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# ON A RESULT CONCERNING A PROPERTY OF CLOSED MANIFOLDS

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(*communicated by Th. M. Rassias*)

*Abstract.* The main result is given in Theorem 2 and it shows that the inequality $\gamma(\widetilde{M}) \leqslant k\gamma(M) - 4(k-1)$ contained in Theorem 1 holds with equality for any closed smooth surface $M^2$ (orientable or not), where $\pi : \widetilde{M} \to M$ is a $k$-covering of $M$ and $\gamma(M)$ is the Morse-Smale characteristic of $M$.

## 1. Introduction

Let us consider $M^m$ be a smooth differentiable manifold which is closed (i.e. $M^m$ is compact and without boundary) and let $\Omega(M)$ be the set of all Morse functions defined on $M$. It is well known that $\Omega(M)$ is dense in $C^\infty(M)$ in the Whitney topology (see for instance [13, pp.213]), therefore one implicitely follows that the set $\Omega(M)$ is not empty. If $f \in \Omega(M)$ denote by $\mu_k(f)$ the number of all critical points of $f$ having their Morse index equal to $k$, where $0 \leqslant k \leqslant m$. Let $\mu(f)$ be the total number of critical points of $f$, i.e.

$$\mu(f) = \sum_{k=0}^{m} \mu_k(f) \tag{1}$$

that is the cardinal number of the critical set $C(f)$ of $f$.

Recall that the number defined by

$$\gamma(M) = \min\{\mu(f) : f \in \Omega(M)\} \tag{2}$$

is called the Morse-Smale characteristic of the manifold $M$. The number $\gamma(M)$ is intensively studied in the papers [1], [2], [3], [15] and for $m \geqslant 7$ it represents a simple homotopy invariant of the manifold $M^m$ (see the paper [8]). In the papers [4], [5] it is naturally considered in a class of invariants associated to a given family of smooth mappings.

In what follows let us consider $\pi : \widetilde{M} \to M$, a $k$-covering of the manifold $M$, where $k \geqslant 2$. If $f \in \Omega(M)$ is a Morse function on $M$, consider the mapping $h : \widetilde{M} \to \mathbf{R}$ given by $h = f \circ \pi$. Because $\pi$ is a locally diffeomorphism, it results that $h \in \Omega(\widetilde{M})$, $C(f) = \pi(C(h))$ and therefore $\mu(h) = k\mu(f)$. Then for any Morse

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function $f \in \Omega(M)$ the following inequality $\gamma(\widetilde{M}) \leqslant k\mu(f)$ holds. Taking into account the definition of Morse-Smale characteristic one obtains

$$\gamma(\widetilde{M}) \leqslant k\gamma(M) \tag{3}$$

We shall show that the above inequality can be improved. First of all we shall prove a result which is given by G. M. Rassias in the paper [15].

THEOREM 1. *Let $M^m$ be a closed smooth manifold and let $\pi : \widetilde{M} \to M$ be a $k$-covering of $M$ $(m \geqslant 2)$. Then the following inequality holds:*

$$\gamma(\widetilde{M}) \leqslant k\gamma(M) - 4(k-1) \tag{4}$$

*Proof.* Let us show for the begining that the manifold $\widetilde{M}$ is compact. For this purpose let us consider the open covering $\{U_j\}_{j\in J}$ of $\widetilde{M}$. Because the covering projection $\pi$ is a locally diffeomorphism it follows that it is an open mapping, therefore $\{\pi(U_j)\}_{j\in J}$ is also an open covering of $M$. From the compactness of $M$ one obtains the finite subcovering $\{\pi(U_{j\nu})\}_{\nu=\overline{1,s}}$ and finally the family $\{\pi^{-1}(\pi(U_{j\nu}))\}_{\nu=\overline{1,s}}$ is a subcovering of $\widetilde{M}$ having $ks$ open sets, thus $\widetilde{M}$ is compact.

Recall that the Morse function $g \in \Omega(M)$ is polar if $\mu_0(g) = \mu_m(g) = 1$, i.e. $g$ has only one minimum point respectively only one maximum point.

Let $f \in \Omega(M)$ be a polar Morse function which is exact, i.e. it satisfies the relation $\mu(f) = \gamma(M)$. Consider the mapping $h : \widetilde{M} \to \mathbf{R}$, defined by $h = f \circ \pi$. Because $\pi$ is a locally diffeomorphism one obtains $h \in \Omega(\widetilde{M})$ and $\mu_j(h) = k\mu_j(f)$, $j = \overline{1,m}$. Writing explicitely these relations it follows $\mu_0(h) = k$, $\mu_j(h) = k\mu_j(f)$, $j = \overline{1,m-1}$, $\mu_m(h) = k$. Applying a wellknown result of M. Morse (see for instance [6]) concerning the elimination of critical points of index $0, 1, m-1, m$, one obtains a Morse function $h_1 \in \Omega(\widetilde{M})$ which is polar and satisfies the following relations

$$\mu_1(h_1) = k\mu_1(f) - (k-1), \quad \mu_{m-1}(h_1) = k\mu_{m-1}(f) - (k-1), \quad \mu_j(h_1) = k\mu_j(f),$$

where $2 \leqslant j \leqslant m-2$. In this case one obtains $\mu(h_1) = k\mu(f) - 4(k-1)$, therefore $\gamma(\widetilde{M}) \leqslant k\mu(f) - 4(k-1)$. Taking into account the principal result of [11] one have the relation $\gamma(M) = \min\{\mu(f) : f \in \Omega^{(1)}(M)\}$, where $\Omega^{(1)}(M)$ is the set of all polar Morse functions on the manifold $M$. From this relation and from the above inequality it follows the assertion of Theorem 1. □

M. Gromov posed the following question [14]: Let $\widetilde{M}_k$, $k \in \mathbf{N}$ be a sequence of manifolds, such that each $\widetilde{M}_k$ is an $a_k$-fold cover of the manifold $M$, where $a_k \to \infty$ as $k \to \infty$. What are the asymptotic properties of the sequence $\gamma(\widetilde{M}_k)$ as $k \to \infty$?

By using the inequality (4) it follows $\gamma(\widetilde{M}_k) \leqslant a_k\gamma(M) - 4(a_k - 1)$. After a simple computation one obtain a partial asymptotic estimation for the above question:

$$\limsup_{k\to\infty} \frac{\gamma(\widetilde{M}_k)}{a_k} \leqslant \gamma(M) - 4.$$

## 2. The main result

The following example shows that, generally, the inequality proved in Theorem 1 is strict. Let $m \geqslant 3$ and the 2-covering $\pi : S^m \to P^m(\mathbf{R})$, where $S^m$ is the $m$-dimensional sphere and $P^m(\mathbf{R})$ is the real projective $m$-dimensional space. It is not difficult to show that $\gamma(S^m) = 2$, $\gamma(P^m(\mathbf{R})) = m+1$. The inequality of Theorem 1 becomes $2 < 2(m+1) - 4$, which is strict since we considered $m \geqslant 3$.

It is natural to ask for obtaining general classes of manifolds for which the inequality (4) contained in Theorem 1 holds with equality. The main result of this paper shows that this property holds for the class of closed smooth surfaces.

THEOREM 2. *If $M^2$ is a closed smooth surface, orientable or not, then*

$$\gamma(\widetilde{M}) = k\gamma(M) - 4(k-1) \tag{5}$$

*i.e. the inequality* (4) *holds with equality.*

*Proof.* We shall use the relation

$$\gamma(M) = 4 - \chi(M) \tag{6}$$

which is given by N.H. Kuiper and it appears in the paper [9] (see also [10] or [7, Proposition 5.6, pp.26]), where $\chi(M)$ is the Euler-Poincaré characteristic of the surface $M^2$. To prove the above relation, let us consider $f \in \Omega(M)$ an exact Morse function, i.e. $f$ satisfies $\mu(f) = \gamma(M)$. Taking into account the result contained in [11, pp.383] or [12, pp.269] it follows that there exists a polar Morse function $h \in \Omega^{(1)}(M)$ with the property $\mu(h) = \mu(f) = \gamma(M)$.

On the other hand the well-known Euler formula from Morse relations give us:

$$\chi(M) = \mu_0(h) - \mu_1(h) + \mu_2(h) = 2 - \mu_1(h)$$

But it is clear that:

$$\mu(h) = \mu_0(h) + \mu_1(h) + \mu_2(h) = 2 + \mu_1(h)$$

therefore by adding these two relations one obtains $\chi(M) + \mu(h) = 4$, i.e. $\gamma(M) = 4 - \chi(M)$.

Let us prove now the relation (5). Consider $f \in \Omega(M)$, $h \in \Omega(\widetilde{M})$, $h = f \circ \pi$, as in the proof of Theorem 1. By using Euler formula it follows:

$$\chi(\widetilde{M}) = \sum_{j=1}^{m}(-1)^j\mu_j(h) = \sum_{j=1}^{m}(-1)^j k\mu_j(f) = k\sum_{j=1}^{m}(-1)^j\mu_j(f) = k\chi(M)$$

Therefore the relation

$$\chi(\widetilde{M}) = k\chi(M) \tag{7}$$

is satisfied without restrictions on dimension of the manifold $M$.

If $m = 2$ it follows $\chi(M) = 4-\gamma(M)$, $\chi(\widetilde{M}) = 4-\gamma(\widetilde{M})$. From the above proved relation (7), one obtains $4-\gamma(\widetilde{M}) = k(4-\gamma(M))$ and finally $\gamma(\widetilde{M}) = k\gamma(M)-4(k-1)$. □

REMARK. Let us note that the method given in the proof of Theorem 2 can be adapted in proving the inequality (4) contained in Theorem 1 in the case when the dimension of manifold $M^m$ is even. For this purpose let us consider the mapping $f$ and $h$ as in Theorem 2. Then

$$\chi(M) + \mu(f) = 4 + 2\alpha(f) \tag{8}$$

where $\alpha(f) = \mu_2(f) + \mu_4(f) + \ldots$ Therefore the following relations holds $\chi(M) + \mu(f) = 4 + 2\alpha(f)$, $\chi(\widetilde{M}) + \gamma(\widetilde{M}) \leqslant 4 + 2\alpha(h) = 4 + 2k\alpha(f)$. The second relation can be written in the form $k\chi(M) + \gamma(\widetilde{M}) \leqslant 4 + 2k\alpha(f)$ and by substraction of $k\chi(M) + k\gamma(M) = 4k + 2k\alpha(f)$ in the both sides of this inequality (4) it follows the inequality contained in Theorem 1.

COROLLARY 3. *Let $M^m$ be a closed smooth manifold and let $G$ be a finite group acting freely on $M$. Then:*

$(i)$

$$\gamma(M/G) \geqslant \frac{1}{|G|}(\gamma(M) + 4(|G| - 1)) \tag{9}$$

$(ii)$ *If $M^2$ is a closed smooth surface the above inequality is verified with equality.*

*Proof.* In these hypotheses $\pi : M \to M/G$ is a $|G|$-covering of $M/G$ and the inequality $(i)$ is a direct consequence of Theorem 1. The assertion $(ii)$ can be easily obtained from Theorem 2. □

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