Mathematical
Inequalities
& Applications
Volume 4, Number 1 (2001), 143–150

WEYL'S THEOREM FOR CLASS A OPERATORS

ATSUSHI UCHIYAMA *

(*communicated by T. Furuta*)

Abstract. In this paper, we show that Weyl's theorem holds for class A operators under a certain condition. We also show that a class A operator whose Weyl spectrum equals to the one-point set $\{0\}$ is always compact and normal.

1. Introduction

We denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. We say that an operator $T \in \mathcal{B}(\mathcal{H})$ belongs to the class A if $|T^2| \geqslant |T|^2$. Class A was first introduced by Furuta-Ito-Yamazaki [5] as a subclass of paranormal operators which includes the classes of p-hyponormal and log-hyponormal operators. The following Theorem A is one of the results associated with class A.

THEOREM A ([5]).
(1) Every log-hyponormal operator is a class A operator.
(2) Every class A operator is a paranormal operator.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Fredholm operator if $T\mathcal{H}$ is closed and both $\mathrm{Ker}T = \{x \in \mathcal{H} : Tx = 0\}$ and $\mathrm{Ker}T^*$ are finite-dimensional. For any Fredholm operator T there corresponds an integer $\mathrm{ind}(T) = \dim\mathrm{Ker}T - \dim\mathrm{Ker}T^*$, which is called the index of T. Let $\mathcal{F}_0$ denote the class of all Fredholm operators in $\mathcal{B}(\mathcal{H})$ with index 0. Then $w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{F}_0\}$ is called the Weyl spectrum of T. It is known that, for $T \in \mathcal{B}(\mathcal{H})$, $w(T)$ is non-empty and $w(T) = \bigcap_{K \in \mathcal{C}(\mathcal{H})} \sigma(T+K)$, where $\sigma(T)$ and $\mathcal{C}(\mathcal{H})$ denote the spectrum of T and the set of all compact operators in $\mathcal{B}(\mathcal{H})$, respectively.

For $T \in \mathcal{B}(\mathcal{H})$, let $\sigma_p(T)$ and $\pi_{00}(T)$ denote the point spectrum and the set of all isolated eigenvalues of finite multiplicity of T, respectively. According to Coburn [3], we say that Weyl's theorem holds for T if $\sigma(T) \setminus w(T) = \pi_{00}(T)$. He showed that Weyl's theorem holds for hyponormal operators and this result was generalized for p-hyponormal operators by Chō-Itoh-Ōshiro [4] and Stampfli [7] proved that if T

Mathematics subject classification (2000): 47A53, 47B20.

Key words and phrases: Class A operators, w-hyponormal operators, continuity of spectra, Weyl's theorem.

* Research Fellow of the Japan Society for Promotion of Science.

Paper MIA-04-11

is hyponormal and $w(T) = \{0\}$, then T is compact and normal. In this paper, we shall prove that Weyl's theorem holds for class A operators which satisfy the condition $\mathrm{Ker}T|_{[T\mathcal{H}]} = \{0\}$ and Stampfli's result above also holds for class A operators. Here we denote the norm closure of a subspace $\mathcal{M} \subseteq \mathcal{H}$ by $[\mathcal{M}]$.

2. Preliminaries

DEFINITION 1. If $\|Tx\|^2 \leqslant \|T^2x\|\ \|x\|$ for all $x \in \mathcal{H}$, then we say that T is paranormal.

The following results are well known.

PROPOSITION 1. *If T is paranormal, then $\|T\| = \sup\{|\lambda|\ ;\ \lambda \in \sigma(T)\}$.*

PROPOSITION 2 (HANSEN'S INEQUALITY [6]). *If $A \geqslant 0$ and $\|B\| \leqslant 1$, then $(B^*AB)^\delta \geqslant B^*A^\delta B$ for all $\delta \in (0,1]$.*

PROPOSITION 3.. *If T is an invertible paranormal operator, then T^{-1} is also paranormal.*

3. Main theorems

LEMMA 1. *If T is a class A operator and $\mathcal{M}$ is an invariant subspace of T, then $T|_{\mathcal{M}}$ is also a class A operator.*

Proof. Let

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \qquad \text{on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp,$$

and P be the projection onto $\mathcal{M}$. Then we have $P\{(T^{*2}T^2)^{\frac{1}{2}} - (T^*T)\}P \geqslant 0$. Hence, we see that $A^*A = P(T^*T)P \leqslant P(T^{*2}T^2)^{\frac{1}{2}}P \leqslant (PT^{*2}T^2P)^{\frac{1}{2}} = (A^{*2}A^2)^{\frac{1}{2}}$ by Hansen's inequality. This implies that A belongs to the class A and the proof is complete.

LEMMA 2 [5]. *If T belongs to the class A, then T is paranormal.*

COROLLARY 1. *If T belongs to the class A and $\sigma(T) = \{0\}$, then $T = 0$.*

Proof. By Lemma 2 and Proposition 1, we have the conclusion.

LEMMA 3. *If T is paranormal, then the restriction $T|_{\mathcal{M}}$ to its invariant subspace $\mathcal{M}$ is also paranormal.*

Proof. Let $x \in \mathcal{M}$ be an arbitrary vector. Then we have,

$$\|T|_{\mathcal{M}}x\|^2 = \|Tx\|^2 \leqslant \|T^2x\|\|x\| = \|(T|_{\mathcal{M}})^2x\|\|x\|.$$

This implies that $T|_{\mathcal{M}}$ is paranormal.

DEFINITION 2. An operator T is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T.

THEOREM 1. *If T is paranormal, then T is isoloid.*

Proof. Let $\lambda \in \sigma(T)$ be an isolated point, then the range of Riesz projection $E = \frac{1}{2\pi i}\int_{\partial D}(zI - T)^{-1}\,dz$ is an invariant closed subspace of T and $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$, where D is a closed disk with its center λ such that $\sigma(T)\cap D = \{\lambda\}$.

If $\lambda = 0$, then $\sigma(T|_{E\mathcal{H}}) = \{0\}$. Since $T|_{E\mathcal{H}}$ is paranormal by Lemma 3, $T|_{E\mathcal{H}} = 0$ by Proposition 1. Therefore 0 is an eigenvalue of T.

If $\lambda \neq 0$, then $T|_{E\mathcal{H}}$ is an invertible paranormal operator and hence $(T|_{E\mathcal{H}})^{-1}$ is also paranormal by Proposition 3. By Proposition 1, we see $\|T|_{E\mathcal{H}}\| = |\lambda|$ and $\|(T|_{E\mathcal{H}})^{-1}\| = \frac{1}{|\lambda|}$. Let $x \in E\mathcal{H}$ be an arbitrary vector. Then $\|x\| \leqslant \|(T|_{E\mathcal{H}})^{-1}\|\|T|_{E\mathcal{H}}x\| = \frac{1}{|\lambda|}\|T|_{E\mathcal{H}}x\| \leqslant \frac{1}{|\lambda|}|\lambda|\|x\| = \|x\|$. This implies that $\frac{1}{\lambda}T|_{E\mathcal{H}}$ is unitary with its spectrum $\sigma(\frac{1}{\lambda}T|_{E\mathcal{H}}) = \{1\}$. Hence $T|_{E\mathcal{H}} = \lambda$ and λ is an eigenvalue of T. This completes the proof.

LEMMA 4. *If T belongs to the class A and λ is a non-zero complex number, then $(T-\lambda)x = 0$ implies that $(T-\lambda)^*x = 0$.*

Proof. We may assume $x \neq 0$. Since $\||T^2|x\| = \|T^2x\| = |\lambda|^2\|x\|$, we have

$$\begin{aligned}|\lambda|^2\|x\|^2 = \|Tx\|^2 &= \langle T^*Tx, x\rangle \\ &\leqslant \langle |T^2|x, x\rangle \quad \text{(since } T \text{ belongs to the class} A) \\ &\leqslant \||T^2|x\|\|x\| \quad \text{(Cauchy-Schwarz inequality)} \\ &= |\lambda|^2\|x\|^2.\end{aligned}$$

Since $|T^2|x$ and x are linearly dependent, we have $|T^2|x = |\lambda|^2x$. Since

$$\|(|T^2| - T^*T)^{\frac{1}{2}}x\|^2 = \langle |T^2|x, x\rangle - \langle T^*Tx, x\rangle = 0,$$

we have $T^*Tx = |T^2|x = |\lambda|^2x$ and therefore $T^*x = \overline{\lambda}x$. This completes the proof.

THEOREM 2. *If T belongs to the class A and* $\mathrm{Ker}T|_{[T\mathcal{H}]} = \{0\}$, *then Weyl's theorem holds for T.*

Proof. First, we shall prove $\sigma(T)\setminus w(T) \subseteq \pi_{00}(T)$.

Let

$$T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} \qquad \text{on } \mathcal{H} = [T\mathcal{H}] \oplus \mathrm{Ker}T^*$$

be a class A operator such that $\mathrm{Ker}A = \{0\}$ and $\lambda \in \sigma(T)\setminus w(T)$. Then $T-\lambda \in \mathcal{F}_0$. This means that $0 < \dim\mathrm{Ker}(T-\lambda) = \dim\mathrm{Ker}(T-\lambda)^* < \infty$ and $(T-\lambda)\mathcal{H}$ is closed. Therefore, it suffices to show that λ is an isolated point of $\sigma(T)$.

If $\lambda \neq 0$, we have $\mathrm{Ker}(T-\lambda) = \mathrm{Ker}(T-\lambda)^*$ by Lemma 4 and it is a reducing subspace of T. Let E be the orthogonal projection onto $\mathrm{Ker}(T-\lambda)$. Then $T = \lambda E \oplus T(1-E)$ on $E\mathcal{H} \oplus (E\mathcal{H})^\perp$ and $\sigma(T) = \{\lambda\} \cup \sigma(T(1-E)|_{(E\mathcal{H})^\perp})$. Since E is a finite rank projection, $\mathrm{ind}(T(1-E) - \lambda(1-E)) = \mathrm{ind}(T-\lambda) = 0$ and since $(T(1-E)-\lambda(1-E))|_{(E\mathcal{H})^\perp}$ is one-to-one, $(T(1-E)-\lambda(1-E))|_{(E\mathcal{H})^\perp}$ is invertible. This implies that $\lambda \notin \sigma(T(1-E)|_{(E\mathcal{H})^\perp})$ and λ is an isolated point of $\sigma(T)$. Hence $\lambda \in \pi_{00}(T)$.

If $\lambda = 0$, then $A \in \mathcal{F}_0$ because S is a finite rank operator and since A is one-to-one A is invertible. Hence $0 \in \pi_{00}(T)$ because $\sigma(T) \subseteq \sigma(A) \cup \{0\}$.

Next, we shall show that $\pi_{00}(T) \subseteq \sigma(T) \setminus w(T)$.

If $\lambda \in \pi_{00}(T) \setminus \{0\}$, then λ is a normal eigenvalue of T by Lemma 4. Hence $T = \lambda \oplus T'$ on $\mathcal{H} = \mathrm{Ker}(T-\lambda) \oplus [\mathrm{Ker}(T-\lambda)]^{\perp}$ and the isolatedness of $\lambda \in \sigma(T)$ implies either λ is an isolated point of $\sigma(T')$ or $\lambda \notin \sigma(T')$. Since T' is paranormal (hence T' is isoloid) with $\mathrm{Ker}(T'-\lambda) = \{0\}$, λ is not an isolated point of $\sigma(T')$ and therefore $T'-\lambda$ is invertible. $T-\lambda \in \mathcal{F}_0$ is immediately from this. Hence we have $\lambda \in \sigma(T) \setminus w(T)$.

If $0 \in \pi_{00}(T)$, then 0 is an isolated point of $\sigma(A)$ or A is invertible. Since A is also a class A operator (and hence isoloid) with $\mathrm{Ker}A = \{0\}$, 0 is not an isolated point of $\sigma(A)$ and therefore A is invertible. It is easy to see that $\mathrm{Ker}T = \{-A^{-1}Su \oplus u\,;\, u \in \mathrm{Ker}T^*\}$ and hence $\dim\mathrm{Ker}T = \dim\mathrm{Ker}T^* < \infty$. The closedness of the range of T follows from the invertibility of A. Hence $0 \in \sigma(T) \setminus w(T)$ and this completes the proof.

REMARK. There are two conditions which are usually assumed by many mathematicians who study paranormal operators. The conditions are following:

1) $T\mathcal{H} \subseteq T^*\mathcal{H}$.

2) $\mathrm{Ker}T \subseteq \mathrm{Ker}T^*$.

We give a following condition:

3) $\mathrm{Ker}T|_{[T\mathcal{H}]} = \{0\}$.

3) is weaker than 1) and 2); indeed, 1) $\Rightarrow$ 2) $\Rightarrow$ 3).

The author showed an example of quasihyponormal operator (hence it is class A operator) which does not satisfy condition 2) and also showed that every p-quasihyponormal operator for $0 < p < 1$ satisfies condition 3). See [8, 9]. Hence we see that if we deal with paranormal operators, then conditions 1) and 2) are rather strong. So we use condition 3) in this paper.

In addition, it is unknown whether paranormal operators satisfy condition 3).

THEOREM 3. *If T belongs to the class A and $w(T) = \{0\}$, then T is compact and normal.*

Proof. Since Weyl's theorem holds for T by Theorem 2 and $w(T) = \{0\}$ by the assumption and Lemma 4, every non-zero spectrum of T is an isolated normal eigenvalue with finite dimensional eigenspace, which reduces T. Hence $\sigma(T) \setminus w(T)$ is a finite set or a countable infinity set whose accumulation point is only 0. Let $\sigma(T) \setminus w(T) = \{\lambda_n\}$ with $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots \geqslant 0$ and let E_n be the orthogonal projection onto $\mathrm{Ker}(T-\lambda_n)$. Then $TE_n = E_nT = \lambda_nE_n$ and $E_nE_m = 0$ if $n \neq m$. Put $E = \bigoplus_n E_n$. Then $T = \bigoplus_n \lambda_nE_n \oplus T|_{(1-E)\mathcal{H}}$ and $\sigma(T|_{(1-E)\mathcal{H}}) = \{0\}$. Since $T|_{(1-E)\mathcal{H}}$ also belongs to the class A because $E\mathcal{H}$ is a reducing subspace of T, $T|_{(1-E)\mathcal{H}} = 0$ by Corollary 1.

Hence $T = \bigoplus_n \lambda_nE_n$ is normal. The compactness of T follows from the finiteness or the countability of $\{\lambda_n\}_n$ satisfying $|\lambda_n| \downarrow 0$ and each E_n is a finite rank projection.

THEOREM 4. *If T belongs to the class A with* $\mathrm{Ker}T|_{[T\mathcal{H}]} = \{0\}$ *and f is a polynomial, then Weyl's theorem holds for $f(T)$.*

Proof. We have only to prove that f is non-constant. Let $\lambda \in \sigma(f(T)) \setminus w(f(T))$ be an arbitrary point and let $\{\mu_1, \dots, \mu_n\}$ be the set of all solutions of the equality $f(z) - \lambda = 0$. Then $\mathrm{Ker}(T - \mu_k) \subseteq \mathrm{Ker}(f(T) - \lambda)$ and $\mathrm{Ker}(T - \mu_k)^* \subseteq \mathrm{Ker}(f(T) - \lambda)^*$ and both $\mathrm{Ker}(T - \mu_k)$ and $\mathrm{Ker}(T - \mu_k)^*$ are finite dimensional for all $1 \leqslant k \leqslant n$. Since

$$\begin{aligned}(T - \mu_k)\mathcal{H} &= (T - \mu_k)[\Pi_{i \neq k}(T - \mu_i)\mathcal{H}] + (T - \mu_k)\mathrm{Ker}(\Pi_{i \neq k}(T - \mu_i)^*) \\ &\subseteq [\Pi_{i=1}^{n}(T - \mu_i)\mathcal{H}] + (T - \mu_k)\mathrm{Ker}(\Pi_{i \neq k}(T - \mu_i)^*) \\ &= [(f(T) - \lambda)\mathcal{H}] + (T - \mu_k)\mathrm{Ker}(\Pi_{i \neq k}(T - \mu_i)^*) \\ &= (f(T) - \lambda)\mathcal{H} + (T - \mu_k)\mathrm{Ker}(\Pi_{i \neq k}(T - \mu_i)^*) \\ &\subseteq (T - \mu_k)\mathcal{H},\end{aligned}$$

we have $(T - \mu_k)\mathcal{H} = (f(T) - \lambda)\mathcal{H} + (T - \mu_k)\mathrm{Ker}(\Pi_{i \neq k}(T - \mu_i)^*)$ and it is closed because $(f(T) - \lambda)\mathcal{H}$ is closed and $(T - \mu_k)\mathrm{Ker}(\Pi_{i \neq k}(T - \mu_i)^*)$ is a finite dimensional subspace. Hence $T - \mu_k$ is a Fredholm operator for every k.

Next, we show that $T - \mu_k \in \mathcal{F}_0$. We may assume that $T - \mu_k$ is not invertible for all k since the assertion is trivial in case that $T - \mu_k$ is invertible. Since $\sum_{i=1}^{n} \mathrm{ind}(T - \mu_i) = \mathrm{ind}(f(T) - \lambda) = 0$, there exists μ_{k_1} such that $\mathrm{ind}(T - \mu_{k_1}) \geqslant 0$. By Lemma 4, if $\mu_{k_1} \neq 0$, then μ_{k_1} is a normal eigenvalue of T and therefore $0 \leqslant \mathrm{ind}(T - \mu_{k_1}) = \mathrm{ind}((T - \mu_{k_1})|_{[\mathrm{Ker}(T - \mu_{k_1})]^{\perp}}) \leqslant 0$. Hence we have $\mu_{k_1} \notin w(T)$. Also if $\mu_{k_1} = 0$, then $0 \leqslant \mathrm{ind}T = \mathrm{ind}A \leqslant 0$ because $\mathrm{Ker}A = \{0\}$ and S is a finite rank operator. So we obtain $\mu_{k_1} = 0 \notin w(T)$. Here A and S are same as in the proof of Theorem 2.

Similarly, since $\sum_{i \neq k_1} \mathrm{ind}(T - \mu_i) = \sum_{i=1}^{n} \mathrm{ind}(T - \mu_i) = 0$, there exists $k_2 \neq k_1$ such that $\mathrm{ind}(T - \mu_{k_2}) \geqslant 0$. By using the previous argument, $\mu_{k_2} \notin w(T)$. By induction, we have $\mu_k \notin w(T)$ for all k.

Since Weyl's theorem holds for T, $\emptyset \neq \{\mu_1, \dots, \mu_n\} \cap \sigma(T) \subseteq \sigma(T) \setminus w(T) = \pi_{00}(T)$ and from this fact it is immediately that $\lambda = f(\mu_k)$ is an isolated point of $\sigma(f(T))$. This implies that $\lambda \in \pi_{00}(f(T))$.

Conversely, let $\lambda \in \pi_{00}(f(T))$ and let $\{\mu_1, \dots, \mu_n\}$ be the set of all solutions of the equality $f(z) - \lambda = 0$. Then we have $\{\mu_1, \dots, \mu_n\} \cap \sigma(T) \subseteq \pi_{00}(T) = \sigma(T) \setminus w(T)$ and hence $T - \mu_k \in \mathcal{F}_0$ for all k. Since $f(T) - \lambda = \alpha \Pi_{i=1}^{n}(T - \mu_i)$ for some $\alpha \neq 0$, $f(T) - \lambda$ is Fredholm and $\mathrm{ind}(f(T) - \lambda) = \sum_{i=1}^{n} \mathrm{ind}(T - \mu_i) = 0$. Hence $\lambda \in \sigma(f(T)) \setminus w(f(T))$. This completes the proof.

In the first part of the above proof, we have shown that $\alpha \Pi_{i=1}^{n}(T - \mu_i) \in \mathcal{F}_0$, $\alpha \neq 0$ implies that $T - \mu_i \in \mathcal{F}_0$ for all i. Since the converse is also true, $\Pi_{i=1}^{n}(T - \mu_i) \in \mathcal{F}_0$ if and only if $T - \mu_i \in \mathcal{F}_0$ for all i. Hence, for a class A operator T with $\mathrm{Ker}T|_{[T\mathcal{H}]} = \{0\}$ and every polynomial f, we have

$$\begin{aligned}\lambda \in w(f(T)) &\Longleftrightarrow \Pi_{i=1}^{n}(T - \mu_i) \notin \mathcal{F}_0 \\ &\Longleftrightarrow T - \mu_i \notin \mathcal{F}_0 \text{ for some } i \\ &\Longleftrightarrow \lambda \in f(w(T)),\end{aligned}$$

where $f(z)-\lambda = \alpha\Pi_{i=1}^{n}(z-\mu_i)$ for some $\alpha \neq 0$. Many mathematicians have shown that $w(f(T)) = f(w(T))$ for every polynomial f implies that $w(f(T)) = f(w(T))$ for every function f which is analytic on a neighborhood of $\sigma(T)$. That assertion and proofs may be well-known, however, for the sake of completes in this paper we will give a proof.

LEMMA 5. *If T satisfies $w(f(T)) = f(w(T))$ for every polynomial f, then $w(h(T)) = h(w(T))$ for every function h of the form $h(z) = f(z)/g(z)$ for some polynomials f and g such that $g(T)^{-1}$ exists.*

Proof. Let f and g be polynomials such that $g(T)^{-1}$ exists and let $h(z) = f(z)/g(z)$. Then we have that $\lambda \in w(h(T))$ is equivalent to $0 \in w(f(T) - \lambda g(T))$. By using above argument, this is also equivalent to $0 \in (f - \lambda g)(w(T))$. Since the last condition is clearly equivalent to $\lambda \in h(w(T))$. Hence the assertion holds.

DEFINITION 3. For a sequence $\{\delta_n\}_{n=1}^{\infty}$ of compact subsets of complex plane $\mathbb{C}$, we define $\limsup_{n\to\infty} \delta_n$ and $\liminf_{n\to\infty} \delta_n$ by

$$\limsup_{n\to\infty} \delta_n = \{\lambda \in \mathbb{C} : \liminf_{n\to\infty} d(\lambda, \delta_n) = 0\},$$
$$\liminf_{n\to\infty} \delta_n = \{\lambda \in \mathbb{C} : \lim_{n\to\infty} d(\lambda, \delta_n) = 0\}.$$

If $\limsup_{n\to\infty} \delta_n = \liminf_{n\to\infty} \delta_n$, then we say that the sequence $\{\delta_n\}_{n=1}^{\infty}$ is convergent and its limit set is given by $\lim_{n\to\infty} \delta_n = \limsup_{n\to\infty} \delta_n = \liminf_{n\to\infty} \delta_n$.

DEFINITION 4. Let A be an operator on a Hilbert space $\mathcal{H}$ and let $\varphi : \mathcal{B}(H) \to \{$ compact subsets of $\mathbb{C}$ $\}$ be a set-valued function. We say that φ is upper-semicontinuous at A if $\limsup_{n\to\infty} \varphi(A_n) \subseteq \varphi(A)$ whenever $\|A_n - A\| \to 0$. Also, we say that φ is lower-semicontinuous at A if $\varphi(A) \subseteq \liminf_{n\to\infty} \varphi(A_n)$ whenever $\|A_n - A\| \to 0$. If φ is upper and lower-semicontinuous at A, we say that φ is continuous at A.

THEOREM 5. *If T satisfies $w(f(T)) = f(w(T))$ for every polynomial f, then $w(h(T)) = h(w(T))$ for every function h which is analytic on a neighborhood of $\sigma(T)$.*

Proof. By the analytic functional calculus, there exist polynomials $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ such that $g_n(T)^{-1}$ exists for all n and $h_n(T) := (f_n/g_n)(T)$ uniformly converges to $h(T)$ as $n \to \infty$.

We shall show that $w(h(T)) = \lim_{n\to\infty} w(h_n(T))$. Since the upper-semicontinuity of the spectrum σ, the essential spectrum σ_e and the Weyl spectrum w are well-known, it suffices to show that $w(h(T)) \subset \liminf_{n\to\infty} w(h_n(T))$.

First, we consider the case that $\lambda \in \sigma_e(h(T))$. Since $\sigma_e(h(T)) \subseteq \sigma_e(h_n(T)) + \sigma_e(h(T) - h_n(T))$ is easily shown by the commutativity of $h(T)$ and $h_n(T)$, we have $d(\lambda, \sigma_e(h_n(T))) \leqslant \|h(T) - h_n(T)\| \to 0$. Hence we have $\lambda \in \liminf_{n\to\infty} \sigma_e(h_n(T)) \subseteq \liminf_{n\to\infty} w(h_n(T))$.

Next, we consider in case that $\lambda \in w(h(T)) \setminus \sigma_e(h(T))$. In this case, $h(T) - \lambda$ is a Fredholm operator with non-zero index.

Assume that $\lambda \notin \liminf_{n\to\infty} w(h_n(T))$. Then, for some $\epsilon > 0$, there exists a subsequence $\{h_{n_k}(T)\}_{k=1}^{\infty}$ such that $d(\lambda, w(h_{n_k}(T))) > \epsilon$ for every $k \geqslant 1$. Therefore $\{h_{n_k}(T) - \lambda\}$ is a sequence of Fredholm operators with the index 0 which converges to a Fredholm operator $h(T) - \lambda$. This contradicts the continuity of the index and we also have $\lambda \in \liminf_{n\to\infty} w(h_n(T))$ in this case.

By above argument, we have

$$\begin{aligned} w(h(T)) &= \lim_{n\to\infty} w(h_n(T)) \\ &= \lim_{n\to\infty} h_n(w(T)) \ \text{ (by Lemma 5)} \\ &= h(w(T)) \ \text{ (since } h_n \to h \text{ uniformly on } \sigma(T)), \end{aligned}$$

and hence the assertion holds.

THEOREM 6. *If T belongs to the class A with* $\operatorname{Ker} T|_{[T\mathcal{H}]} = \{0\}$ *and f is an analytic function on a neighborhood of $\sigma(T)$, then Weyl's theorem holds for $f(T)$.*

Proof. Let $\lambda \in \sigma(f(T)) \setminus w(f(T))$ be an arbitrary point. We shall show that $\lambda \in \pi_{00}(f(T))$. Since $0 < \dim\operatorname{Ker}(T-\lambda) < \infty$ by assumption, it suffices to show that $\lambda \in \sigma(f(T))$ is isolated. By Theorem 5 we have $\lambda \in \sigma(f(T)) \setminus f(w(T)) = f(\sigma(T)) \setminus f(w(T))$, hence $\{z \in \sigma(T) : f(z) = \lambda\} \subseteq \sigma(T) \setminus w(T) = \pi_{00}(T)$ because Weyl's theorem holds for T. Isolatedness of $\lambda \in \sigma(f(T))$ is immediately from this.

Next, we show the converse. Let $\lambda \in \pi_{00}(f(T))$ be an arbitrary point. Isolatedness of $\lambda \in \sigma(f(T))$ implies that every point in $\{z \in \sigma(T) : f(z) = \lambda\}$ is isolated in $\sigma(T)$ and therefore is in $\pi_{00}(T) = \sigma(T) \setminus w(T)$ since T is isoloid. Put $\{z \in \sigma(T) : f(z) = \lambda\} = \{\mu_i\}_{i=1}^{n}$. Then $f(z) - \lambda = \{\Pi_{i=1}^{n}(z-\mu_i)\}g(z)$ for some analytic function on a neighborhood of $\sigma(T)$ with no zeros. Hence $T - \mu_i \in \mathcal{F}_0$ for every i and $g(T)$ is invertible. This implies that $f(T) - \lambda \in \mathcal{F}_0$. Hence $\lambda \in \sigma(f(T)) \setminus w(f(T))$. This completes the proof.

DEFINITION 5. We say that an operator T with the polar decomposition $T = U|T|$ is w-hyponormal if the Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ of T satisfies $|\widetilde{T}^*| \leqslant |T| \leqslant |\widetilde{T}|$. An operator T is called p-hyponormal if $(TT^*)^p \leqslant (T^*T)^p$. An invertible operator T is called log-hyponormal if $\log(TT^*) \leqslant \log(T^*T)$.

In [1], Aluthge and Wang showed the following theorems.

THEOREM B.
(1) If T is a p-hyponormal operator for $p > 0$, then T is w-hyponormal.
(2) If T is a log-hyponormal operator, then T is w-hyponormal.
(3) If T is a w-hyponormal operator, then $|T^2| \geqslant |T|^2$ and $|T^*|^2 \geqslant |T^{*2}|$ hold.

THEOREM C. *If T and T^* are w-hyponormal with* $\operatorname{Ker} T \subseteq \operatorname{Ker} T^*$, *then T is normal.*

Finally, we show that Theorem C holds without the kernel condition $\operatorname{Ker} T \subseteq \operatorname{Ker} T^*$.

THEOREM 7. *If T belongs to the class A and T^* is w-hyponormal, then T is normal.*

Proof. Since T^* is w-hyponormal, by (3) of Theorem B, T^* belongs to the class A and $|T^2| \leqslant |T|^2$. Hence $|T^2| = |T|^2$ because T belongs to the class A. Hence, we have $T^{*2}T^2 = |T^2|^2 = |T|^4 = (T^*T)^2$ and $P\{T^*T - TT^*\}P = 0$, where P is the orthogonal projection onto $[T\mathcal{H}]$. Let

$$T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix} \qquad \text{on } \mathcal{H} = [T\mathcal{H}] \oplus \mathrm{Ker}T^*$$

be a 2×2 matrix representation of T. Then $P\{T^*T - TT^*\}P = 0$ implies that $A^*A = AA^* + SS^*$. Since T^* belongs to the class A, we have

$$\begin{aligned} AA^* + SS^* = PTT^*P \leqslant & P|T^{*2}|P = (A^2A^{*2} + ASS^*A^*)^{\frac{1}{2}} \\ &= (A\{AA^* + SS^*\}A^*)^{\frac{1}{2}} = AA^* \\ &\quad (\text{since } AA^* + SS^* = A^*A). \end{aligned}$$

Hence $S = 0$ and A is normal. This implies that T is normal.

Acknowledgment. I am grateful to Professor Muneo Chō for his valuable comments and helpful suggestions.

REFERENCES

[1] A. ALUTHGE AND D. WANG, *An operator inequality which implies paranormality*, Math. Inequal. Appl. **2** (1999), 307–315.
[2] S. K. BERBERIAN, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279.
[3] L. A. COBURN, *Weyl's theorem for non-normal operators*, Michigan Math. J. **13** (1966), 285–288.
[4] M. CHŌ, M. ITOH AND S. OSHIRO, *Weyl's theorem holds for p-hyponormal operators*, Glasgow Math. J. **39** (1997), 217–220.
[5] T. FURUTA AND M. ITO AND T. YAMAZAKI, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Scientiae Mathematicae **1** (1998), 389–403.
[6] F. HANSEN, *An inequality*, Math. Ann. **246** (1980), 249–250.
[7] J. G. STAMPFLI, *Hyponormal operators*, Pacific J. Math. **12** (1962), 1453–1458.
[8] A. UCHIYAMA,, *Berger-Shaw's theorem for p-hyponormal operators*, Integral Equations and Operator Theory **33** (1999), 221–230.
[9] A. UCHIYAMA,, *Inequalities of Putnam and Berger-Shaw for p-quasihyponormal operators*, Integral Equations and Operator Theory **34** (1999), 91–106.
[10] D. XIA, *On the non-normal operators – semihyponormal operators*, Sci. Sinica. **23** (1980), 700–713.

(Received May 10, 2000)

Atsushi Uchiyama
Mathematical Institute
Tôhoku University
Sendai 980-8578, Japan
e-mail: uchiyama@math.tohoku.ac.jp