Pexiderized quadratic inequalities :

$$\|f(x + y) + f(x - y) - 2g(x) - 2g(y)\| \leq \varphi(x, y), \tag{1.1}$$

$$\|f(x + y) + g(x - y) - 2h(x) - 2k(y)\| \leq \varphi(x, y). \tag{1.2}$$

2. Stability of the inequality (1.1).

In this section, we prove the stability of the functional inequality (1.1).

THEOREM 2.1. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \rightarrow [0, \infty)$ be a mapping such that*

$$\tilde{\varphi}(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty \tag{a}$$

or

$$\tilde{\varphi}(x, y) := \sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) < \infty \tag{a'}$$

for all $x, y \in V \setminus \{0\}$. Suppose that the functions $f, g : V \rightarrow X$ satisfy

$$\|f(x + y) + f(x - y) - 2g(x) - 2g(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V \setminus \{0\}. \tag{2.1}$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \tilde{\varphi}\left(\frac{5x}{2}, \frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{5x}{2}, \frac{x}{2}\right) + \tilde{\varphi}(2x, 2x) + 2\tilde{\varphi}(2x, x) \\ &\quad + \tilde{\varphi}\left(\frac{3x}{2}, \frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{x}{2}\right) + \tilde{\varphi}(x, x) + \tilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}\right) \\ \|g(x) - \frac{f(0)}{2} - Q(x)\| &\leq \frac{1}{4}(\tilde{\varphi}(5x, 3x) + \tilde{\varphi}(5x, x) + \tilde{\varphi}(4x, 4x) + 2\tilde{\varphi}(4x, 2x) \\ &\quad + \tilde{\varphi}(3x, 3x) + \tilde{\varphi}(3x, x) + \tilde{\varphi}(2x, 2x) + \tilde{\varphi}(x, x)) + \frac{\varphi(x, x)}{4} \end{aligned} \tag{2.2}$$

for all $x \in V \setminus \{0\}$. The function Q is given by

$$Q(x) = \begin{cases} \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} \\ \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n} & \text{if } \varphi \text{ satisfies (a)} \\ \lim_{n \rightarrow \infty} 4^n \left(f\left(\frac{x}{2^n}\right) - f(0)\right) \\ \lim_{n \rightarrow \infty} 4^n \left(g\left(\frac{x}{2^n}\right) - \frac{f(0)}{2}\right) & \text{if } \varphi \text{ satisfies (a')} \end{cases} \tag{2.3}$$

for all $x \in V$.

Proof. Suppose that φ satisfies the condition (a). From (2.1), we easily obtain

$$\|f(2x) + f(0) - 2g(x) - 2g(x)\| \leq \varphi(x, x), \quad (2.4)$$

$$\|f(4x) + f(0) - 4g(2x)\| \leq \varphi(2x, 2x), \quad (2.5)$$

$$\|f(4x) + f(2x) - 2g(3x) - 2g(x)\| \leq \varphi(3x, x), \quad (2.6)$$

$$\|f(6x) + f(0) - 4g(3x)\| \leq \varphi(3x, 3x), \quad (2.7)$$

$$\|f(6x) + f(2x) - 2g(4x) - 2g(2x)\| \leq \varphi(4x, 2x), \quad (2.8)$$

$$\|f(6x) + f(4x) - 2g(5x) - 2g(x)\| \leq \varphi(5x, x), \quad (2.9)$$

$$\|f(8x) + f(0) - 4g(4x)\| \leq \varphi(4x, 4x) \text{ and} \quad (2.10)$$

$$\|f(8x) + f(2x) - 2g(5x) - 2g(3x)\| \leq \varphi(5x, 3x) \quad (2.11)$$

for all $x \in V \setminus \{0\}$.

From (2.4), (2.6) and (2.7), we have

$$\begin{aligned} & \|f(6x) - 2f(4x) - f(2x) + 2f(0)\| \\ & \leq 2\|f(4x) + f(2x) - 2g(3x) - 2g(x)\| \\ & \quad + \|f(6x) + f(0) - 4g(3x)\| + \|f(2x) + f(0) - 4g(x)\| \\ & \leq 2\varphi(3x, x) + \varphi(3x, 3x) + \varphi(x, x) \end{aligned} \quad (2.12)$$

for all $x \in V \setminus \{0\}$.

From (2.5), (2.8) and (2.10), we have

$$\begin{aligned} & \|f(8x) - 2f(6x) + f(4x) - 2f(2x) + 2f(0)\| \\ & \leq 2\|f(6x) + f(2x) - 2g(4x) - 2g(2x)\| \\ & \quad + \|f(8x) + f(0) - 4g(4x)\| + \|f(4x) + f(0) - 4g(2x)\| \\ & \leq 2\varphi(4x, 2x) + \varphi(4x, 4x) + \varphi(2x, 2x) \end{aligned} \quad (2.13)$$

for all $x \in V \setminus \{0\}$.

From (2.4), (2.7), (2.9) and (2.11), we have

$$\begin{aligned} & \|2f(8x) - 3f(6x) - 2f(4x) + 3f(2x)\| \\ & \leq 2\|f(6x) + f(4x) - 2g(5x) - 2g(x)\| \\ & \quad + 2\|f(8x) + f(2x) - 2g(5x) - 2g(3x)\| \\ & \quad + \|f(6x) + f(0) - 4g(3x)\| + \|f(2x) + f(0) - 4g(x)\| \\ & \leq 2\varphi(5x, x) + 2\varphi(5x, 3x) + \varphi(3x, 3x) + \varphi(x, x) \end{aligned} \quad (2.14)$$

for all $x \in V \setminus \{0\}$. From (2.13) and (2.14), we get

$$\begin{aligned} \|f(6x) - 4f(4x) + 7f(2x) - 4f(0)\| & \leq 2\varphi(5x, 3x) + 2\varphi(5x, x) + 2\varphi(4x, 4x) \\ & \quad + 4\varphi(4x, 2x) + \varphi(3x, 3x) + 2\varphi(2x, 2x) + \varphi(x, x) \end{aligned} \quad (2.15)$$

for all $x \in V \setminus \{0\}$. From (2.12) and (2.15), we get

$$\|2f(4x) - 8f(2x) + 6f(0)\| \leq 2\varphi(5x, 3x) + 2\varphi(5x, x) + 2\varphi(4x, 4x) + 4\varphi(4x, 2x) \\ + 2\varphi(3x, 3x) + 2\varphi(3x, x) + 2\varphi(2x, 2x) + 2\varphi(x, x)$$

for all $x \in V \setminus \{0\}$. Hence we have

$$\|f(x) - f(0) - \frac{f(2x) - f(0)}{4}\| \leq \frac{\varphi(\frac{5x}{2}, \frac{3x}{2}) + \varphi(\frac{5x}{2}, \frac{x}{2}) + \varphi(2x, 2x) + 2\varphi(2x, x)}{4} \\ + \frac{\varphi(\frac{3x}{2}, \frac{3x}{2}) + \varphi(\frac{3x}{2}, \frac{x}{2}) + \varphi(x, x) + \varphi(\frac{x}{2}, \frac{x}{2})}{4} \quad (2.16)$$

for all $x \in V \setminus \{0\}$.

Replacing x by $2^n x$ and dividing by 4^n in (2.16), we have

$$\|\frac{f(2^n x) - f(0)}{4^n} - \frac{f(2^{n+1} x) - f(0)}{4^{n+1}}\| \leq \frac{\varphi(2^n \cdot \frac{5x}{2}, 2^n \cdot \frac{3x}{2}) + \varphi(2^n \cdot \frac{5x}{2}, 2^n \cdot \frac{x}{2})}{4^{n+1}} \\ + \frac{\varphi(2^n \cdot 2x, 2^n \cdot 2x) + 2\varphi(2^n \cdot 2x, 2^n \cdot x)}{4^{n+1}} \\ + \frac{\varphi(2^n \cdot \frac{3x}{2}, 2^n \cdot \frac{3x}{2}) + \varphi(2^n \cdot \frac{3x}{2}, 2^n \cdot \frac{x}{2})}{4^{n+1}} \\ + \frac{\varphi(2^n \cdot x, 2^n \cdot x) + \varphi(2^n \cdot \frac{x}{2}, 2^n \cdot \frac{x}{2})}{4^{n+1}}$$

for all $n \in N$ and $x \in V \setminus \{0\}$. Induction argument implies

$$\|f(x) - f(0) - \frac{f(2^n x) - f(0)}{4^n}\| \leq \tilde{\varphi}(\frac{5x}{2}, \frac{3x}{2}) + \tilde{\varphi}(\frac{5x}{2}, \frac{x}{2}) + \tilde{\varphi}(2x, 2x) + 2\tilde{\varphi}(2x, x) \\ + \tilde{\varphi}(\frac{3x}{2}, \frac{3x}{2}) + \tilde{\varphi}(\frac{3x}{2}, \frac{x}{2}) + \tilde{\varphi}(x, x) + \tilde{\varphi}(\frac{x}{2}, \frac{x}{2}) \quad (2.17)$$

for all $n \in N$ and $x \in V \setminus \{0\}$. Hence

$$\|\frac{f(2^n x) - f(0)}{4^n} - \frac{f(2^{m+n} x) - f(0)}{4^{m+n}}\| \leq \frac{\tilde{\varphi}(2^n \cdot \frac{5x}{2}, 2^n \cdot \frac{3x}{2}) + \tilde{\varphi}(2^n \cdot \frac{5x}{2}, 2^n \cdot \frac{x}{2})}{4^n} \\ + \frac{\tilde{\varphi}(2^n \cdot 2x, 2^n \cdot 2x) + 2\tilde{\varphi}(2^n \cdot 2x, 2^n \cdot x)}{4^n} \\ + \frac{\tilde{\varphi}(2^n \cdot \frac{3x}{2}, 2^n \cdot \frac{3x}{2}) + \tilde{\varphi}(2^n \cdot \frac{3x}{2}, 2^n \cdot \frac{x}{2})}{4^n} \\ + \frac{\tilde{\varphi}(2^n \cdot x, 2^n \cdot x) + \tilde{\varphi}(2^n \cdot \frac{x}{2}, 2^n \cdot \frac{x}{2})}{4^n}$$

for all $m, n \in N$ and $x \in V \setminus \{0\}$. This shows that $\{\frac{f(2^n x) - f(0)}{4^n}\}$ is a Cauchy sequence for $x \in V \setminus \{0\}$ and thus converges.

Therefore we can define $Q : V \rightarrow X$ such that

$$Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x) - f(0)}{4^n} \text{ for } x \in V.$$

Replacing x by $2^n x$ and dividing by 4^{n+1} in (2.4), we have

$$\left\| \frac{f(2^{n+1}x) - f(0)}{4^{n+1}} - \frac{g(2^n x) - \frac{f(0)}{2}}{4^n} \right\| \leq \frac{\varphi(2^n x, 2^n x)}{4^{n+1}} \quad (2.18)$$

for all $x \in V$. Taking the limit in (2.18) as $n \rightarrow \infty$, we get

$$Q(x) = \lim_{n \rightarrow \infty} \frac{g(2^n x) - \frac{f(0)}{2}}{4^n} \text{ for } x \in V.$$

Taking the limit in (2.17) as n to ∞ , we obtain

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \tilde{\varphi}\left(\frac{5x}{2}, \frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{5x}{2}, \frac{x}{2}\right) + \tilde{\varphi}(2x, 2x) + 2\tilde{\varphi}(2x, x) \\ &\quad + \tilde{\varphi}\left(\frac{3x}{2}, \frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{x}{2}\right) + \tilde{\varphi}(x, x) + \tilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}\right) \end{aligned} \quad (2.19)$$

for all $x \in V \setminus \{0\}$. From (2.4) and (2.19), we easily get

$$\begin{aligned} \left\| g(x) - \frac{f(0)}{2} - Q(x) \right\| &\leq \frac{1}{4} (\tilde{\varphi}(5x, 3x) + \tilde{\varphi}(5x, x) + \tilde{\varphi}(4x, 4x) + 2\tilde{\varphi}(4x, 2x) \\ &\quad + \tilde{\varphi}(3x, 3x) + \tilde{\varphi}(3x, x) + \tilde{\varphi}(2x, 2x) + \tilde{\varphi}(x, x)) + \frac{\varphi(x, x)}{4} \end{aligned}$$

for all $x \in V \setminus \{0\}$. From (2.1) and the definition of Q , we easily obtain

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in V$. Taking the limit in (2.17) and (2.26) as n to ∞ , we obtain (2.2). If Q' is another quadratic function satisfying (2.2), then

$$\begin{aligned} \|Q(x) - Q'(x)\| &\leq \left\| \frac{Q(2^n x)}{4^n} - \frac{f(2^n x)}{4^n} + \frac{f(0)}{4^n} \right\| + \left\| \frac{f(2^n x)}{4^n} - \frac{f(0)}{4^n} - \frac{Q'(2^n x)}{4^n} \right\| \\ &\leq \frac{\tilde{\varphi}(2^n \cdot \frac{5x}{2}, 2^n \cdot \frac{3x}{2}) + \tilde{\varphi}(2^n \cdot \frac{5x}{2}, 2^n \cdot \frac{x}{2}) + \tilde{\varphi}(2^n \cdot 2x, 2^n \cdot 2x)}{2 \cdot 4^{n-1}} \\ &\quad + \frac{2\tilde{\varphi}(2^n \cdot 2x, 2^n \cdot x) + \tilde{\varphi}(2^n \cdot \frac{3x}{2}, 2^n \cdot \frac{3x}{2}) + \tilde{\varphi}(2^n \cdot \frac{3x}{2}, 2^n \cdot \frac{x}{2})}{2 \cdot 4^{n-1}} \\ &\quad + \frac{\tilde{\varphi}(2^n \cdot x, 2^n \cdot x) + \tilde{\varphi}(2^n \cdot \frac{x}{2}, 2^n \cdot \frac{x}{2})}{2 \cdot 4^{n-1}} \end{aligned}$$

for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. Therefore we have

$$Q(x) = Q'(x) \text{ for all } x \in V.$$

For the case that φ satisfies the condition (a'), the proof is analogous to the case (a).

COROLLARY 2.1. Let $p \neq 2$, $\theta > 0$ be real numbers. Suppose that the functions $f, g : V \rightarrow X$ satisfy

$$\|f(x+y) + f(x-y) - 2g(x) - 2g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \text{ for all } x, y \in V \setminus \{0\}.$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \frac{2(5^p + 2 \cdot 4^p + 2 \cdot 3^p + 2 \cdot 2^p + 2)}{2^p|4 - 2^p|} \theta \|x\|^p, \\ \|g(x) - \frac{f(0)}{2} - Q(x)\| &\leq \left(\frac{5^p + 2 \cdot 4^p + 2 \cdot 3^p + 2 \cdot 2^p + 2}{2|4 - 2^p|} + \frac{1}{2} \right) \theta \|x\|^p \end{aligned}$$

for all $x \in V \setminus \{0\}$. The function Q is given by (2.3).

THEOREM 2.2. Let $\varphi : V \times V \rightarrow [0, \infty)$ be a mapping such that

$$\tilde{\varphi}(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty \quad (\text{b})$$

or

$$\tilde{\varphi}(x, y) := \sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) < \infty \quad (\text{b}')$$

for all $x, y \in V$. Suppose that the functions $f, g : V \rightarrow X$ satisfy

$$\|f(x+y) + f(x-y) - 2g(x) - 2g(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V. \quad (2.20)$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \tilde{\varphi}(x, x) + 2\tilde{\varphi}(x, 0) + \tilde{\varphi}(0, 0) \\ \|g(x) - g(0) - Q(x)\| &\leq \frac{1}{2} \tilde{\varphi}(2x, 0) + \tilde{\varphi}(x, x) + \frac{1}{2} \tilde{\varphi}(0, 0) \end{aligned} \quad (2.21)$$

for all $x \in V$. The function Q is given by (2.3).

Proof. If φ satisfies the condition (b'), we easily know the fact that

$$\tilde{\varphi}(0, 0) = 0.$$

Suppose that φ satisfies the condition (b). Let x be an arbitrary fixed element of V . From (2.20), we have

$$\left\| \frac{f(0)}{2} - g(0) \right\| \leq \frac{1}{4} \varphi(0, 0), \quad (2.22)$$

$$\|f(x) - g(x) - g(0)\| \leq \frac{1}{2} \varphi(x, 0), \quad (2.23)$$

$$\left\| \frac{1}{4}(f(2x) + f(0)) - g(x) \right\| \leq \frac{1}{4} \varphi(x, x). \quad (2.24)$$

From (2.22), (2.23) and (2.24), we get

$$\begin{aligned} \|f(x) - f(0) - \frac{1}{4}(f(2x) - f(0))\| &\leq \|f(x) - g(x) - g(0)\| \\ &+ \left\| \frac{1}{4}(f(2x) + f(0)) - g(x) \right\| + \left\| \frac{f(0)}{2} - g(0) \right\| \\ &\leq \frac{1}{4}\varphi(x, x) + \frac{1}{2}\varphi(x, 0) + \frac{1}{4}\varphi(0, 0). \end{aligned}$$

Induction argument implies

$$\begin{aligned} \|f(x) - f(0) - \frac{1}{4^n}(f(2^n x) - f(0))\| &\leq \sum_{i=0}^{n-1} \frac{\varphi(2^i x, 2^i x) + 2\varphi(2^i x, 0) + \varphi(0, 0)}{4^{i+1}} \\ &\leq \tilde{\varphi}(x, x) + 2\tilde{\varphi}(x, 0) + \tilde{\varphi}(0, 0). \end{aligned} \quad (2.25)$$

From (2.25), we can easily show that $\left\{ \frac{f(2^n x) - f(0)}{4^n} \right\}$ is a Cauchy sequence and thus converges. From this, we can define $Q : V \rightarrow X$ such that

$$Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x) - f(0)}{4^n} = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Replacing x by $2^n x$ and dividing by 4^n in (2.23), we get

$$\left\| \frac{1}{4^n} f(2^n x) - \frac{1}{4^n} g(2^n x) - \frac{1}{4^n} g(0) \right\| \leq \frac{\varphi(2^n x, 0)}{2 \cdot 4^n} \quad (2.26)$$

for all $n \in N$. Taking the limit in (2.26), we have

$$Q(x) = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n}. \quad (2.27)$$

From (2.20), (2.27) and the definition of Q , we get

$$Q(x+y) + Q(x-y) - Q(x) - Q(y) = 0.$$

Replacing x by $2x$ and dividing by 4 in (2.23), we have

$$\frac{1}{4} \|f(2x) - g(2x) - g(0)\| \leq \frac{1}{8} \varphi(2x, 0). \quad (2.28)$$

From (2.22), (2.23), (2.24) and (2.28), we get

$$\begin{aligned} \|g(x) - g(0) - \frac{1}{4}(g(2x) - g(0))\| &\leq \frac{1}{4} \|f(2x) - g(2x) - g(0)\| \\ &+ \left\| \frac{1}{4}(f(2x) + f(0)) - g(x) \right\| + \left\| \frac{f(0)}{2} - g(0) \right\| \\ &\leq \frac{1}{8} \varphi(2x, 0) + \frac{1}{4} \varphi(x, x) + \frac{1}{8} \varphi(0, 0). \end{aligned}$$

By the similar method as in (2.25), we have

$$\|g(x) - g(0) - \frac{1}{4^n}(g(2^n x) - g(0))\| \leq \frac{1}{2} \tilde{\varphi}(2x, 0) + \tilde{\varphi}(x, x) + \frac{1}{2} \tilde{\varphi}(0, 0) \quad (2.29)$$

for all $n \in N$. From (2.25) and (2.29), we obtain (2.21).

COROLLARY 2.2. Let $p < 2$, $\theta > 0$ be real numbers. Let $\psi : V \rightarrow [0, \infty)$ be a mapping such that $\psi(x) = \|x\|^p$ for $x \neq 0$. Suppose that the functions $f, g : V \rightarrow X$ satisfy

$$\|f(x+y) + f(x-y) - 2g(x) - 2g(y)\| \leq \theta(\psi(x) + \psi(y)) \text{ for all } x, y \in V.$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \frac{4\theta}{4-2^p}\psi(x) + \frac{4\theta}{3}\psi(0) \\ \|g(x) - g(0) - Q(x)\| &\leq \frac{(2^p+4)\theta}{2(4-2^p)}\psi(x) + \frac{\theta}{2}\psi(0) \end{aligned}$$

for all $x \in V$. The function Q is given by (2.3).

COROLLARY 2.3. Let $p \geq 0 (p \neq 2)$, $\theta > 0$ be real numbers. Suppose that the functions $f, g : V \rightarrow X$ satisfy

$$\|f(x+y) + f(x-y) - 2g(x) - 2g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \text{ for all } x, y \in V.$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \frac{4\theta}{|2^p-4|}\|x\|^p \\ \|g(x) - g(0) - Q(x)\| &\leq \frac{(2^p+4)\theta}{|2(2^p-4)|}\|x\|^p \end{aligned}$$

for all $x \in V$. The function Q is given by (2.3).

3. Stability of the inequality (1.2).

In this section, we prove the stability of the Pexiderized quadratic inequality (1.2).

THEOREM 3.1 (EVEN FUNCTION). Let φ be the mapping as in Theorem 2.1. Suppose that the even functions $f, g, h, k : V \rightarrow X$ satisfy

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V \setminus \{0\}. \quad (3.1)$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq M_1(x) \\ \|g(x) - g(0) - Q(x)\| &\leq M_1(x) \text{ and} \\ \left\| \frac{h(x) + k(x)}{2} - \frac{f(0) + g(0)}{4} - Q(x) \right\| &\leq \frac{M_1(2x)}{4} \end{aligned}$$

for all $x \in V \setminus \{0\}$, where

$$\begin{aligned} M_1(x) = & \frac{1}{2} \left[\tilde{\varphi}\left(\frac{5x}{2}, -\frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{5x}{2}\right) + \tilde{\varphi}\left(\frac{5x}{2}, -\frac{x}{2}\right) + \tilde{\varphi}\left(\frac{x}{2}, \frac{5x}{2}\right) \right. \\ & + \tilde{\varphi}(2x, -2x) + \tilde{\varphi}(2x, 2x) + 2\tilde{\varphi}(2x, -x) + 2\tilde{\varphi}(x, 2x) \\ & + \tilde{\varphi}\left(\frac{3x}{2}, -\frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, -\frac{x}{2}\right) + \tilde{\varphi}\left(\frac{x}{2}, \frac{3x}{2}\right) \\ & + \tilde{\varphi}(x, -x) + \tilde{\varphi}(x, x) + \tilde{\varphi}\left(\frac{x}{2}, -\frac{x}{2}\right) + \tilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}\right) \\ & \left. + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(\frac{x}{2}, -\frac{x}{2}\right) \right]. \end{aligned}$$

The function Q is given by

$$Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n}$$

for all $x \in V$ if φ satisfies (a), and

$$Q(x) = \lim_{n \rightarrow \infty} 4^n \left(f\left(\frac{x}{2^n}\right) - f(0) \right) = \lim_{n \rightarrow \infty} 4^n \left(g\left(\frac{x}{2^n}\right) - g(0) \right)$$

for all $x \in V$ if φ satisfies (a').

Proof. From (3.1), we can easily obtain

$$\|f(x-y) + g(x+y) - 2h(x) - 2k(-y)\| \leq \varphi(x, -y) \quad \text{and} \quad (3.2)$$

$$\|f(x+y) + g(-x+y) - 2h(y) - 2k(x)\| \leq \varphi(y, x) \quad (3.3)$$

for all $x, y \in V \setminus \{0\}$. Let

$$2U(x) = f(x) + g(x) \quad \text{and} \quad 2V(x) = h(x) + k(x) \quad (3.4)$$

for all $x \in V$. Because f, g, h, k are even functions, from (3.2) and (3.3), we obtain

$$\|U(x+y) + U(x-y) - 2V(x) - 2V(y)\| \leq \frac{1}{2}\varphi(x, -y) + \frac{1}{2}\varphi(y, x)$$

for all $x, y \in V \setminus \{0\}$. By Theorem 2.1 and (3.4), there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} & \left\| \frac{f(x) + g(x)}{2} - \frac{f(0) + g(0)}{2} - Q(x) \right\| \\ & \leq \frac{1}{2} \left[\tilde{\varphi}\left(\frac{5x}{2}, -\frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{5x}{2}\right) + \tilde{\varphi}\left(\frac{5x}{2}, -\frac{x}{2}\right) + \tilde{\varphi}\left(\frac{x}{2}, \frac{5x}{2}\right) \right. \\ & \quad + \tilde{\varphi}(2x, -2x) + \tilde{\varphi}(2x, 2x) + 2\tilde{\varphi}(2x, -x) + 2\tilde{\varphi}(x, 2x) \\ & \quad + \tilde{\varphi}\left(\frac{3x}{2}, -\frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, \frac{3x}{2}\right) + \tilde{\varphi}\left(\frac{3x}{2}, -\frac{x}{2}\right) + \tilde{\varphi}\left(\frac{x}{2}, \frac{3x}{2}\right) \\ & \quad \left. + \tilde{\varphi}(x, -x) + \tilde{\varphi}(x, x) + \tilde{\varphi}\left(\frac{x}{2}, -\frac{x}{2}\right) + \tilde{\varphi}\left(\frac{x}{2}, \frac{x}{2}\right) \right] \quad (3.5) \end{aligned}$$

$$\left\| \frac{h(x) + k(x)}{2} - \frac{f(0) + g(0)}{4} - Q(x) \right\| \leq \frac{M_1(2x)}{4}$$

for all $x \in V \setminus \{0\}$ and

$$Q(x) = \lim_{n \rightarrow \infty} \frac{U(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{V(2^n x)}{4^n} \quad (3.6)$$

for all $x \in V$. Since f, g, h, k are even functions, replacing y by x on the both sides of (3.1), we have

$$\|f(2x) + g(0) - 2h(x) - 2k(x)\| \leq \varphi(x, x) \quad (3.7)$$

for all $x \in V \setminus \{0\}$. Similarly we have

$$\|f(0) + g(2x) - 2h(x) - 2k(x)\| \leq \varphi(x, -x) \quad (3.8)$$

for all $x \in V \setminus \{0\}$. From (3.7) and (3.8), we get

$$\begin{aligned} \|f(2x) - f(0) - g(2x) + g(0)\| &\leq \|f(2x) + g(0) - 2h(x) - 2k(x)\| \\ &\quad + \|f(0) + g(2x) - 2h(x) - 2k(x)\| \\ &\leq \varphi(x, x) + \varphi(x, -x) \end{aligned} \quad (3.9)$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^{n-1}x$ and dividing by 4^n in (3.9), we have

$$\left\| \frac{f(2^n x) - f(0)}{4^n} - \frac{g(2^n x) - g(0)}{4^n} \right\| \leq \frac{\varphi(2^{n-1}x, 2^{n-1}x) + \varphi(2^{n-1}x, -2^{n-1}x)}{4^n}$$

for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. From this and (3.6), we obtain

$$\lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n} = Q(x) \quad \text{for all } x \in V.$$

From (3.5) and (3.9), we get

$$\begin{aligned} &\|f(x) - f(0) - Q(x)\| \\ &\leq \left\| \frac{f(x) + g(x)}{2} - \frac{f(0) + g(0)}{2} - Q(x) \right\| + \frac{1}{2} \|f(x) - f(0) - g(x) + g(0)\| \\ &\leq M_1(x) \end{aligned}$$

for all $x \in V \setminus \{0\}$. Similarly we get

$$\|g(x) - g(0) - Q(x)\| \leq M_1(x)$$

for all $x \in V \setminus \{0\}$. By the similar method, we obtain the remaining results.

COROLLARY 3.1 (EVEN FUNCTION). *Let $p \neq 2$, $\theta > 0$ be real numbers. Suppose that the even functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad \text{for all } x, y \in V \setminus \{0\}. \quad (3.10)$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \left(\frac{2 \cdot 5^p + 4 \cdot 4^p + 4 \cdot 3^p + 4 \cdot 2^p + 4}{2^p |4 - 2^p|} + \frac{2}{2^p} \right) \theta \|x\|^p \\ \|g(x) - g(0) - Q(x)\| &\leq \left(\frac{2 \cdot 5^p + 4 \cdot 4^p + 4 \cdot 3^p + 4 \cdot 2^p + 4}{2^p |4 - 2^p|} + \frac{2}{2^p} \right) \theta \|x\|^p \text{ and} \\ \left\| \frac{h(x) + k(x)}{2} - \frac{f(0) + g(0)}{4} - Q(x) \right\| &\leq \left(\frac{5^p + 2 \cdot 4^p + 2 \cdot 3^p + 2 \cdot 2^p + 2}{2|4 - 2^p|} + \frac{1}{2} \right) \theta \|x\|^p \end{aligned}$$

for all $x \in V \setminus \{0\}$. The function Q is given by

$$Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{h(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{k(2^n x)}{4^n}$$

for all $x \in V$ if $p < 2$ and

$$Q(x) = \lim_{n \rightarrow \infty} 4^n \left(f\left(\frac{x}{2^n}\right) - f(0) \right) = \lim_{n \rightarrow \infty} 4^n \left(g\left(\frac{x}{2^n}\right) - g(0) \right)$$

for all $x \in V$ if $p > 2$.

Proof. Applying Theorem 3.1, we can easily prove the results in the corollary except the fact that

$$Q(x) = \lim_{n \rightarrow \infty} \frac{h(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{k(2^n x)}{4^n}$$

for all $x \in V$ if $p < 2$. Replacing x, y by $2^n x, x$, respectively and dividing by 4^n in (3.10), we have

$$\frac{1}{4^n} \|f((2^n + 1)x) + g((2^n - 1)x) - 2h(2^n x) - 2k(x)\| \leq \frac{(2^{np} + 1)\theta}{4^n} \|x\|^p \quad (3.11)$$

for all $x \in V \setminus \{0\}$ and $n \in N$. From (3.7), (3.9), (3.10) and (3.11), we get

$$\begin{aligned} &\left\| \frac{2h(2^n x)}{4^n} - 2Q(x) - \frac{2(h(x) - Q(x))}{4^n} \right\| \\ &= \frac{1}{4^n} \|2h(2^n x) - 2h(x) - 2(4^n - 1)Q(x)\| \\ &\leq \frac{1}{4^n} \| -f((2^n + 1)x) - g((2^n - 1)x) + 2h(2^n x) + 2k(x) \| \\ &\quad + \|f(2x) + g(0) - 2h(x) - 2k(x)\| + \|f((2^n + 1)x) - f(0) - (2^n + 1)^2 Q(x)\| \\ &\quad + \|g((2^n - 1)x) - g(0) - (2^n - 1)^2 Q(x)\| + \| -f(2x) + f(0) + 4Q(x) \| \\ &\leq \left(\frac{2^{np} + 3}{4^n} + M \frac{(2^n + 1)^p + 2^p + (2^n - 1)^p}{4^n} \right) \theta \|x\|^p \end{aligned}$$

for all $x \in V \setminus \{0\}$ and $n \in N$, where $M = \frac{2 \cdot 5^p + 4 \cdot 4^p + 4 \cdot 3^p + 2 \cdot 2^p + 12}{2^p(4 - 2^p)}$. Taking the limit, we have

$$\lim_{n \rightarrow \infty} \frac{h(2^n x)}{4^n} = Q(x)$$

for all $x \in V$.

Remark. Define the even functions $f, g, h, k : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = 0, g(x) = 0, h(x) = L$ and $k = -L$ for all $x \in \mathbb{R} \setminus \{0\}$, $f(0) = g(0) = h(0) = k(0) = 0$, where L is a positive real number. Then f, g, h, k are satisfy (3.10) for arbitrary positive real number L . From Theorem 3.1, $Q(x) = 0$ for all $x \in \mathbb{R}$ but we cannot find any fixed M, M' such that

$$\|h(x) - h(0) - Q(x)\| \leq M\theta\|x\|^p + M' \text{ for all } x \in \mathbb{R} \setminus \{0\}$$

for arbitrary positive real number L .

THEOREM 3.2 (EVEN FUNCTION). *Let φ be the mapping as in Theorem 2.2. Suppose that the even functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V. \quad (3.12)$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq M_3(x), \\ \|g(x) - g(0) - Q(x)\| &\leq M_3(x), \\ \|h(x) - h(0) - Q(x)\| &\leq M_3(x) + \frac{1}{2}[\varphi(x, 0) + \varphi(0, 0)] \text{ and} \\ \|k(x) - k(0) - Q(x)\| &\leq M_3(x) + \frac{1}{2}[\varphi(0, x) + \varphi(0, 0)] \end{aligned}$$

for all $x \in V$, where

$$M_3(x) = \frac{1}{2}[\tilde{\varphi}(x, x) + 2\tilde{\varphi}(0, x) + 2\tilde{\varphi}(x, 0) + \tilde{\varphi}(x, -x) + 2\tilde{\varphi}(0, 0) + \varphi(\frac{x}{2}, \frac{x}{2}) + \varphi(\frac{x}{2}, -\frac{x}{2})].$$

The function Q is given by

$$Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{h(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{k(2^n x)}{4^n}.$$

for all $x \in V$ if φ satisfies (b), and

$$\begin{aligned} Q(x) &= \lim_{n \rightarrow \infty} 4^n (f(\frac{x}{2^n}) - f(0)) = \lim_{n \rightarrow \infty} 4^n (g(\frac{x}{2^n}) - g(0)) \\ &= \lim_{n \rightarrow \infty} 4^n (h(\frac{x}{2^n}) - h(0)) = \lim_{n \rightarrow \infty} 4^n (k(\frac{x}{2^n}) - k(0)) \end{aligned}$$

for all $x \in V$ if φ satisfies (b').

Proof. Since f, g, h, k are the even functions, from (3.12), we can easily obtain

$$\|f(x) + g(x) - 2h(x) - 2k(0)\| \leq \varphi(x, 0) \quad (3.13)$$

$$\|f(2x) + g(0) - 2h(x) - 2k(x)\| \leq \varphi(x, x) \quad (3.14)$$

$$\|f(x) + g(x) - 2h(0) - 2k(x)\| \leq \varphi(0, x) \quad (3.15)$$

$$\|f(0) + g(2x) - 2h(x) - 2k(x)\| \leq \varphi(x, -x) \quad (3.16)$$

$$\|f(0) + g(0) - 2h(0) - 2k(0)\| \leq \varphi(0, 0) \quad (3.17)$$

for all $x \in V$. From (3.13), (3.14) and (3.15), we get

$$\begin{aligned} \|f(2x) - 2f(x) - 2g(x) + g(0) + 2h(0) + 2k(0)\| &\leq \|f(2x) + g(0) - 2h(x) - 2k(x)\| \\ &\quad + \|f(x) + g(x) - 2h(0) - 2k(0)\| + \|f(x) + g(x) - 2h(x) - 2k(0)\| \\ &\leq \varphi(x, x) + \varphi(0, x) + \varphi(x, 0) \end{aligned} \quad (3.18)$$

for all $x \in V$. From (3.13), (3.15) and (3.16), we get

$$\begin{aligned} \|g(2x) - 2f(x) - 2g(x) + f(0) + 2h(0) + 2k(0)\| &\leq \|f(0) + g(2x) - 2h(x) - 2k(x)\| \\ &\quad + \|f(x) + g(x) - 2h(x) - 2k(0)\| + \|f(x) + g(x) - 2h(0) - 2k(x)\| \\ &\leq \varphi(x, -x) + \varphi(x, 0) + \varphi(0, x) \end{aligned} \quad (3.19)$$

for all $x \in V$. From (3.17), (3.18) and (3.19), we get

$$\begin{aligned} &\|4(f(x) + g(x) - f(0) - g(0)) - (f(2x) + g(2x) - f(0) - g(0))\| \\ &\leq \|f(2x) - 2f(x) - 2g(x) + g(0) + 2h(0) + 2k(0)\| \\ &\quad + \|g(2x) - 2f(x) - 2g(x) + f(0) + 2h(0) + 2k(0)\| \\ &\quad + 2\|f(0) + g(0) - 2h(0) - 2k(0)\| \\ &\leq \varphi(x, x) + 2\varphi(0, x) + 2\varphi(x, 0) + \varphi(x, -x) + 2\varphi(0, 0) \end{aligned}$$

for all $x \in V$. Induction argument implies

$$\begin{aligned} \|f(x) + g(x) - f(0) - g(0) - \frac{f(2^n x) + g(2^n x) - f(0) - g(0)}{4^n}\| \\ \leq \tilde{\varphi}(x, x) + 2\tilde{\varphi}(0, x) + 2\tilde{\varphi}(x, 0) + \tilde{\varphi}(x, -x) + 2\tilde{\varphi}(0, 0) \end{aligned} \quad (3.20)$$

for all $n \in \mathbb{N}$ and $x \in V$. From (3.20), we can easily show that $\left\{ \frac{f(2^n x) + g(2^n x) - f(0) - g(0)}{4^n} \right\}$ is a Cauchy sequence and thus converges. Define $Q : V \rightarrow X$ by

$$2Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x) + g(2^n x) - f(0) - g(0)}{4^n}$$

for all $x \in V$. From (3.20) and the definition of Q , we have

$$\begin{aligned} \|f(x) + g(x) - f(0) - g(0) - 2Q(x)\| \\ \leq \tilde{\varphi}(x, x) + 2\tilde{\varphi}(0, x) + 2\tilde{\varphi}(x, 0) + \tilde{\varphi}(x, -x) + 2\tilde{\varphi}(0, 0) \end{aligned} \quad (3.21)$$

for all $x \in V$. From (3.14), (3.16) and (3.21), we get

$$\begin{aligned} \|2f(x) - 2f(0) - 2Q(x)\| &\leq \|f(x) + g(x) - f(0) - g(0) - 2Q(x)\| \\ &\quad + \|f(x) + g(0) - 2h\left(\frac{x}{2}\right) - 2k\left(\frac{x}{2}\right)\| + \|f(0) + g(x) - 2h\left(\frac{x}{2}\right) - 2k\left(\frac{x}{2}\right)\| \\ &\leq 2M_3(x) \end{aligned}$$

for all $x \in V$. Replacing x by $2^n x$ and dividing by 4^n in (3.13), we have

$$\left\| \frac{f(2^n x) + g(2^n x)}{4^n} - \frac{2h(2^n x) + 2k(0)}{4^n} \right\| \leq \frac{\varphi(2^n x, 0)}{4^n} \quad (3.22)$$

for all $n \in \mathbb{N}$ and $x \in V$. Taking the limit in (3.22), we have

$$Q(x) = \lim_{n \rightarrow \infty} \frac{h(2^n x)}{4^n}$$

for all $x \in V$. By the similar method, we obtain the remainder of the proof.

COROLLARY 3.2 (EVEN FUNCTION). *Let $p < 2$, $\theta > 0$ be real numbers. Let ψ be the mapping as in Corollary 2.2. Suppose that the even functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \theta(\psi(x) + \psi(y)) \text{ for all } x, y \in V.$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \left(\frac{2}{2^p} + \frac{4}{4-2^p}\right)\theta\psi(x) + \frac{4}{3}\theta\psi(0), \\ \|g(x) - g(0) - Q(x)\| &\leq \left(\frac{2}{2^p} + \frac{4}{4-2^p}\right)\theta\psi(x) + \frac{4}{3}\theta\psi(0), \\ \|h(x) - h(0) - Q(x)\| &\leq \left(\frac{1}{2} + \frac{2}{2^p} + \frac{4}{4-2^p}\right)\theta\psi(x) + \frac{17}{6}\theta\psi(0) \text{ and} \\ \|k(x) - k(0) - Q(x)\| &\leq \left(\frac{1}{2} + \frac{2}{2^p} + \frac{4}{4-2^p}\right)\theta\psi(x) + \frac{17}{6}\theta\psi(0) \end{aligned}$$

for all $x \in V$. The function Q is given by the equation in Corollary 3.1.

COROLLARY 3.3 (EVEN FUNCTIONS). *Let $p \geq 0$ ($p \neq 2$), $\theta > 0$ be real numbers. Suppose that the even functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \theta(\|x\|^p + \|y\|^p) \text{ for all } x, y \in V.$$

Then there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x)\| &\leq \left(\frac{2}{2^p} + \frac{4}{|2^p-4|}\right)\theta\|x\|^p \\ \|g(x) - g(0) - Q(x)\| &\leq \left(\frac{2}{2^p} + \frac{4}{|2^p-4|}\right)\theta\|x\|^p \\ \|h(x) - h(0) - Q(x)\| &\leq \left(\frac{1}{2} + \frac{2}{2^p} + \frac{4}{|2^p-4|}\right)\theta\|x\|^p \text{ and} \\ \|k(x) - k(0) - Q(x)\| &\leq \left(\frac{1}{2} + \frac{2}{2^p} + \frac{4}{|2^p-4|}\right)\theta\|x\|^p \end{aligned}$$

for all $x \in V$. The function Q is given by the equation in Corollary 3.1.

The following lemma is easily obtained from Theorem 2.2 and Theorem 3.2 in [17].

LEMMA 3.1. Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \rightarrow [0, \infty)$ be a mapping such that

$$\hat{\varphi}(x, y) := \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \varphi(3^i x, 3^i y) < \infty \text{ for all } x, y \in V \setminus \{0\}, \tag{c}$$

or

$$\hat{\varphi}(x, y) := \sum_{i=0}^{\infty} 3^i \varphi\left(\frac{x}{3^{i+1}}, \frac{y}{3^{i+1}}\right) < \infty \text{ for all } x, y \in V \setminus \{0\}. \tag{c'}$$

Let $f, g, h : V \rightarrow X$ be mappings such that

$$\|f(x + y) - g(x) - h(y)\| \leq \varphi(x, y)$$

for all $x, y \in V \setminus \{0\}$. Then there exists a unique mapping $T : V \rightarrow X$ such that

$$T(x + y) = T(x) + T(y) \text{ for all } x, y \in V$$

and

$$\begin{aligned} \|f(x) - T(x) - f(0)\| \leq & \hat{\varphi}\left(\frac{x}{2}, \frac{-x}{2}\right) + \hat{\varphi}\left(\frac{-x}{2}, \frac{x}{2}\right) + \hat{\varphi}\left(\frac{x}{2}, \frac{x}{2}\right) + 2\hat{\varphi}\left(\frac{-x}{2}, \frac{-x}{2}\right) \\ & + \hat{\varphi}\left(\frac{-x}{2}, \frac{3x}{2}\right) + \hat{\varphi}\left(\frac{3x}{2}, \frac{-x}{2}\right) + \hat{\varphi}\left(\frac{3x}{2}, \frac{3x}{2}\right) \end{aligned}$$

for all $x \in V \setminus \{0\}$. The function $T : V \rightarrow X$ is given by

$$T(x) = \begin{cases} \lim_{n \rightarrow \infty} \frac{f(3^n x)}{3^n} & \text{if } \varphi \text{ satisfies (c)} \\ \lim_{n \rightarrow \infty} 3^n [f\left(\frac{x}{3^n}\right) - f(0)] & \text{if } \varphi \text{ satisfies (c')} \end{cases}$$

for all $x \in V$.

THEOREM 3.3 (ODD FUNCTION). Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \rightarrow [0, \infty)$ be a mapping as in Lemma 3.1. Suppose that the odd functions $f, g, h, k : V \rightarrow X$ satisfy

$$\|f(x + y) + g(x - y) - 2h(x) - 2k(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V \setminus \{0\}. \tag{3.23}$$

Then there exist two unique additive mappings $T, T' : V \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq M_2(x, x), \tag{3.24}$$

$$\|g(x) - T'(x)\| \leq M_2(x, -x), \tag{3.25}$$

$$\begin{aligned} \|h(x) - \frac{T(x) + T'(x)}{2}\| \leq & \frac{M_2(2x, 2x) + M_2(2x, -2x)}{4} \\ & + \frac{\varphi(x, x) + \varphi(x, -x)}{4}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} \|k(x) - \frac{T(x) - T'(x)}{2}\| \leq & \frac{M_2(2x, 2x) + M_2(2x, -2x)}{4} \\ & + \frac{\varphi(x, x) + \varphi(x, -x)}{4} \end{aligned} \tag{3.27}$$

for all $x \in V \setminus \{0\}$, where

$$M_2(x, y) = \hat{\varphi}\left(\frac{x}{2}, \frac{-y}{2}\right) + \hat{\varphi}\left(\frac{-x}{2}, \frac{y}{2}\right) + \hat{\varphi}\left(\frac{x}{2}, \frac{y}{2}\right) + 2\hat{\varphi}\left(\frac{-x}{2}, \frac{-y}{2}\right) \\ + \hat{\varphi}\left(\frac{-x}{2}, \frac{3y}{2}\right) + \hat{\varphi}\left(\frac{3x}{2}, \frac{-y}{2}\right) + \hat{\varphi}\left(\frac{3x}{2}, \frac{3y}{2}\right).$$

The functions T, T' are given by

$$T(x) = \lim_{n \rightarrow \infty} \frac{f(3^n x)}{3^n}, \quad T'(x) = \lim_{n \rightarrow \infty} \frac{g(3^n x)}{3^n} \quad \text{if } \varphi \text{ satisfies (c),} \\ T(x) = \lim_{n \rightarrow \infty} 3^n f(3^{-n} x), \quad T'(x) = \lim_{n \rightarrow \infty} 3^n g(3^{-n} x) \quad \text{if } \varphi \text{ satisfies (c')}$$

for all $x \in V$. Furthermore, the following properties hold:

$$\lim_{n \rightarrow \infty} \frac{h(3^n x)}{3^n} = \frac{1}{2}(T(x) + T'(x)), \quad \lim_{n \rightarrow \infty} \frac{k(3^n x)}{3^n} = \frac{1}{2}(T(x) - T'(x)) \quad \text{if } \varphi \text{ satisfies (c),} \\ \lim_{n \rightarrow \infty} 3^n h(3^{-n} x) = \frac{1}{2}(T(x) + T'(x)), \quad \lim_{n \rightarrow \infty} 3^n k(3^{-n} x) = \frac{1}{2}(T(x) - T'(x)) \quad \text{if } \varphi \text{ satisfies (c')}$$

for all $x \in V$.

Proof. Let φ satisfy the condition (c). Since f, g, h, k are odd functions, from (3.23), we can easily obtain

$$\|f(x - y) + g(x + y) - 2h(x) + 2k(y)\| \leq \varphi(x, -y), \quad (3.28)$$

$$\|f(x + y) - g(x - y) - 2h(y) - 2k(x)\| \leq \varphi(y, x) \quad (3.29)$$

for all $x, y \in V \setminus \{0\}$. From (3.23) and (3.29), we obtain

$$\|2f(x + y) - 2h(x) - 2h(y) - 2k(x) - 2k(y)\| \leq \varphi(x, y) + \varphi(y, x) \quad (3.30)$$

for all $x, y \in V \setminus \{0\}$. Define $U, W : V \rightarrow X$ by

$$U(x) = 2f(x) \quad \text{and} \quad W(x) = 2h(x) + 2k(x) \quad \text{for } x \in V.$$

From (3.30) and the definition of U and W , we know that

$$\|U(x + y) - W(x) - W(y)\| \leq \varphi(x, y) + \varphi(y, x)$$

for all $x, y \in V \setminus \{0\}$. Applying Lemma 3.1, there exists a unique additive mapping $T : V \rightarrow X$ satisfying (3.24) and

$$\lim_{n \rightarrow \infty} \frac{f(3^n x)}{3^n} = T(x) \quad \text{for all } x \in V.$$

By the similar method as the above, there exists a unique additive mapping $T' : V \rightarrow X$ satisfying (3.25) and

$$\lim_{n \rightarrow \infty} \frac{g(3^n x)}{3^n} = T'(x) \quad \text{for all } x \in V.$$

From (3.23) and (3.28), we have

$\|f(x+y) + f(x-y) + g(x+y) + g(x-y) - 4h(x)\| \leq \varphi(x, y) + \varphi(x, -y)$
 $x, y \in V \setminus \{0\}$. Replacing y by x and dividing by 4 in the above inequality, we have

$$\left\| \frac{f(2x) + g(2x)}{4} - h(x) \right\| \leq \frac{1}{4} [\varphi(x, x) + \varphi(x, -x)] \quad (3.31)$$

for all $n \in N$ and $x, y \in V \setminus \{0\}$. From (3.24), (3.25) and (3.31), we get

$$\begin{aligned} & \left\| h(x) - \frac{T(x) + T'(x)}{2} \right\| \\ & \leq \left\| \frac{f(2x) - T(2x)}{4} \right\| + \left\| \frac{g(2x) - T'(2x)}{4} \right\| + \left\| \frac{f(2x) + g(2x)}{4} - h(x) \right\| \\ & \leq \frac{1}{4} (M_2(2x, 2x) + M_2(2x, -2x)) + \frac{1}{4} (\varphi(x, x) + \varphi(x, -x)) \end{aligned}$$

for all $x \in V \setminus \{0\}$. From the above inequality, we easily obtain

$$\lim_{n \rightarrow \infty} \frac{h(3^n x)}{3^n} = \frac{1}{2} (T(x) + T'(x)) \quad \text{for all } x \in V.$$

Similarly we can obtain (3.27) and

$$\lim_{n \rightarrow \infty} \frac{k(3^n x)}{3^n} = \frac{1}{2} (T(x) - T'(x)) \quad \text{for all } x \in V.$$

The remainder of the proof is easily obtained by the similar method.

COROLLARY 3.4 (ODD FUNCTION). *Let $p \neq 1$, $\theta > 0$ be real numbers. Suppose that the odd functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \theta (\|x\|^p + \|y\|^p) \quad \text{for all } x, y \in V \setminus \{0\}. \quad (3.23)$$

Then there exist two unique additive mappings $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - T(x)\| & \leq \frac{4\theta(3 + 3^p)}{2^p|3 - 3^p|} \|x\|^p, \\ \|g(x) - T'(x)\| & \leq \frac{4\theta(3 + 3^p)}{2^p|3 - 3^p|} \|x\|^p \\ \|h(x) - \frac{T(x) + T'(x)}{2}\| & \leq \left(\frac{6 + 2 \cdot 3^p}{|3 - 3^p|} + 1 \right) \theta \|x\|^p \quad \text{and} \\ \|k(x) - \frac{T(x) - T'(x)}{2}\| & \leq \left(\frac{6 + 2 \cdot 3^p}{|3 - 3^p|} + 1 \right) \theta \|x\|^p \end{aligned}$$

for all $x \in V \setminus \{0\}$. The functions T, T' are given by the equation as in Theorem 3.3.

REMARK. In Corollary 3.4, we can easily obtain the following properties:

$$\begin{aligned} T &= T' & \text{if } f = g \text{ or } f = h \text{ or } g = h, \\ T' &= 0 & \text{if } h = k, \\ T &= T' = 0 & \text{if } f = g \text{ and } h = k, \\ T &= T' = 0 & \text{if } f = h \text{ and } g = k. \end{aligned}$$

The following lemma is easily obtained from Theorem 2.2 and Theorem 3.2 in [18].

LEMMA 3.2. Let $\varphi : V \times V \rightarrow [0, \infty)$ be a mapping such that

$$\check{\varphi}(x, y) := \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi(2^i x, 2^i y) < \infty \text{ for all } x, y \in V, \tag{d}$$

or

$$\check{\varphi}(x, y) := \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) < \infty \text{ for all } x, y \in V. \tag{d'}$$

Let $f, g, h : V \rightarrow X$ be mappings such that

$$\|f(x + y) - g(x) - h(y)\| \leq \varphi(x, y)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \rightarrow X$ such that

$$\begin{aligned} \|T(x) - f(x) + f(0)\| &\leq \check{\varphi}(x, 0) + \check{\varphi}(0, x) + \check{\varphi}(x, x) + M, \\ \|T(x) - g(x) + g(0)\| &\leq \check{\varphi}(x, -x) + \check{\varphi}(x, 0) + \check{\varphi}(2x, -x) + M, \\ \|T(x) - h(x) + h(0)\| &\leq \check{\varphi}(-x, x) + \check{\varphi}(0, x) + \check{\varphi}(-x, 2x) + M \end{aligned}$$

for all $x \in V$, where $M = \|f(0) - g(0) - h(0)\|$. The function T is given by

$$T(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n} = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^n} = \lim_{n \rightarrow \infty} \frac{h(2^n x)}{2^n} \text{ for all } x \in V$$

if φ satisfies the condition (d),

$$T(x) = \begin{cases} \lim_{n \rightarrow \infty} 2^n [f(2^{-n} x) - f(0)] \\ \lim_{n \rightarrow \infty} 2^n [g(2^{-n} x) - g(0)] \\ \lim_{n \rightarrow \infty} 2^n [h(2^{-n} x) - h(0)] \end{cases} \text{ for all } x \in V$$

if φ satisfies the condition (d').

THEOREM 3.4 (ODD FUNCTION). Let $\varphi : V \times V \rightarrow [0, \infty)$ be a mapping as in Lemma 3.2. Suppose that the odd functions $f, g, h, k : V \rightarrow X$ satisfy

$$\|f(x + y) + g(x - y) - 2h(x) - 2k(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V.$$

Then there exist two unique additive mappings $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - T(x)\| &\leq M_4(x, x), \\ \|g(x) - T'(x)\| &\leq M_4(x, -x) \\ \|h(x) - \frac{1}{2}(T(x) + T'(x))\| &\leq M_5(x), \\ \|k(x) - \frac{1}{2}(T(x) - T'(x))\| &\leq M_5(x) \end{aligned}$$

for all $x \in V$, where

$$\begin{aligned} M_4(x, y) &= \check{\phi}(x, 0) + \check{\phi}(0, y) + \check{\phi}(x, y), \\ M_5(x) &= \frac{1}{4}[\check{\phi}(x, -x) + 2\check{\phi}(x, 0) + \check{\phi}(2x, -x) + \check{\phi}(-x, x) \\ &\quad + \check{\phi}(0, x) + \check{\phi}(-x, 2x) + \check{\phi}(x, x) + \check{\phi}(2x, x) \\ &\quad + \check{\phi}(-x, -x) + \check{\phi}(0, -x) + \check{\phi}(-x, -2x)]. \end{aligned}$$

The functions T, T' are given by

$$\begin{aligned} T(x) &= \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}, & T'(x) &= \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^n} \quad \text{if } \phi \text{ satisfies } (d), \\ T(x) &= \lim_{n \rightarrow \infty} 2^n f(2^{-n} x), & T'(x) &= \lim_{n \rightarrow \infty} 2^n g(2^{-n} x) \quad \text{if } \phi \text{ satisfies } (d') \end{aligned}$$

for all $x \in V$.

Proof. Applying Lemma 3.2, we can obtain two unique additive mappings $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - T(x)\| &\leq M_4(x, x), \\ \|g(x) - T'(x)\| &\leq M_4(x, -x), \\ \|h(x) + k(x) - T(x)\| &\leq \frac{1}{2}[\check{\phi}(x, -x) + \check{\phi}(x, 0) + \check{\phi}(2x, -x) \\ &\quad + \check{\phi}(-x, x) + \check{\phi}(0, x) + \check{\phi}(-x, 2x)], \end{aligned} \tag{3.32}$$

$$\begin{aligned} \|h(x) - k(x) - T'(x)\| &\leq \frac{1}{2}[\check{\phi}(x, x) + \check{\phi}(x, 0) + \check{\phi}(2x, x) \\ &\quad + \check{\phi}(-x, -x) + \check{\phi}(0, -x) + \check{\phi}(-x, -2x)] \end{aligned} \tag{3.33}$$

for all $x \in V$ by the similar method as in Theorem 3.3. From (3.32) and (3.33), we get

$$\begin{aligned} \|h(x) - \frac{1}{2}(T(x) + T'(x))\| &\leq M_5(x), \\ \|k(x) - \frac{1}{2}(T(x) - T'(x))\| &\leq M_5(x) \end{aligned}$$

for all $x \in V$. The remainder of the proof is analogous to the proof of Theorem 3.3.

COROLLARY 3.5 (ODD FUNCTION). *Let $p \neq 1$, $\theta > 0$ be real numbers. Let $\psi : V \rightarrow [0, \infty)$ be a mapping such that $\psi(x) = \|x\|^p$ for $x \neq 0$ and $\psi(0) = 0$ if $p > 1$. Suppose that the odd functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \theta(\psi(x) + \psi(y)) \quad \text{for all } x, y \in V.$$

Then there exist two unique additive mappings $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - T(x)\| &\leq \frac{4\theta}{|2^p - 2|} \|x\|^p + 2\psi(0)\theta, \\ \|g(x) - T'(x)\| &\leq \frac{4\theta}{|2^p - 2|} \|x\|^p + 2\psi(0)\theta, \\ \|h(x) - \frac{1}{2}(T(x) + T'(x))\| &\leq \frac{(4 + 2^p)\theta}{|2^p - 2|} \|x\|^p + \psi(0)\theta \\ \|k(x) - \frac{1}{2}(T(x) - T'(x))\| &\leq \frac{(4 + 2^p)\theta}{|2^p - 2|} \|x\|^p + \psi(0)\theta \end{aligned}$$

for all $x \in V$. The functions T, T' are given by

$$\begin{aligned} T(x) &= \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}, \quad T'(x) = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^n} \quad \text{if } p < 1, \\ T(x) &= \lim_{n \rightarrow \infty} 2^n f(2^{-n} x), \quad T'(x) = \lim_{n \rightarrow \infty} 2^n g(2^{-n} x) \quad \text{if } p > 1 \end{aligned}$$

for all $x \in V$.

Now we prove the stability of the general Pexiderized quadratic equation.

THEOREM 3.5. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \rightarrow X$ be a mapping satisfying one of the conditions in Theorem 2.1 and one of the conditions in Theorem 3.3. Suppose that the functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x + y) + g(x - y) - 2h(x) - 2k(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V \setminus \{0\}. \quad (3.34)$$

Then there exist exactly one quadratic function $Q : V \rightarrow X$ and two unique additive functions $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x) - T(x)\| &\leq \frac{M_1(x) + M_1(-x) + M_2(x, x) + M_2(-x, -x)}{2}, \\ \|g(x) - g(0) - Q(x) - T'(x)\| &\leq \frac{M_1(x) + M_1(-x) + M_2(x, -x) + M_2(-x, x)}{2}, \\ \left\| \frac{h(x) + k(x)}{2} - \frac{f(0) + g(0)}{4} - Q(x) - \frac{T(x)}{2} \right\| &\leq \frac{M_1(2x) + M_1(-2x)}{8} \\ &\quad + \frac{M_2(2x, 2x) + M_2(2x, -2x) + M_2(-2x, 2x) + M_2(-2x, -2x)}{8} \\ &\quad + \frac{\varphi(x, x) + \varphi(x, -x) + \varphi(-x, x) + \varphi(-x, -x)}{8} \end{aligned}$$

for all $x \in V \setminus \{0\}$, where $M_1(x)$ is as in Theorem 3.1 and $M_2(x, y)$ is as in Theorem 3.3.

The function Q is given by

$$Q(x) = \begin{cases} \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n} & \text{if } \varphi \text{ satisfies (a)} \\ \lim_{n \rightarrow \infty} \frac{4^n}{2} \left(f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right) - 2f(0) \right) \\ = \lim_{n \rightarrow \infty} \frac{4^n}{2} \left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right) - 2g(0) \right) & \text{if } \varphi \text{ satisfies (a')} \end{cases}$$

for all $x \in V$ and the functions T, T' are given by

$$\begin{aligned}
 T(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{f(3^n x) - f(-3^n x)}{2 \cdot 3^n} & \text{if } \varphi \text{ satisfies (c)} \\ \lim_{n \rightarrow \infty} 3^n (f(\frac{x}{3^n}) - f(0)) & \text{if } \varphi \text{ satisfies (c')}, \end{cases} \\
 T'(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{g(3^n x) - g(-3^n x)}{2 \cdot 3^n} & \text{if } \varphi \text{ satisfies (c)} \\ \lim_{n \rightarrow \infty} 3^n (g(\frac{x}{3^n}) - g(0)) & \text{if } \varphi \text{ satisfies (c')}, \end{cases} \\
 \frac{T(x) + T'(x)}{2} &= \begin{cases} \lim_{n \rightarrow \infty} \frac{h(3^n x) - h(-3^n x)}{2 \cdot 3^n} & \text{if } \varphi \text{ satisfies (c)} \\ \lim_{n \rightarrow \infty} \frac{3^n}{2} (h(\frac{x}{3^n}) - h(-\frac{x}{3^n})) & \text{if } \varphi \text{ satisfies (c')}, \end{cases} \\
 \frac{T(x) - T'(x)}{2} &= \begin{cases} \lim_{n \rightarrow \infty} \frac{k(3^n x) - k(-3^n x)}{2 \cdot 3^n} & \text{if } \varphi \text{ satisfies (c)} \\ \lim_{n \rightarrow \infty} \frac{3^n}{2} (k(\frac{x}{3^n}) - k(-\frac{x}{3^n})) & \text{if } \varphi \text{ satisfies (c')} \end{cases}
 \end{aligned}$$

for all $x \in V$.

Proof. Assume that φ satisfies the condition (a) in Theorem 2.1 and the condition (c) in Theorem 3.3. Let f_1, g_1, h_1, k_1 be even parts and f_2, g_2, h_2, k_2 be odd parts of f, g, h, k , respectively. From (3.34), we get

$$\begin{aligned}
 & \|f_1(x+y) + g_1(x-y) - 2h_1(x) - 2k_1(y)\| \\
 & \leq \frac{1}{2} \|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \\
 & \quad + \frac{1}{2} \|f(-x-y) + g(-x+y) - 2h(-x) - 2k(-y)\| \\
 & \leq \frac{\varphi(x, y) + \varphi(-x, -y)}{2}
 \end{aligned}$$

for all $x, y \in V \setminus \{0\}$. By Theorem 3.1, there exists exactly one quadratic function $Q : V \rightarrow X$ such that

$$\|f_1(x) - f(0) - Q(x)\| \leq \frac{M_1(x) + M_1(-x)}{2} \tag{3.35}$$

for all $x \in V \setminus \{0\}$. From (3.34), we similarly get

$$\|f_2(x+y) + g_2(x-y) - 2h_2(x) - 2k_2(y)\| \leq \frac{\varphi(x, y) + \varphi(-x, -y)}{2} \text{ for all } x \in V \setminus \{0\}.$$

Applying Theorem 3.3, there exists a unique additive mapping $T : V \rightarrow X$ such that

$$\|f_2(x) - T(x)\| \leq \frac{M_2(x, x) + M_2(-x, -x)}{2} \tag{3.36}$$

for all $x \in V \setminus \{0\}$. From (3.35) and (3.36), we get

$$\begin{aligned}
 & \|f(x) - f(0) - Q(x) - T(x)\| \\
 & \leq \|f_1(x) - f(0) - Q(x)\| + \|f_2(x) - T(x)\| \\
 & \leq \frac{M_1(x) + M_1(-x)}{2} + \frac{M_2(x, -x) + M_2(-x, -x)}{2}
 \end{aligned} \tag{3.37}$$

for all $x \in V \setminus \{0\}$. Replacing x by $2^n x$ and dividing 4^n in (3.37), we get

$$\begin{aligned} \left\| \frac{f(2^n x)}{4^n} - \frac{f(0)}{4^n} - \frac{Q(2^n x)}{4^n} - \frac{T(2^n x)}{4^n} \right\| &\leq \|f_1(x) - f(0) - Q(x)\| + \|f_2(x) - T(x)\| \\ &\leq \frac{M_1(2^n x) + M_1(-2^n x)}{2 \cdot 4^n} + \frac{M_2(2^n x, 2^n x) + M_2(-2^n x, -2^n x)}{2 \cdot 4^n} \end{aligned}$$

for all $x \in V$. Taking the limit in the above inequality as $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = Q(x)$$

for all $x \in V \setminus \{0\}$.

By the similar method, we obtain the remaining results.

COROLLARY 3.6. *Let $p \neq 1, 2$, $\theta > 0$ be real numbers. Suppose that the functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \theta(\|x\|^p + \|y\|^p) \text{ for all } x, y \in V \setminus \{0\}. \quad (3.34)$$

Then there exist exactly one quadratic function $Q : V \rightarrow X$ and two unique additive functions $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x) - T(x)\| &\leq \left(\frac{2 \cdot 5^p + 4 \cdot 4^p + 4 \cdot 3^p + 4 \cdot 2^p + 4}{2^p |4 - 2^p|} + \frac{2}{2^p} + \frac{4(3 + 3^p)}{2^p |3 - 3^p|} \right) \theta \|x\|^p, \\ \|g(x) - g(0) - Q(x) - T'(x)\| &\leq \left(\frac{2 \cdot 5^p + 4 \cdot 4^p + 4 \cdot 3^p + 4 \cdot 2^p + 4}{2^p |4 - 2^p|} + \frac{2}{2^p} + \frac{4(3 + 3^p)}{2^p |3 - 3^p|} \right) \theta \|x\|^p, \\ \left\| \frac{h(x) + k(x)}{2} - \frac{f(0) + g(0)}{4} - Q(x) - \frac{T(x)}{2} \right\| &\leq \left[\frac{5^p + 2 \cdot 4^p + 2 \cdot 3^p + 2 \cdot 2^p + 2}{2 |4 - 2^p|} + \frac{3}{2} + \frac{6 + 2 \cdot 3^p}{|3 - 3^p|} \right] \theta \|x\|^p \end{aligned}$$

for all $x \in V \setminus \{0\}$.

From Theorem 3.2 and Theorem 3.4, we can easily obtain the following theorem.

THEOREM 3.6. *Let $\varphi : V \times V \rightarrow X$ be a mapping satisfying one of the conditions in Theorem 3.2 and one of the conditions in Theorem 3.4. Suppose that the functions $f, g, h, k : V \rightarrow X$ satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \varphi(x, y) \text{ for all } x, y \in V.$$

Then there exist exactly one quadratic function $Q : V \rightarrow X$ and two unique additive

functions $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x) - T(x)\| &\leq \frac{M_3(x) + M_3(-x) + M_4(x, x) + M_4(-x, -x)}{2}, \\ \|g(x) - g(0) - Q(x) - T'(x)\| &\leq \frac{M_3(x) + M_3(-x) + M_4(x, -x) + M_4(-x, x)}{2}, \\ \|h(x) - h(0) - Q(x) - \frac{1}{2}(T(x) + T'(x))\| &\leq \frac{M_3(x) + M_3(-x) + M_5(x) + M_5(-x)}{2} \\ &\quad + \frac{\varphi(x, 0) + 2\varphi(0, 0) + \varphi(-x, 0)}{4}, \text{ and} \\ \|k(x) - k(0) - Q(x) - \frac{1}{2}(T(x) - T'(x))\| &\leq \frac{M_3(x) + M_3(-x) + M_5(x) + M_5(-x)}{2} \\ &\quad + \frac{\varphi(0, x) + 2\varphi(0, 0) + \varphi(0, -x)}{4} \end{aligned}$$

for all $x \in V$ where $M_3(x)$ is as in Theorem 3.2 and $M_4(x, y), M_5(x)$ are as in Theorem 3.4. The function Q is given by

$$Q(x) = \begin{cases} \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{g(2^{2n} x)}{4^n} & \text{if } \varphi \text{ satisfies (b)} \\ \lim_{n \rightarrow \infty} \frac{4^n}{2} (f(\frac{x}{2^n}) + f(-\frac{x}{2^n}) - 2f(0)) \\ = \lim_{n \rightarrow \infty} \frac{4^n}{2} (g(\frac{x}{2^n}) + g(-\frac{x}{2^n}) - 2g(0)) & \text{if } \varphi \text{ satisfies (b')} \end{cases}$$

and the functions T, T' are given by

$$\begin{aligned} T(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}, & \text{if } \varphi \text{ satisfies (d)} \\ \lim_{n \rightarrow \infty} 2^n (f(2^{-n} x) - f(0)), & \text{if } \varphi \text{ satisfies (d')} \end{cases} \\ T'(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{g(2^n x) - g(-2^n x)}{2^{n+1}}, & \text{if } \varphi \text{ satisfies (d)} \\ \lim_{n \rightarrow \infty} 2^n (g(2^{-n} x) - g(0)), & \text{if } \varphi \text{ satisfies (d')} \end{cases} \end{aligned}$$

COROLLARY 3.7. Let $p \neq 1, 2$, $\theta > 0$ be real numbers. Let $\psi : V \rightarrow [0, \infty)$ be a mapping such that $\psi(x) = \|x\|^p$ for $x \neq 0$ and $\psi(0) = 0$ if $p > 1$. Suppose that the functions $f, g, h, k : V \rightarrow X$ satisfy

$$\|f(x + y) + g(x - y) - 2h(x) - 2k(y)\| \leq \theta(\psi(x) + \psi(y)) \text{ for all } x, y \in V.$$

Then there exist exactly one quadratic function $Q : V \rightarrow X$ and two unique additive functions $T, T' : V \rightarrow X$ such that

$$\begin{aligned} \|f(x) - f(0) - Q(x) - T(x)\| &\leq (M + M_1)\theta\|x\|^p + \frac{10}{3}\theta\psi(0), \\ \|g(x) - g(0) - Q(x) - T'(x)\| &\leq (M + M_1)\theta\|x\|^p + \frac{10}{3}\theta\psi(0), \\ \|h(x) - h(0) - Q(x) - \frac{1}{2}(T(x) + T'(x))\| &\leq (\frac{1}{2} + M + M_2)\theta\|x\|^p + \frac{23}{6}\theta\psi(0) \text{ and} \\ \|k(x) - k(0) - Q(x) - \frac{1}{2}(T(x) - T'(x))\| &\leq (\frac{1}{2} + M + M_2)\theta\|x\|^p + \frac{23}{6}\theta\psi(0) \end{aligned}$$

for all $x \in V$ where $M = \frac{2}{2^p} + \frac{4}{|4-2^p|}$, $M_1 = \frac{4}{|2^p-2|}$ and $M_2 = \frac{4+2^p}{|2^p-2|}$. The functions Q, T, T' are given by

$$\begin{aligned}
 Q(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{4^n} & \text{if } p < 2 \\ \lim_{n \rightarrow \infty} \frac{4^n}{2} (f(2^{-n}x) + f(-2^{-n}x) - 2f(0)) & \\ \lim_{n \rightarrow \infty} \frac{4^n}{2} (g(2^{-n}x) + g(-2^{-n}x) - 2g(0)) & \text{if } 2 < p, \end{cases} \\
 T(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}, & \text{if } p < 1 \\ \lim_{n \rightarrow \infty} 2^{n-1} (f(2^{-n}x) - f(-2^{-n}x)), & \text{if } 1 < p \end{cases} \\
 T'(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{g(2^n x) - g(-2^n x)}{2^{n+1}}, & \text{if } p < 1 \\ \lim_{n \rightarrow \infty} 2^{n-1} (g(2^{-n}x) - g(-2^{-n}x)), & \text{if } 1 < p \end{cases}
 \end{aligned}$$

The following corollary, which is due to S. M. Jung [15], can be easily obtained from Corollary 3.7.

COROLLARY 3.8. *Suppose that the functions $f, g, h, k : V \rightarrow X$ satisfy*

$$f(x + y) + g(x - y) - 2h(x) - 2k(y) = 0 \text{ for all } x, y \in V.$$

Then there exist exactly one quadratic function $Q : V \rightarrow X$ and two unique additive functions $T, T' : V \rightarrow X$ such that

$$\begin{aligned}
 f(x) &= Q(x) + T(x) + f(0) \\
 g(x) &= Q(x) + T'(x) + g(0) \\
 h(x) &= Q(x) + \frac{1}{2}(T(x) + T'(x)) + h(0) \\
 k(x) &= Q(x) + \frac{1}{2}(T(x) - T'(x)) + k(0)
 \end{aligned}$$

for all $x \in V$.

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