Mathematical
Inequalities
& Applications
Volume 4, Number 1 (2001), 119–126

KOROVKIN TYPE ERROR ESTIMATES FOR MEYER–KÖNIG AND ZELLER OPERATORS

OCTAVIAN AGRATINI

(*communicated by J. Pečarić*)

Abstract. In this paper we construct a linear and positive approximation process of discrete type which includes as a particular case the Meyer-König and Zeller operators. Based on several inequalities we prove that the sequence converges to the identity operator. We obtain inequalities regarding estimations of the remainder which are given by using the moduli of smoothness of first and second order as well as the Lipschitz type maximal function. Also we establish that our operators have the variation diminishing property.

1. Introduction

The operators of Meyer-König and Zeller in the slight modification of Cheney and Sharma [6]

$$(M_nf)(x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n+k}\right), \quad 0 \leqslant x < 1, \quad (M_nf)(1) = f(1), \tag{1}$$

and

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1},$$

were the subject of several investigations in approximation theory.

Among the first remarkable papers with this topic we would like to mention here the results obtained by A. Lupaş and M.W. Müller [12]. In present days U. Abel [1] has obtained the complete asymptotic expansion for M_n operators. These operators are defined on the set $B^*[0,1]$ of all functions f which are bounded on $[0,1]$ and continuous on $[0,1)$. The integral analogue of M_n operators were more deeply studied by M. W. Müller [13], V. Totik [16], Wenzhong Chen [5], see also [10].

In time, we point out that many generalizations were given, one of the most recent being obtained in 1998 by O. Dŏgru [7].

In the present paper we define another sequence of generalized linear and positive operators which includes the sequence obtained in [7]. We prove the convergence of the sequence to the identity operator and under some additional assumptions we study

---

Mathematics subject classification (2000): 41A36, 26D07.

Key words and phrases: linear and positive operators, moduli of smoothness, variation diminishing property, Lipschitz type maximal function.

Paper MIA-04-09

the degree of approximation in terms of the moduli of smoothness of first and second order. We establish that the variation diminishing property takes place and also we give a local error estimate using Lipschitz type maximal function of order $\alpha \in (0,1]$.

2. Construction of the operators

Let's define the sequences of real numbers $(\alpha_n), (\beta_k), (\gamma_{n,k})$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, having the following properties

$$1 \leqslant \alpha_n = 1 + \mathscr{O}\left(\frac{1}{n}\right), \quad 0 \leqslant \beta_k \leqslant \beta_{k+1} + 1, \quad 0 \leqslant \gamma_{n,k} = \mathscr{O}\left(\frac{1}{n}\right). \tag{2}$$

Let a be a real number on the interval $(0,1)$. Assume that a sequence of functions (φ_n) satisfies the following conditions:

1° Every function φ_n is analytic on a domain Ω containing the disk $D = \{z \in \mathbb{C} : |z| \leqslant a\}$.

2° $\varphi_n^{(0)}(0) = \varphi_n(0) > 0$ and $\varphi_n^{(k)}(0) = \frac{d^k}{dx^k}\varphi_n(x)\Big|_{x=0} > 0, \quad k = 1,2,\ldots$

3° $\varphi_n^{(k)}(0) = \alpha_n(k+n+\beta_k)(1+\gamma_{n,k})\varphi_n^{(k-1)}(0)$, $k = 1,2,\ldots$, such as the conditions presented at (2) are fulfilled.

We introduce the sequence of operators

$$(D_nf)(x) = \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty} w_{n,k}(x) f\left(\frac{k}{k+n+\beta_k}\right), \tag{3}$$

where

$$w_{n,k}(x) = \varphi_n^{(k)}(0)\frac{x^k}{k!} \quad \text{and} \quad f \in C[0,a].$$

It is clear that D_n is a linear and positive operator. If $\beta_k = 0$, $k \in \mathbb{N}$, and $\varphi_n(x) = (1-x)^{-n-1}$ after a few calculations we obtain $\alpha_n = 1$, $\gamma_{n,k} = 0$ and D_n becomes the operator M_n defined by (1).

By using (2) and taking into account the definition of the sequence $\gamma_{n,k}$ we deduce that there exists a constant $c > 0$ so that

$$\gamma_{n,k} \leqslant \frac{c}{n}, \quad \text{for any} \quad k \in \mathbb{N}. \tag{4}$$

3. Approximation properties

Further the convergence of the (D_nf) sequence to the function f will be proved. In the sequel we denote by e_i the i-th monomial, $i = 0,1,2$. It is obvious that

$$(D_ne_0)(x) = \frac{1}{\varphi_n(x)}\sum_{k=0}^{\infty} w_{n,k}(x) = 1. \tag{5}$$

LEMMA 3.1. *If the operator D_n is defined by (3) then the following inequalities*

$$0 \leqslant (D_ne_1)(x) - x \leqslant x(\alpha_n - 1) + \frac{x\alpha_n c}{n}$$

hold.

Proof. By using (4) we can write successively

$$(D_n e_1)(x) = \frac{1}{\varphi_n(x)} \sum_{k=1}^{\infty} \alpha_n (1+\gamma_{n,k}) \frac{\varphi_n^{(k-1)}(0)}{(k-1)!} x^k =$$

$$= \frac{x\alpha_n}{\varphi_n(x)} \sum_{k=1}^{\infty} (1+\gamma_{n,k}) w_{n,k-1}(x) \leqslant x\alpha_n \left(1+\frac{c}{n}\right)(D_n e_0)(x).$$

In this way we obtain

$$(D_n e_1)(x) - x \leqslant x(\alpha_n - 1) + \frac{x\alpha_n c}{n}. \tag{6}$$

On the other hand the requirement 3° allows us to write

$$\begin{aligned}(D_n e_1)(x) &= \frac{1}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{k}{n+k+\beta_k} \varphi_n^{(k)}(0) \frac{x^k}{k!} = \frac{\alpha_n x}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{x^k}{k!} (1+\gamma_{n,k+1}) \varphi_n^{(k)}(0) = \\ &= \alpha_n x (D_n e_0)(x) + \frac{\alpha_n x}{\varphi_n(x)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \gamma_{n,k+1} \varphi_n^{(k)}(0)\end{aligned}$$

and consequently $(D_n e_1)(x) - x = x(\alpha_n - 1) + \dfrac{\alpha_n x}{\varphi_n(x)} \displaystyle\sum_{k=0}^{\infty} w_{n,k}(x)\gamma_{n,k+1}$.

Following (2), each term of the right side is non-negative, thus

$$(D_n e_1)(x) - x \geqslant 0. \tag{7}$$

The relations (6) and (7) finish the proof.

LEMMA 3.2. *If the operator D_n is defined by (3) then the following inequalities*

$$0 \leqslant (D_n e_2)(x) - x^2 \leqslant (\alpha_n^2 - 1)x^2 + \alpha_n x \left(\frac{2c\alpha_n x + 1}{n} + \frac{c(\alpha_n c x + 1)}{n^2} \right)$$

hold.

Proof. By using again the requirement 3° we have

$$(D_n e_2)(x) = \frac{1}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{k^2}{(k+n+\beta_k)^2} w_{n,k}(x) =$$

$$= \frac{x^2}{\varphi_n(x)} \sum_{k=2}^{\infty} \alpha_n^2 \frac{k-1+n+\beta_{k-1}}{k+n+\beta_k} (1+\gamma_{n,k})(1+\gamma_{n,k-1}) w_{n,k-2}(x) +$$

$$+ \frac{x}{\varphi_n(x)} \sum_{k=1}^{\infty} \frac{\alpha_n(1+\gamma_{n,k})}{k+n+\beta_k} w_{n,k-1}(x) \leqslant \frac{\alpha_n^2 x^2}{\varphi_n(x)} \left(1+\frac{c}{n}\right)^2 \sum_{k=2}^{\infty} w_{n,k-2}(x) +$$

$$+ \frac{\alpha_n x}{n\varphi_n(x)} \left(1+\frac{c}{n}\right) \sum_{k=1}^{\infty} w_{n,k-1}(x) = \alpha_n^2 \left(1+\frac{c}{n}\right)^2 x^2 + \frac{\alpha_n}{n}\left(1+\frac{c}{n}\right) x.$$

In this way we have obtained

$$(D_n e_2)(x) - x^2 \leqslant (\alpha_n^2 - 1)x^2 + \frac{1}{n}(2c^2\alpha_n^2 x^2 + \alpha_n x) + \frac{1}{n^2}(\alpha_n^2 c^2 x^2 + c\alpha_n x). \tag{8}$$

Because of $e_2 = (e_1 - xe_0)^2 + 2xe_1 - x^2 e_0$ we get

$$(L_n e_2)(x) - x^2 = D_n((e_1 - xe_0)^2; x) + 2xD_n(e_1 - xe_0; x) \geqslant 0. \tag{9}$$

We have also used the positivity of our operators and the inequality (7).

By taking into account the relations (8) and (9) the proof is complete.

THEOREM 3.3. *If the operator D_n is defined by (3) then*

$$\lim_{n\to\infty} \|D_n f - f\| = 0, \quad \text{for every} \quad f \in C[0,a],$$

where $\|\cdot\|$ is the uniform norm.

Proof. Applying the hypothesis 3°, Lemma 3.1 and Lemma 3.2 guarantee $\lim\limits_{n\to\infty} D_n e_i = e_i$, $i = 1, 2$. This fact together with (5) lead us to the desired result in concordance with the well-known theorem of Bohman-Korovkin.

In order to investigate other properties of our operators we recall that the first, respectively second, modulus of smoothness of a function $f \in C(K)$, (K a compact interval of the real axis) are given for $h \geqslant 0$ by

$$\omega_1(f;h) = \sup\{|f(x+\delta) - f(x)| : x, x+\delta \in K, \ 0 \leqslant \delta \leqslant h\},$$

$$\omega_2(f;h) = \sup\{|f(x-\delta) - 2f(x) + f(x+\delta)| : x, x \pm h \in K, \ 0 \leqslant \delta \leqslant h\}.$$

Let $B(K)$ denote the Banach space of bounded and real-valued functions on K. Our result requires the following proposition due to H. H. Gonska [9].

THEOREM 3.4. *If $K = [a,b]$ and $L : C(K) \to B(K)$ is a positive linear operator, then for $f \in C(K)$, $x \in K$ and each $h > 0$ the following holds*

$$|L(f,x) - f(x)| \leqslant \left[\frac{3}{2}(\|L\| + 1) + L((e_1 - x)^2; x)\max\{h^{-2}, (b-a)^{-2}\}\right]\omega_2(f;h) +$$

$$+2|L(e_1 - x; x)\max\{h^{-1}, (b-a)^{-1}\}\omega_1(f;h) + |L(e_0;x) - 1|[\|f\|_K + \omega_1(f;h)].$$

Here the moduli of smoothness are taken over K.

We recall that if the operator L maps an element $f \in C(K)$ into an element $g \in B(K)$ we can denote this by $g(x) = L(f,x) = (Lf)(x)$, $x \in K$.

In our case $K = [0,a]$ and $D_n(C[0,a]) \subset C[0,a]$. The relation (5) implies $\|D_n\| = \sup\limits_{\|f\| \leqslant 1} \|D_n f\| = 1$. If we set $\mu_{n,s}(x) := D_n((e_1 - xe_0)^s; x)$ the s-th order central moment of our operator then Lemma 3.1 respectively Lemma 3.2 lead us to the following relations

$$|\mu_{n,1}(x)| = |D_n(e_1 - xe_0; x)| \leqslant x\left(\frac{\alpha_n(n+c)}{n} - 1\right);$$

$$\mu_{n,2}(x) = D_n((e_1 - xe_0)^2; x) = (D_n e_2)(x) - x^2 + 2x(x - (D_n e_1)(x)) \leqslant$$
$$\leqslant (D_n e_2)(x) - x^2 = x^2\left(\frac{\alpha_n^2(n+c)^2}{n^2} - 1\right) + \frac{x\alpha_n(n+c)}{n^2}.$$

Choosing

$$h_n = \left(\frac{\alpha_n(n+c)}{n} - 1\right)^{1/2}, \tag{10}$$

we have obtained

$$|\mu_{n,1}(x)| \leqslant xh_n^2 \quad \text{and} \quad \mu_{n,2}(x) \leqslant x^2 h_n^2(h_n^2+2) + \frac{x}{n}(h_n^2+1). \tag{11}$$

The relation (2) guarantees $h_n = \mathscr{O}\left(\frac{1}{\sqrt{n}}\right)$ and for sufficiently large n we have $h_n \leqslant a$ in other words $\max\{h_n^{-j}, a^{-j}\} = h_n^{-j}$, $j = 1, 2$.

By using Theorem 3.4 and the above relations it results

THEOREM 3.5. *Let* (D_n) *be given by (3). The following property*

$$|(D_n f)(x) - f(x)| \leqslant (3 + x^2(2 + h_n^2) + x(1 + h_n^{-2})n^{-1})\omega_2(f; h_n) + 2xh_n\omega_1(f; h_n)$$

holds, where h_n *is defined by (10).*

In [8] is established the inequality

$$\omega_1(f;\delta) \leqslant \left(3 + \frac{I(K)}{\delta}\right)\omega_2(f;\delta) + \frac{6\delta}{I(K)}\|f\|, \quad 0 < \delta \leqslant I(K), \quad f \in C(K),$$

where $I(K)$ represents the length of the interval compact K.

By using this fact and knowing that $0 \leqslant x \leqslant a < 1$ Theorem 3.5 implies the following

COROLLARY 3.6. *Let* (D_n) *be given by (3). The following property*

$$|(D_n f)(x) - f(x)| \leqslant \left(3 + a^2(4 + h_n^2) + a\left(\frac{h_n^2+1}{nh_n} + 6h_n\right)\right)\omega_2(f; h_n) + 12h_n^2\|f\|$$

holds, where h_n *is defined by (10).*

Also, we can establish a quantitative estimation only in terms of the first modulus. To do this, from (11) we notice that $|\mu_{n,1}(x)| \leqslant xh_n$ and $\mu_{n,2}(x) \leqslant 3h_n^2x^2 + \dfrac{2x}{n}$. Now we apply some classical results concerning the linear and positive operators which reproduce the monomial e_0, see for example F. Altomare [3], Theorem 5.1.2. After a few calculations we get

THEOREM 3.7. *Let* (D_n) *be given by (3) and* $f \in C[0,a]$. *We have*
(i) $|(D_n f)(x) - f(x)| \leqslant 2\omega_1(f; \delta_n)$,
(ii) If f *is differentiable on* $[0,a]$ *and* $f' \in C[0,a]$ *then*

$$|(D_n f)(x) - f(x)| \leqslant h_n x|f'(x)| + 2\delta_n\omega_1(f'; \delta_n),$$

where $x \in [0,a]$, $\delta_n = \left(3h_n^2x^2 + \dfrac{2x}{n}\right)^{1/2}$ *and* h_n *is defined by (10).*

REMARKS. In the particular case $D_n \equiv M_n$ presented in the previous paragraph the results stated by (5) and Lemma 3.1 mean the well-known relations $(M_n e_0)(x) = 1$, $(M_n e_1)(x) = x$. Because $\gamma_{n,k} = 0$ in (4) we can choose any constant $c > 0$ and Lemma 3.2 implies

$$0 \leqslant (M_n e_2)(x) - x^2 \leqslant cx \left(\frac{2x + c^{-1}}{n} + \frac{cx + 1}{n^2} \right).$$

In time, many similar bounds for the numbers $(M_n e_2)(x) - x^2$ have been obtained, such as

$$(M_n e_2)(x) - x^2 \leqslant \frac{x(1-x)^2}{n+1} + \frac{x^2(1-x)(2-x)}{(n+1)^2} \quad \text{(Sikkema [15], 1970)}$$

$$(M_n e_2)(x) - x^2 \leqslant \frac{x(1-x)^2}{n+1} \left(1 + \frac{2x}{n+1} \right) \quad \text{(Becker-Nessel [4], 1978)}$$

In 1984 Alkemade [2] was the first who succeeded in deriving an explicit expression for the second moment in terms of a hypergeometric series. In this special case from (10) we obtain $h_n = \sqrt{c/n}$. Starting from Theorem 3.5 and Theorem 3.7 we are led to the following properties verified by M_n for any constant $c > 0$:

$$|(M_n f)(x) - f(x)| \leqslant \left(3 + \left(2 + \frac{c}{n} \right) x^2 + \left(\frac{1}{c} + \frac{1}{n} \right) x \right) \omega_2 \left(f; \sqrt{\frac{c}{n}} \right) + 2x\sqrt{\frac{c}{n}}\, \omega_1 \left(f; \sqrt{\frac{c}{n}} \right),$$

$$|(M_n f)(x) - f(x)| \leqslant 2\omega_1 \left(f; \sqrt{\frac{3cx^2 + 2x}{n}} \right).$$

Returning to D_n we will present another property. Let $f : \Omega \to \mathbb{R}$ be an arbitrary function where Ω is an interval of the form $[a, b]$ or $[a, \infty)$. If $\xi^{(n)}(\xi_0, \xi_1, \ldots, \xi_n)$ is a system of points in Ω such that $\xi_0 < \xi_1 < \cdots < \xi_n$ then $V[f, \xi^{(n)}]$ denotes the number of changes of sign in the finite sequence of ordinates $f(\xi_k)$, where zeroes are disregarded. $V_\Omega[f]$ stands for the number of changes of sign of f in the domain Ω and is defined as follows $V_\Omega[f] := \sup V[f, \xi^{(n)}]$ where the supremum is taken for all finite systems $\xi^{(n)}$. Let $\mathscr{S}$ be a set of real functions defined on Ω. We consider an operator Λ transforming any $f \in \mathscr{S}$ in a Λf function defined on another interval K of the real axis. According to I.J. Schoenberg [14] we say that the Λ operator is a variation diminishing operator if

$$V_K[\Lambda f] \leqslant V_\Omega[f] \quad \text{for each} \quad f \in \mathscr{S}.$$

THEOREM 3.8. *The operator D_n defined by (3) is a variation diminishing operator, that is*

$$V_{[0,a']}[D_n f] \leqslant V_{[0,a]}[f], \quad f \in C[0, a],$$

where $0 \leqslant a' \leqslant a$.

Proof. If $V_{[0,a]}[f]$ is not finite, then obviously our statement holds. Assume now that $V_{[0,a]}[f]$ is finite and that $f \neq 0$. Further we set by $V[\{\alpha_k\},\ k=1,2,\dots]$ the number of sign changes of the sequence $\alpha_1,\alpha_2,\dots$, where zeroes are disregarded. Because $w_{n,k}>0$, $x\in[0,a]$, we can write

$$V_{[0,a']}[D_nf] \leqslant V\left[\left\{f\left(\frac{k}{n+k+\beta_k}\right)\right\},\ k=0,1,2,\dots\right] \leqslant V_{[0,a]}[f]$$

and this completes the proof.

From (3) it is clear that we can extend (D_n) over all measurable and bounded functions f on $[0,1]$. We will characterize the local convergence for the positive linear operator D_n by the elements of the Lipschitz class $Lip\alpha$. Here the local behaviour of a function will be measured by the Lipschitz-type maximal function of order α introduced by B. Lenze [11] as

$$\widetilde{\omega}_\alpha(f,x) := \sup_{t\neq x, t\in[0,a]} \frac{|f(x)-f(t)|}{|x-t|^\alpha}, \quad x\in[0,a], \quad \alpha\in(0,1]. \tag{12}$$

This function is homogeneous and subadditive. The finiteness of $\widetilde{\omega}_\alpha(f,\cdot)$ gives a local control for the smoothness of f. Boundedness of $\widetilde{\omega}_\alpha(f,\cdot)$ is roughly speaking equivalent to $f\in Lip\alpha$ on $[0,1]$. We have the following local direct estimate.

THEOREM 3.9. *Let $\alpha\in(0,1]$ and $f:[0,1]\to\mathbb{R}$ be measurable and bounded. Then for all $x\in[0,a]$ we have*

$$|f(x)-(D_nf)(x)| \leqslant x^{\alpha/2}\left(\frac{h_n^\alpha+1}{n^{\alpha/2}} + x^{\alpha/2}h_n^\alpha(h_n^\alpha+2^{\alpha/2})\right)\widetilde{\omega}_\alpha(f,x),$$

where h_n is defined by (10).

Proof. From (12) we have for all $x\in[0,a]$ and $k\geqslant 0$ integer the following inequality

$$\left|f(x)-f\left(\frac{k}{k+n+\beta_k}\right)\right| \leqslant \widetilde{\omega}_\alpha(f,x)\left|x-\frac{k}{k+n+\beta_k}\right|^\alpha$$

and we obtain

$$|f(x)-(D_nf)(x)| = \left|\sum_{k=0}^\infty w_{n,k}(x)\left(f(x)-f\left(\frac{k}{k+n+\beta_k}\right)\right)\right| \leqslant$$

$$\leqslant \widetilde{\omega}_\alpha(f,x)\sum_{k=0}^\infty w_{n,k}(x)\left|x-\frac{k}{k+n+\beta_k}\right|^\alpha.$$

Applying Hölder's inequality with $r:=2/\alpha$ and $1/s:=1-1/r$ we have

$$|f(x)-(D_nf)(x)| \leqslant \widetilde{\omega}_\alpha(f,x)\sum_{k=0}^\infty w_{n,k}(x)\left|x-\frac{k}{k+n+\beta_k}\right|^\alpha \leqslant$$

$$\leqslant \widetilde{\omega}_\alpha(f,x)\left(\sum_{k=0}^{\infty} w_{n,k}(x)\left(x-\frac{k}{k+n+\beta_k}\right)^2\right)^{\frac{\alpha}{2}}\left(\sum_{k=0}^{\infty} w_{n,k}(x)\right)^{1-\frac{\alpha}{2}}$$

$$= \widetilde{\omega}_\alpha(f,x)\mu_{n,2}^{\alpha/2}(x).$$

Using both the relation (11) and the known inequality $(A+B)^\tau \leqslant A^\tau + B^\tau$, $\tau \in (0,1]$, $A \geqslant 0$, $B \geqslant 0$, our assertion follows.

REFERENCES

[1] U. ABEL, *The complete asymptotic expansion for the Meyer-König and Zeller Operators*, Journal of Math. Analysis and Applications **208**(1997), 109–119.
[2] J. A. H. ALKEMADE, *The second moment for the Meyer-König and Zeller operators*, J. Approx. Theory, **40**(1984), 261–273.
[3] F. ALTOMARE AND M. CAMPITI, *Korovkin-Type Approximation Theory and Its Applications*, de Gruyter Series Studies in Mathematics, Vol.17, Walter de Gruyter, Berlin-New York, 1994.
[4] M. BECKER AND J. NESSEL, *A global approximation theorem for Meyer-König and Zeller operators*, Math. Z., **160**(1978), 195–206.
[5] WENZHONG CHEN, *On the integral type Meyer-König and Zeller operators*, Approx. Theory and Its Applications, **2**(1986), 3, 7–18.
[6] E. W. CHENEY AND A. SHARMA, *Bernstein power series*, Canad. J. Math., **16**(1964), 241–253.
[7] OGÜN DǑGRU, *Approximation order and asymptotic approximation for generalized Meyer-König and Zeller operators*, Mathematica Balkanica, New Series vol. **12**(1998), 3–4, 359–367.
[8] I. GAVREA AND I. RAŞA, *Remarks on some quantitative Korovkin-type results*, Revue d'Analyse Num. et de Theorie de l'Approx., **22**(1993), 2, 173–176.
[9] H. H. GONSKA, *Quantitative Korovkin type theorems on simultaneous approximation*, Math. Z., **186**(1984), 419–433.
[10] XU JIHUA AND LI LUOQING, *Converse theorems on approximation by integral type Meyer-König and Zeller Operators*, in "Progress in Approximation Theory" (P. Nevai and A. Pinkus Eds.) Academic Press, Boston, 1991, 899–911.
[11] B. LENZE, *On Lipschitz-type maximal functions and their smoothness spaces*, Proc. Netherland Acad. Sci. A **91**(1988), 53–63.
[12] A. LUPAŞ AND M. W. MÜLLER, *Approximation properties of the M_n-operators*, Aequationes Math., **5**(1970), 19–37.
[13] M. W. MÜLLER, *L_p-approximation by the method of integral Meyer-König and Zeller operators*, Studia Math., **63**(1978), 81–88.
[14] I. J. SCHOENBERG, *On variation diminishing approximation methods*, In: On Numerical Approximation (Proc. Sympos. Conducted by the MRC; ed. R. E. Langer), Madison, Univ. of Wisconsin Press 1959, 249–274.
[15] P. C. SIKKEMA, *On some linear positive operators*, Indag. Math., **32**(1970), 327–337.
[16] V. TOTIK, *Approximation by Meyer-König and Zeller type operators*, Math. Z., **182**(1983), 425–446.

(Received November 26, 1999)

Octavian Agratini
Faculty of Mathematics and Informatics
Babeş-Bolyai University
Kogălniceanu, 1
3400 Cluj-Napoca
ROMANIA
e-mail: agratini@math.ubbcluj.ro