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ESTIMATIONS IN HÖLDER'S TYPE INEQUALITIES

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(*communicated by S. Saitoh*)

Abstract. Using a technique due to Ozeki, we give an upper bound of

$$\left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q} - \lambda \sum a_k b_k$$

for $\lambda > 0$, for $p > 1$, $q > 1$ satisfying $1/p + 1/q = 1$, and for n-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of positive numbers under certain conditions. This yields a complement of Hölder's inequality. The estimation with a parameter λ enables us to unify the discussions on difference and ratio inequalities derived from Hölder's inequality.

1. Introduction

The Hölder's inequality is one of the most important inequalities in analysis: If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are n-tuples of nonnegative numbers, then for any $p > 1$, $q > 1$ satisfying $1/p + 1/q = 1$,

$$\left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q} \geqslant \sum a_k b_k.$$

Recently, using Ozeki's method, the first author [5] obtained an upper bound of

$$S_p(a,b) := \left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q} - \left(\sum a_k b_k\right) \tag{1.1}$$

under the conditions

$$\begin{aligned} m_1 \leqslant a_k \leqslant M_1, \qquad m_2 \leqslant b_k \leqslant M_2 \quad (k = 1, 2, \ldots, n), \\ 0 < m_1 < M_1 \quad \text{and} \quad 0 < m_2 < M_2, \end{aligned} \tag{1.2}$$

and moreover in [6] the estimation was shown to be the best possible.

Consider $S_p(a,b)$ as a function defined on the product $[m_1, M_1]^n \times [m_2, M_2]^n$. Then

(i) $S_p(a,b)$ is a separately convex function with respect to a and b, so that its maximum is attained at an extreme point, that is, a vertex of the corresponding $2n$-dimensional cube [8].

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(ii) Let $\underline{c} = (\underline{c}_1, \ldots, \underline{c}_n)$ and $\overline{c} = (\overline{c}_1, \ldots, \overline{c}_n)$ be the rearrangements of $c = (c_1, \ldots, c_n)$ in decreasing order and in increasing order, respectively. Then $\sum \underline{a}_k \overline{b}_k$ $(= \sum \overline{a}_k \underline{b}_k) \leqslant \sum a_k b_k$ [4, p. 261], so that

$$S_p(\underline{a}, \overline{b}) \ (= S_p(\overline{a}, \underline{b})) \geqslant S_p(a, b).$$

Hence the maximum of $S_p(a,b)$ is attained at a point such that a and b are monotone in mutually opposite orders.

We note that the use of Ozeki's method is to apply Ozeki's properties (i) and (ii).

In [10], (cf. [7], [9, p. 121]) Ozeki himself presented a complementary Cauchy's inequality:

$$\sum a_k^2 \sum b_k^2 - \left(\sum a_k b_k\right)^2 \leqslant \frac{n^2}{3}(M_1 M_2 - m_1 m_2)^2$$

based on the fact that the left-hand side possesses the properties (i) and (ii).

In this paper we discuss the estimation of the following difference

$$S_{p,\lambda}(a,b) := \left(\sum a_k^p\right)^{1/p} \left(\sum b_k^q\right)^{1/q} - \lambda \sum a_k b_k \tag{1.3}$$

with a parameter $\lambda > 0$, and obtain an upper bound $nM_1M_2F(\lambda)$ ($F(\lambda)$ is defined in Lemma 3.1) of $S_{p,\lambda}(a,b)$, which yields a complement of Hölder's inequality. The upper bound $nM_1M_2F(\lambda)$ is shown the best possible, in some reasonable sense, by applying Sion's minimax theorem ([1], [12]). As applications, we derive some inequalities which are given by $p = q = 2$ or $m_2/M_2 \to 1$. Furthermore, taking λ satisfying $F(\lambda) = 0$, we obtain a ratio inequality (reverse Hölder's inequality) which is equivalent to the one given by S. A. Gheorghiu [3].

2. Preliminaries

In this section we state some useful facts, and define some constants for our discussion.

LEMMA 2.1. *Let u, w, μ and ν be positive numbers. Then*

$$u^{1/p} w^{1/q} \leqslant \left(\frac{1}{p\mu}\right)^{1/p} \left(\frac{1}{q\nu}\right)^{1/q} (\mu u + \nu w) \tag{2.1}$$

and the equality holds for

$$p\mu u = q\nu w. \tag{2.2}$$

Proof. The Young's inequality says that

$$u^{1/p} w^{1/q} \leqslant \frac{u}{p} + \frac{w}{q}. \tag{2.3}$$

Hence we have

$$u^{1/p} w^{1/q} = \left(\frac{1}{p\mu}\right)^{1/p} \left(\frac{1}{q\nu}\right)^{1/q} (p\mu u)^{1/p} (q\nu w)^{1/q} \leqslant \left(\frac{1}{p\mu}\right)^{1/p} \left(\frac{1}{q\nu}\right)^{1/q} (\mu u + \nu w).$$

Since the equality in (2.3) holds for $u = w$, the equality in (2.1) holds for $p\mu u = qvw$. $\square$

LEMMA 2.2. *Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be n-tuples of real numbers satisfying (1.2). Then for any $\lambda > 0$, the maximum of $S_{p,\lambda}(a,b)$ is attained at a point $(a^{(s)}, b^{(t)}) \in [m_1, M_1]^n \times [m_2, M_2]^n$ such that*

$$a^{(s)} = (\overbrace{M_1, \dots, M_1}^{s}, \overbrace{m_1, \dots, m_1}^{n-s}) \quad \text{and} \quad b^{(t)} = (\overbrace{m_2, \dots, m_2}^{t}, \overbrace{M_2, \dots, M_2}^{n-t}), \tag{2.4}$$

where s and t are integers satisfying $0 \leqslant s \leqslant n$ and $0 \leqslant t \leqslant n$.

Proof. Let

$$\| a \|_r = \left(\sum a_k^r\right)^{1/r} \quad \text{and} \quad \langle a, b\rangle = \sum a_k b_k.$$

Then

$$S_{p,\lambda}(a,b) = \| a \|_p \| b \|_q - \lambda \langle a, b\rangle$$

is a separately convex function with respect to a and b, and

$$S_{p,\lambda}(\underline{a}, \overline{b}) \ (= S_{p,\lambda}(\overline{a}, \underline{b})) \geqslant S_{p,\lambda}(a,b).$$

Hence $S_{p,\lambda}(a,b)$ has Ozeki's properties (i) and (ii) as well as $S_p(a,b)$, so that its maximum is attained at an extreme point. Note that $e = (e_1, \dots, e_n)$ is an extreme point of $[m, M]^n$ if and only if $e_k = m$ or M for each k. Since we may confine ourselves to the case that a is of decreasing order and b is of increasing order, we can obtain the maximum of $S_{p,\lambda}(a,b)$ for a and b of the form in (2.4). $\square$

Now we define several constants. Let $0 < \alpha < 1$, $0 < \beta < 1$, and let

$$K_\alpha = \frac{1-\alpha^p}{1-\alpha}, \quad K_\beta = \frac{1-\beta^q}{1-\beta} \quad \text{and} \quad K = \left(\frac{K_\alpha}{p}\right)^{1/p} \left(\frac{K_\beta}{q}\right)^{1/q}. \tag{2.5}$$

Furthermore, put

$$\tilde{K}_\alpha = \frac{K_\alpha}{\alpha^{p/q}}, \quad \tilde{K}_\beta = \frac{K_\beta}{\beta^{q/p}} \quad \text{and} \quad \tilde{K} = \frac{K}{\alpha^{1/q}\beta^{1/p}} \left(= \left(\frac{\tilde{K}_\alpha}{p}\right)^{1/p} \left(\frac{\tilde{K}_\beta}{q}\right)^{1/q}\right). \tag{2.6}$$

Then since $1/p + 1/q = 1$, we see that K is a weighted geometric mean of $\frac{K_\alpha}{p}$ and $\frac{K_\beta}{q}$, and that $\tilde{K}$ is that of $\frac{\tilde{K}_\alpha}{p}$ and $\frac{\tilde{K}_\beta}{q}$. By the mean-value theorem there exist real numbers θ_α and θ_β such that

$$\alpha < \theta_\alpha < 1, \quad \beta < \theta_\beta < 1, \quad K_\alpha = p\theta_\alpha^{p-1} \quad \text{and} \quad K_\beta = q\theta_\beta^{q-1}. \tag{2.7}$$

Hence we have

$$\alpha^{p-1} < \frac{K_\alpha}{p} < 1, \quad \beta^{q-1} < \frac{K_\beta}{q} < 1 \quad \text{and} \quad \alpha^{1/q}\beta^{1/p} < \theta_\alpha^{1/q}\theta_\beta^{1/p} = K < 1. \tag{2.8}$$

Furthermore, since $p/q = p-1$ and $q/p = q-1$, we have

$$1 < \frac{\tilde{K}_\alpha}{p} < \alpha^{1-p}, \quad 1 < \frac{\tilde{K}_\beta}{q} < \beta^{1-q} \quad \text{and} \quad 1 < \tilde{K} < \alpha^{-1/q}\beta^{-1/p}. \tag{2.9}$$

LEMMA 2.3. *Let* $0 < \alpha < 1$, $0 < \beta < 1$ *and* $\lambda > 0$. *Then*
(1) *The equation* (*of* $\tau > 0$)

$$(1-\alpha)(\lambda - K\tau^{1/q}) = (1-\beta)(\lambda - K\tau^{-1/p}) \tag{2.10}$$

has a unique positive solution, which we denote by $\tau = \tau_*$. *Define a constant* c_λ *by*

$$c_\lambda = (1-\alpha)(\lambda - K\tau_*^{1/q}) \ \Big(= (1-\beta)(\lambda - K\tau_*^{-1/p})\Big). \tag{2.11}$$

Then $K \leqslant \lambda$ *if and only if* $c_\lambda \geqslant 0$.
(2) *The equation*

$$(1-\alpha)(K\tau^{1/q} - \beta\lambda) = (1-\beta)(K\tau^{-1/p} - \alpha\lambda) \tag{2.12}$$

is equivalent to (2.10), that is, it has the same solution $\tau = \tau_*$ *as a unique one. Define the constant* $\tilde{c}_\lambda$ *by*

$$\tilde{c}_\lambda = (1-\alpha)(K\tau_*^{1/q} - \beta\lambda) \ \Big(= (1-\beta)(K\tau_*^{-1/p} - \alpha\lambda)\Big). \tag{2.13}$$

Then $\lambda \leqslant \tilde{K}$ *if and only if* $\tilde{c}_\lambda \geqslant 0$.

Proof. Put

$$L(\tau) = (1-\alpha)(\lambda - K\tau^{1/q}) - (1-\beta)(\lambda - K\tau^{-1/p}). \tag{2.14}$$

Then $L(\tau)$ is strictly decreasing on $(0,\infty)$, and $\lim_{\tau\to+0} L(\tau) = +\infty$, $\lim_{\tau\to\infty} L(\tau) = -\infty$. Hence the equation $L(\tau) = 0$, or equivalently, (2.10) has a unique positive solution τ_*. Next note that the signs of $(1-\alpha)(\lambda - K\tau_*^{1/q})$ and $(1-\beta)(\lambda - K{\tau_*}^{-1/p})$ are the same, or the both are zero. Hence, if $c_\lambda \geqslant 0$, then from $1-\alpha > 0$ and $1-\beta > 0$ we see that $\lambda - K\tau_*^{1/q} \geqslant 0$ and $\lambda - K\tau_*^{-1/p} \geqslant 0$, that is, $(K/\lambda)^p \leqslant \tau_* \leqslant (\lambda/K)^q$. This implies $K \leqslant \lambda$. On the other hand, if $c_\lambda < 0$, then by the same method, we obtain $K > \lambda$. To see (2), we note that the equation (2.12) is obtained by subtracting the both sides of (2.10) respectively from $(1-\alpha)(1-\beta)\lambda$. The remaining fact is obtained similarly as in (1). □

We need the following theorem ([1], [12]) later to guarantee the best possibility of our estimation of (1.3) (cf. (3.1)) in some sense.

THEOREM S. (Sion's minimax theorem) *Let* f *be a real-valued function defined on the product* $X \times Y$ *of* X *and* Y, *each of which is a subset in a respective topological vector space. Assume that*

(1) *X is compact convex and Y is convex.*

(2) *For all τ in Y, $f(\cdot,\tau)$ is quasiconcave on X, i.e., for $\lambda \in \mathbb{R} \quad (=(-\infty,\infty))$,*

$$\{z; z \in X, f(z,\tau) \geqslant \lambda\}$$

is convex or empty in X.

(3) *For all z in X, $f(z,\cdot)$ is quasiconvex on Y, i.e., for $\lambda \in \mathbb{R}$,*

$$\{\tau; \tau \in Y, f(z,\tau) \leqslant \lambda\}$$

is convex or empty in Y.

(4) *For all τ in Y, $f(\cdot,\tau)$ is upper semicontinuous on X and for all z in X, $f(z,\cdot)$ is lower semicontinuous on Y.*

Then

$$\inf_{\tau\in Y}\sup_{z\in X} f(z,\tau) = \sup_{z\in X}\inf_{\tau\in Y} f(z,\tau).$$

3. Main results

Our problem is calculating an upper bound of $S_{p,\lambda}(a,b)$ under the assumption (1.2). By Lemma 2.2 the problem is reduced to that of computing the maximum of $S_{p,\lambda}(a,b)$ among points $(a^{(s)},b^{(t)}) \in [m_1,M_1]^n \times [m_2,M_2]^n$ given in (2.4). In this section we really compute an upper bound of $S_{p,\lambda}(a^{(s)},b^{(t)})$, which is not always the maximum, but the best possible in a reasonable sense (see Remark 3.3).

LEMMA 3.1. *Let $a^{(s)}$ and $b^{(t)}$ be n-tuples given in Lemma 2.2 (2.4). Put $\alpha = m_1/M_1$, $\beta = m_2/M_2$, $\tau_\alpha = \frac{qK_\alpha}{pK_\beta}\alpha^{-p}$ and $\tau_\beta = \frac{qK_\alpha}{pK_\beta}\beta^{q}$. Then for any $\lambda > 0$*

$$S = S_{p,\lambda}(a^{(s)},b^{(t)}) \leqslant nM_1M_2F(\lambda), \tag{3.1}$$

where $F(\lambda) = F(\lambda;\alpha,\beta,p)$ is the constant defined as follows.

Case I: $0 \leqslant t \leqslant s \leqslant n$

$$F(\lambda) = \begin{cases} 1-\lambda & \text{if } 0<\lambda<\min\{\frac{K_\alpha}{p},\frac{K_\beta}{q}\} \\ \{\frac{1}{K_\alpha}+\frac{1}{K_\beta}(\frac{K}{\lambda})^q-1\}\lambda & \text{if } \frac{K_\alpha}{p}(=\min\{\frac{K_\alpha}{p},\frac{K_\beta}{q}\}) \leqslant \lambda < K \\ \{\frac{1}{K_\alpha}(\frac{K}{\lambda})^p+\frac{1}{K_\beta}-1\}\lambda & \text{if } \frac{K_\beta}{q}(=\min\{\frac{K_\alpha}{p},\frac{K_\beta}{q}\}) \leqslant \lambda < K \\ \beta(1-\lambda) & \text{if } K \leqslant \lambda, \tau_* < \tau_\beta \\ (\frac{1}{K_\alpha}+\frac{1}{K_\beta}-1)\lambda - c_\lambda(\frac{1}{1-\alpha^p}+\frac{1}{1-\beta^q}-1) & \text{if } K \leqslant \lambda, \tau_\beta \leqslant \tau_* \leqslant \tau_\alpha \\ \alpha(1-\lambda) & \text{if } K \leqslant \lambda, \tau_\alpha < \tau_*. \end{cases} \tag{3.2}$$

Case II: $0 \leqslant s \leqslant t \leqslant n$

$$F(\lambda) = \begin{cases} \beta(1-\lambda) & \textit{if } 0 < \lambda \leqslant \tilde{K}, \tau_* < \tau_\beta \\ (\frac{1}{K_\alpha} + \frac{1}{K_\beta} - 1)\lambda - c_\lambda(\frac{1}{1-\alpha^p} + \frac{1}{1-\beta^q} - 1) & \textit{if } 0 < \lambda \leqslant \tilde{K}, \tau_\beta \leqslant \tau_* \leqslant \tau_\alpha \\ \alpha(1-\lambda) & \textit{if } 0 < \lambda \leqslant \tilde{K}, \tau_\alpha < \tau_* \\ \{\frac{\alpha^p}{K_\alpha} + \frac{1}{K_\beta}(\frac{K}{\lambda})^q - \alpha\}\beta\lambda & \textit{if } \tilde{K} < \lambda \leqslant \frac{\tilde{K}_\alpha}{p} (= \max\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\}) \\ \{\frac{1}{K_\alpha}(\frac{K}{\lambda})^p + \frac{\beta^q}{K_\beta} - \beta\}\alpha\lambda & \textit{if } \tilde{K} < \lambda \leqslant \frac{\tilde{K}_\beta}{q} (= \max\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\}) \\ \alpha\beta(1-\lambda) & \textit{if } \max\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\} < \lambda. \end{cases} \tag{3.3}$$

Proof. Without loss of generality we may assume $M_1 = M_2 = 1$, so that $m_1 = \alpha$ and $m_2 = \beta$.

Case I: Let $0 \leqslant t \leqslant s \leqslant n$, and let

$$a^{(s)} = (\overbrace{1,\ldots,1}^{t}, \overbrace{1,\ldots,1}^{s-t}, \overbrace{\alpha,\ldots,\alpha}^{n-s}), \quad b^{(t)} = (\overbrace{\beta,\ldots,\beta}^{t}, \overbrace{1,\ldots,1}^{s-t}, \overbrace{1,\ldots,1}^{n-s}).$$

Then

$$S = \{s + (n-s)\alpha^p\}^{1/p}\{t\beta^q + (n-t)\}^{1/q} - \{t\beta + (s-t) + (n-s)\alpha\}\lambda.$$

Putting $x = n - s$ and $y = t$, we have

$$\begin{aligned} S &= S[x,y] \qquad (3.4) \\ &= (n - x + x\alpha^p)^{1/p}(\beta^q y + n - y)^{1/q} - (\beta y + n - x - y + \alpha x)\lambda \\ &= \{n - (1-\alpha^p)x\}^{1/p}\{n - (1-\beta^q)y\}^{1/q} + \{(1-\alpha)x + (1-\beta)y - n\}\lambda \\ &\qquad \text{for integers } x, y,\ x \geqslant 0,\ y \geqslant 0 \text{ and } x + y \leqslant n. \end{aligned}$$

We set

$$\Delta := \{(x,y); x \geqslant 0, y \geqslant 0, x + y \leqslant n\}.$$

Then we wish to determine the maximum of S over the *integer* points of Δ, a subset of Δ. However, it will be very involved. Therefore we shall consider S over the *whole* points of Δ instead. That is, we shall determine

$$\max S := \max\{S[x,y]; (x,y) \in \Delta\}.$$

To this end, we want to introduce a subsidiary function (defined later) of $S[x,y]$. First replace u, w, μ and ν in Lemma 2.1 by

$$n - (1-\alpha^p)x,\ n - (1-\beta^q)y,\ \frac{\mu_1}{K_\alpha} \text{ and } \frac{\nu_1}{K_\beta} \quad \text{for } \mu_1 > 0,\ \nu_1 > 0,$$

respectively. Then from (2.1), we have

$$\begin{aligned} &\{n - (1-\alpha^p)x\}^{1/p}\{n - (1-\beta^q)y\}^{1/q} \qquad (3.5) \\ &\leqslant \left(\frac{K_\alpha}{p\mu_1}\right)^{1/p}\left(\frac{K_\beta}{q\nu_1}\right)^{1/q}\left[\frac{\mu_1}{K_\alpha}\{n - (1-\alpha^p)x\} + \frac{\nu_1}{K_\beta}\{n - (1-\beta^q)y\}\right] \\ &= K\left[\frac{n}{K_\alpha}\left(\frac{\mu_1}{\nu_1}\right)^{1/q} + \frac{n}{K_\beta}\left(\frac{\mu_1}{\nu_1}\right)^{-1/p} - (1-\alpha)\left(\frac{\mu_1}{\nu_1}\right)^{1/q} x - (1-\beta)\left(\frac{\mu_1}{\nu_1}\right)^{-1/p} y\right]. \end{aligned}$$

Hence, if we put $\tau = \frac{\mu_1}{\nu_1}$, then it follows from (3.4) and (3.5) that

$$S \leqslant nK\left(\frac{\tau^{1/q}}{K_\alpha}+\frac{\tau^{-1/p}}{K_\beta}\right)-n\lambda+(1-\alpha)(\lambda-K\tau^{1/q})x+(1-\beta)(\lambda-K\tau^{-1/p})y, \quad (3.6)$$

$$\text{for integers } x,y, \quad (x,y)\in\Delta.$$

Now we denote by $T_\lambda(x,y;\tau)$ the right-hand side of (3.6) as a subsidiary function for $S[x,y]$, and put

$$\phi(\tau) = \max_{(x,y)\in\Delta} T_\lambda(x,y;\tau) \quad \text{for } \tau > 0.$$

Then we shall calculate the minimum ϕ of $\phi(\tau)$, that is,

$$\phi = \min_{\tau>0}\phi(\tau) = \min_{\tau>0}\max_{(x,y)\in\Delta} T_\lambda(x,y;\tau) \quad (3.7)$$

as an upper bound $F(\lambda)$ of $S = S[x,y]$, in place of $\max S[x,y]$ itself. Since $T_{\lambda,\tau}(x,y) = T_\lambda(x,y;\tau)$ is an affine function on the triangle Δ, it attains its maximum $\phi(\tau)$ at one of the vertices $(n,0),(0,n)$ and $(0,0)$ of Δ, i.e.,

$$\phi(\tau) = \max_{(x,y)\in\Delta} T_{\lambda,\tau}(x,y) = \max\{T_{\lambda,\tau}(n,0), T_{\lambda,\tau}(0,n), T_{\lambda,\tau}(0,0)\}. \quad (3.8)$$

For convenience sake we define several functions. Let

$$\begin{aligned} G_\lambda(\tau) &= \frac{T_{\lambda,\tau}(n,0)}{n} = K\left(\frac{\tau^{1/q}}{K_\alpha}+\frac{\tau^{-1/p}}{K_\beta}\right)-\lambda+(1-\alpha)(\lambda-K\tau^{1/q}) \\ &= K\left(\frac{\alpha^p\tau^{1/q}}{K_\alpha}+\frac{\tau^{-1/p}}{K_\beta}\right)-\alpha\lambda, \end{aligned} \quad (3.9)$$

$$\begin{aligned} H_\lambda(\tau) &= \frac{T_{\lambda,\tau}(0,n)}{n} = K\left(\frac{\tau^{1/q}}{K_\alpha}+\frac{\tau^{-1/p}}{K_\beta}\right)-\lambda+(1-\beta)(\lambda-K\tau^{-1/p}) \\ &= K\left(\frac{\tau^{1/q}}{K_\alpha}+\frac{\beta^q\tau^{-1/p}}{K_\beta}\right)-\beta\lambda, \end{aligned} \quad (3.10)$$

$$I_\lambda(\tau) = \frac{T_{\lambda,\tau}(0,0)}{n} = K\left(\frac{\tau^{1/q}}{K_\alpha}+\frac{\tau^{-1/p}}{K_\beta}\right)-\lambda, \quad (3.11)$$

and let

$$\begin{aligned} g_\lambda(\tau) &= G_\lambda(\tau)-I_\lambda(\tau) = (1-\alpha)(\lambda-K\tau^{1/q}), \\ h_\lambda(\tau) &= H_\lambda(\tau)-I_\lambda(\tau) = (1-\beta)(\lambda-K\tau^{-1/p}). \end{aligned}$$

Then we have easily the following facts.

(i) $G'_\lambda(\tau)=0$ has a unique solution $\tau=\tau_\alpha=\frac{qK_\alpha}{pK_\beta}\alpha^{-p}$, moreover $G_\lambda(\tau)$ is decreasing for $(0<)\tau\leqslant\tau_\alpha$, increasing for $\tau_\alpha<\tau$ and hence $\min_{\tau>0}G_\lambda(\tau)=G_\lambda(\tau_\alpha)=\alpha(1-\lambda)$.

(ii) $H'_\lambda(\tau)=0$ has a unique solution $\tau=\tau_\beta=\frac{qK_\alpha}{pK_\beta}\beta^q$, moreover $H_\lambda(\tau)$ is decreasing for $(0<)\tau \leqslant \tau_\beta$, increasing for $\tau_\beta < \tau$ and hence $\min_{\tau>0} H_\lambda(\tau) = H_\lambda(\tau_\beta) = \beta(1-\lambda)$.

(iii) $I'_\lambda(\tau) = 0$ has a unique solution $\tau = \tau_I = \frac{qK_\alpha}{pK_\beta}$, moreover $I_\lambda(\tau)$ is decreasing for $(0<)\tau \leqslant \tau_I$, increasing for $\tau_I < \tau$ and hence $\min_{\tau>0} I_\lambda(\tau) = I_\lambda(\tau_I) = 1-\lambda$.

(iv) $g_\lambda(\tau)$ is strictly decreasing and $\lim_{\tau\to+0} g_\lambda(\tau) = (1-\alpha)\lambda$, $\lim_{\tau\to\infty} g_\lambda(\tau) = -\infty$, and hence the equation $g_\lambda(\tau) = 0$ has a unique solution $\tau_g = (\frac{\lambda}{K})^q$.

(v) $h_\lambda(\tau)$ is strictly increasing and $\lim_{\tau\to+0} h_\lambda(\tau) = -\infty$, $\lim_{\tau\to\infty} h_\lambda(\tau) = (1-\beta)\lambda$, and hence the equation $h_\lambda(\tau) = 0$ has a unique solution $\tau_h = (\frac{K}{\lambda})^p$.

Note that (3.8) can be rewritten as follows:

$$\phi(\tau) = n\max\{G_\lambda(\tau), H_\lambda(\tau), I_\lambda(\tau)\}.$$

We denote by ϕ_0 the minimum ϕ/n of the function $\phi(\tau)/n$. Recall that the equation $G_\lambda(\tau) = H_\lambda(\tau)$, i.e., $g_\lambda(\tau) = h_\lambda(\tau)$ has a unique solution τ_* from Lemma 2.3(1), and that the constant c_λ is defined (in (2.11)) as

$$c_\lambda = (1-\alpha)(\lambda - K\tau_*^{1/q}) \ \left(= (1-\beta)(\lambda - K\tau_*^{-1/p})\right).$$

Now in order to compute ϕ_0, we divide Case I again into two cases according to $c_\lambda \geqslant 0$ and $c_\lambda < 0$.

[I-1] Let $c_\lambda \geqslant 0$. Then $K \leqslant \lambda$ as an equivalent condition from Lemma 2.3(1). In this case, at least one of $g_\lambda(\tau)$ and $h_\lambda(\tau)$ is nonnegative (for any $\tau > 0$), for if not, then $\lambda - K\tau^{1/q} < 0$ and $\lambda - K\tau^{-1/p} < 0$, which implies that $\left(\frac{\lambda}{K}\right)^q \leqslant \tau \leqslant \left(\frac{K}{\lambda}\right)^p$, or $\lambda < K$, a contradiction. Hence, either $G_\lambda(\tau)$ or $H_\lambda(\tau)$ is not smaller than $I_\lambda(\tau)$. Furthemore, since $L(\tau) = g_\lambda(\tau) - h_\lambda(\tau) = G_\lambda(\tau) - H_\lambda(\tau) \geqslant 0$ or < 0 according to $0 < \tau \leqslant \tau_*$ or $\tau_* < \tau$ (from the proof of Lemma 2.3 (1)), we have

$$\phi(\tau) = n\max\{G_\lambda(\tau), H_\lambda(\tau)\} = \begin{cases} nG_\lambda(\tau) & \text{if } 0 < \tau \leqslant \tau_* \\ nH_\lambda(\tau) & \text{if } \tau_* < \tau. \end{cases} \tag{3.12}$$

Note that $\tau_\beta = \frac{qK_\alpha}{pK_\beta}\beta^q \leqslant \frac{qK_\alpha}{pK_\beta}\alpha^{-p} = \tau_\alpha$. Now by a simple computation or by tracing the graphs of $G_\lambda(\tau)$ and $H_\lambda(\tau)$, we can obtain the minimum ϕ_0 of $\phi(\tau)/n$ as follows:

$$\phi_0 = \begin{cases} G_\lambda(\tau_\alpha) & \text{if } \tau_\alpha < \tau_* \\ G_\lambda(\tau_*) \ (= H_\lambda(\tau_*)) & \text{if } \tau_\beta \leqslant \tau_* \leqslant \tau_\alpha \\ H_\lambda(\tau_\beta) & \text{if } \tau_* < \tau_\beta. \end{cases} \tag{3.13}$$

We now want to express ϕ_0 in terms of $\lambda, \alpha, \beta, K_\alpha, K_\beta$ and τ_* (or c_λ). If $\tau_\beta \leqslant \tau_* \leqslant \tau_\alpha$, then by the definition of c_λ, we have

$$K\tau_*^{1/q} = \lambda - \frac{c_\lambda}{1-\alpha} \quad \text{and} \quad K\tau_*^{-1/p} = \lambda - \frac{c_\lambda}{1-\beta},$$

so that, from (3.9) (or (3.10)), we have

$$\begin{aligned}\phi_0 &= G_\lambda(\tau_*)\ (= H_\lambda(\tau_*))\\ &= K\left(\frac{\tau_*^{1/q}}{K_\alpha}+\frac{\tau_*^{-1/p}}{K_\beta}\right)-\lambda+(1-\alpha)(\lambda-K\tau_*^{1/q})\\ &= \left(\frac{1}{K_\alpha}+\frac{1}{K_\beta}-1\right)\lambda-c_\lambda\left(\frac{1}{1-\alpha^p}+\frac{1}{1-\beta^q}-1\right).\end{aligned}$$

If $\tau_\alpha < \tau_*$, then $\phi_0 = G_\lambda(\tau_\alpha) = \alpha(1-\lambda)$ from (i). If $\tau_* < \tau_\beta$, then $\phi_0 = H_\lambda(\tau_\beta) = \beta(1-\lambda)$ from (ii). Hence, puttig $F(\lambda) = \phi_0$ in each subcase, we have the later half (the case $K \leqslant \lambda$) of (3.2).

[I-2] Let $c_\lambda < 0$. Then $(0 <)\lambda < K$ as an equivalent condition from Lemma 2.3(1). Recall that from (iv) and (v) $g_\lambda(\tau) = G_\lambda(\tau) - I_\lambda(\tau)$ and $h_\lambda(\tau) = H_\lambda(\tau) - I_\lambda(\tau)$ are strictly decreasing and strictry increasing, respectively, and $\tau_g = (\frac{\lambda}{K})^q$ and $\tau_h = (\frac{K}{\lambda})^q$ are unique solutions of $g_\lambda(\tau) = 0$ and $h_\lambda(\tau) = 0$, respectively. Hence from $\lambda < K$ we can see that $\tau_g < \tau_h$, and furthermore that

$$\phi(\tau) = \begin{cases} nG_\lambda(\tau) & \text{if } 0 < \tau \leqslant \tau_g \\ nI_\lambda(\tau) & \text{if } \tau_g < \tau < \tau_h \\ nH_\lambda(\tau) & \text{if } \tau_h \leqslant \tau\,. \end{cases} \tag{3.14}$$

By an elementary computation or by tracing the graphs of $G_\lambda(\tau)$, $H_\lambda(\tau)$ and $I_\lambda(\tau)$, we can obtain the minimum ϕ_0 of $\phi(\tau)/n$ as follows:

$$\phi_0 = \begin{cases} I_\lambda(\tau_g) & \text{if } \tau_I \leqslant \tau_g \\ I_\lambda(\tau_I) & \text{if } \tau_g < \tau_I < \tau_h \\ I_\lambda(\tau_h) & \text{if } \tau_h \leqslant \tau_I\,. \end{cases} \tag{3.15}$$

Now we want to express ϕ_0 in terms of K_α, K_β, K and λ. First let $\tau_I \leqslant \tau_g(< \tau_h)$. Then we have

$$\frac{\tau_I}{\tau_g} = \frac{qK_\alpha/pK_\beta}{(\lambda/K)^q} = \left(\frac{K_\alpha}{p\lambda}\right)^q \leqslant 1.$$

Hence, as an equivalent condition to $\tau_I \leqslant \tau_g(< \tau_h)$, we have $\frac{K_\alpha}{p} \leqslant \lambda$, or more precisely

$$\frac{K_\alpha}{p}\left(= \min\left\{\frac{K_\alpha}{p}, \frac{K_\beta}{q}\right\}\right) \leqslant \lambda < K,$$

because K lies between $\frac{K_\alpha}{p}$ and $\frac{K_\beta}{q}$ as a weighted geometric mean of them (cf. (2.5)). In this case, from (3.11) we have

$$\phi_0 = I_\lambda(\tau_g) = K\left(\frac{\tau_g^{1/q}}{K_\alpha}+\frac{\tau_g^{-1/p}}{K_\beta}\right)-\lambda = \left\{\frac{1}{K_\alpha}+\frac{1}{K_\beta}\left(\frac{K}{\lambda}\right)^q-1\right\}\lambda.$$

Secondly let $(\tau_g <)\tau_h \leqslant \tau_I$. Then similarly as before, we have $\frac{K_\beta}{q} \leqslant \lambda$, or more precisely

$$\frac{K_\beta}{q}\left(= \min\left\{\frac{K_\alpha}{p}, \frac{K_\beta}{q}\right\}\right) \leqslant \lambda < K$$

as an equivalent condition to $\tau_h \leqslant \tau_I$. In this case, from (3.11) we have

$$\phi_0 = I_\lambda(\tau_h) = K\left(\frac{\tau_h^{1/q}}{K_\alpha} + \frac{\tau_h^{-1/p}}{K_\beta}\right) - \lambda = \left\{\frac{1}{K_\alpha}\left(\frac{K}{\lambda}\right)^p + \frac{1}{K_\beta} - 1\right\}\lambda.$$

Finally let $\tau_g < \tau_I < \tau_h$, Then by the above argument, we observe that

$$(0 <)\lambda < \min\left\{\frac{K_\alpha}{p}, \frac{K_\beta}{q}\right\} \ (\leqslant K)$$

as an equivalent condition to $\tau_g < \tau_I < \tau_h$. In this case, from (iii) we have

$$\phi_0 = I_\lambda(\tau_I) = 1 - \lambda.$$

Now, putting $F(\lambda) = \phi_0$ in each subcase, we obtain the first half $(0 < \lambda \leqslant K)$ of (3.2).

Case II: Let $0 \leqslant s \leqslant t \leqslant n$, and let

$$a^{(s)} = (\overbrace{1,\ldots,1}^{s}, \overbrace{\alpha,\ldots,\alpha}^{t-s}, \overbrace{\alpha,\ldots,\alpha}^{n-t}), \quad b^{(t)} = (\overbrace{\beta,\ldots,\beta}^{s}, \overbrace{\beta,\ldots,\beta}^{t-s}, \overbrace{1,\ldots,1}^{n-t}).$$

Then

$$S = \{s + (n-s)\alpha^p\}^{1/p}\{(n-t) + t\beta^q\}^{1/q} - \{s\beta + (t-s)\alpha\beta + (n-t)\alpha\}\lambda.$$

Putting $x = s$ and $y = n - t$, we have

$$\begin{aligned} S = \tilde{S}[x,y] &= \{n\alpha^p + (1-\alpha^p)x\}^{1/p}\{n\beta^q + (1-\beta^q)y\}^{1/q} \\ &\quad -\{\beta(1-\alpha)x + \alpha(1-\beta)y + n\alpha\beta\}\lambda \end{aligned} \tag{3.16}$$

for integers x, y, $x \geqslant 0$, $y \geqslant 0$ and $x + y \leqslant n$.

As in Case I, by using (2.1) of Lemma 2.1, we have

$$\begin{aligned} &\{n\alpha^p + (1-\alpha^p)x\}^{1/p}\{n\beta^q + (1-\beta^q)y\}^{1/q} \\ &\leqslant K\left\{\tfrac{\alpha^p\tau^{1/q}}{K_\alpha}n + \tfrac{\beta^q\tau^{-1/p}}{K_\beta}n + (1-\alpha)x\tau^{1/q} + (1-\beta)y\tau^{-1/p}\right\}. \end{aligned} \tag{3.17}$$

Hence it follows from (3.16) and (3.17) that

$$\begin{aligned} S &\leqslant nK\left(\tfrac{\alpha^p\tau^{1/q}}{K_\alpha} + \tfrac{\beta^q\tau^{-1/p}}{K_\beta}\right) - n\alpha\beta\lambda \\ &\quad +(1-\alpha)(K\tau^{1/q} - \beta\lambda)x + (1-\beta)(K\tau^{-1/p} - \alpha\lambda)y \end{aligned} \tag{3.18}$$

for integers x, y, $(x,y) \in \Delta$.

Similarly as in Case I, we denote by $\tilde{T}_\lambda(x,y;\tau)$ the right-hand side of (3.18) as a subsidiary function for $\tilde{S}[x,y]$ and put

$$\tilde{\phi}(\tau) = \max_{(x,y)\in\Delta} \tilde{T}_\lambda(x,y;\tau) \quad \text{for } \tau > 0.$$

Then we shall calculate the minimum $\tilde{\phi}$ of $\tilde{\phi}(\tau)$, that is,

$$\tilde{\phi} = \min_{\tau>0} \tilde{\phi}(\tau) = \min_{\tau>0} \max_{(x,y)\in\Delta} \tilde{T}_\lambda(x,y;\tau)$$

instead of $\max \tilde{S}[x,y]$, as an upper bound $F(\lambda)$ of S. Since $\tilde{T}_{\lambda,\tau}(x,y) = \tilde{T}_\lambda(x,y,\tau)$ is an affine function on Δ, it attains its maximum $\tilde{\phi}(\tau)$ at one of the vertices $(n,0),(0,n)$ and $(0,0)$ of Δ as in Case I, i.e.,

$$\tilde{\phi}(\tau) = \max_{(x,y)\in\Delta} \tilde{T}_{\lambda,\tau}(x,y) = \max\{\tilde{T}_{\lambda,\tau}(n,0), \tilde{T}_{\lambda,\tau}(0,n), \tilde{T}_{\lambda,\tau}(0,0)\}. \tag{3.19}$$

Here we define the following functions as Case I. Let

$$\begin{aligned} \tilde{G}_\lambda(\tau) &= \tilde{T}_{\lambda,\tau}(n,0)/n, \\ \tilde{H}_\lambda(\tau) &= \tilde{T}_{\lambda,\tau}(0,n)/n, \\ \tilde{I}_\lambda(\tau) &= \tilde{T}_{\lambda,\tau}(0,0)/n = K\left(\frac{\alpha^p\tau^{1/q}}{K_\alpha} + \frac{\beta^q\tau^{-1/p}}{K_\beta}\right) - \alpha\beta\lambda, \end{aligned} \tag{3.20}$$

and let

$$\begin{aligned} \tilde{g}_\lambda(\tau) &= \tilde{G}_\lambda(\tau) - \tilde{I}_\lambda(\tau) = (1-\alpha)(K\tau^{1/q} - \beta\lambda), \\ \tilde{h}_\lambda(\tau) &= \tilde{H}_\lambda(\tau) - \tilde{I}_\lambda(\tau) = (1-\beta)(K\tau^{-1/p} - \alpha\lambda). \end{aligned}$$

Then we easily see the following facts.

(vi) $\tilde{g}_\lambda(\tau)$ is strictly increasing and $\lim_{\tau\to+0} \tilde{g}_\lambda(\tau) = -(1-\alpha)\beta\lambda$, $\lim_{\tau\to\infty} \tilde{g}_\lambda(\tau) = \infty$, and hence the equation $\tilde{g}_\lambda(\tau) = 0$ has a unique solution $\tau_{\tilde{g}} = (\frac{\beta\lambda}{K})^q$,

(vii) $\tilde{h}_\lambda(\tau)$ is strictly decreasing and $\lim_{\tau\to+0} \tilde{h}_\lambda(\tau) = \infty$, $\lim_{\tau\to\infty} \tilde{h}_\lambda(\tau) = -\alpha(1-\beta)\lambda$, and hence the equation $\tilde{h}_\lambda(\tau) = 0$ has a unique solution $\tau_{\tilde{h}} = (\frac{K}{\alpha\lambda})^p$.

(viii) $\tilde{I}'_\lambda(\tau) = 0$ has a unique solution $\tau = \tau_{\tilde{I}} = \frac{q\beta^q K_\alpha}{p\alpha^p K_\beta}$, moreover $I_\lambda(\tau)$ is decreasing for $(0<)\tau \leqslant \tau_{\tilde{I}}$, increasing for $\tau_{\tilde{I}} < \tau$ and hence $\min_{\tau>0} \tilde{I}_\lambda(\tau) = \tilde{I}_\lambda(\tau_{\tilde{I}}) = \alpha\beta(1-\lambda)$.

Note that (3.19) can be rewritten as follows:

$$\tilde{\phi}(\tau) = n \max\left\{\tilde{G}_\lambda(\tau), \tilde{H}_\lambda(\tau), \tilde{I}_\lambda(\tau)\right\}.$$

We denote by $\tilde{\phi}_0$ the minimum $\tilde{\phi}/n$ of the function $\tilde{\phi}(\tau)/n$. Recall that the equation $\tilde{G}_\lambda(\tau) = \tilde{H}_\lambda(\tau)$, i.e., $\tilde{g}_\lambda(\tau) = \tilde{h}_\lambda(\tau)$ has a unique solution τ_* from Lemma 2.3(2), and that the constant $\tilde{c}_\lambda$ is defined (cf. (2.13)) as

$$\tilde{c}_\lambda = (1-\alpha)(K{\tau_*}^{1/q} - \beta\lambda)\ \left(= (1-\beta)(K{\tau_*}^{-1/p} - \alpha\lambda)\right).$$

Now in order to compute $\tilde{\phi}_0$, we divide Case II again into two cases according to $\tilde{c}_\lambda \geqslant 0$ and $\tilde{c}_\lambda < 0$.

[II-1] Let $\tilde{c}_\lambda \geqslant 0$. Then $(0<)\lambda \leqslant \tilde{K}$ as an equivalent condition from Lemma 2.3(2). In this case we can see that at least one of $\tilde{g}_\lambda(\tau)$ and $\tilde{h}_\lambda(\tau)$ is nonnegative (for any $\tau > 0$), i.e., either $\tilde{G}_\lambda(\tau)$ or $\tilde{H}_\lambda(\tau)$ is not smaller than $\tilde{I}_\lambda(\tau)$. Furthermore, we can see that $\tilde{G}_\lambda(\tau) = \tilde{T}_{\lambda,\tau}(n,0)/n = H_\lambda(\tau)$ and $\tilde{H}_\lambda(\tau) = \tilde{T}_{\lambda,\tau}(0,n)/n = G_\lambda(\tau)$, so that

$$\tilde{\phi}(\tau) = n\max\left\{\tilde{G}_\lambda(\tau), \tilde{H}_\lambda(\tau)\right\} = n\max\left\{G_\lambda(\tau), H_\lambda(\tau)\right\} (= \phi(\tau)). \tag{3.21}$$

Hence we can reduce the further discussion to one in Case [I-1]. Consequently, we obtain the same value ϕ as the minimum $\tilde{\phi}$ of $\tilde{\phi}(\tau) = \max\left\{\tilde{T}_{\lambda,\tau}(x,y); (x,y) \in \Delta\right\}$.

[II-2] Let $\tilde{c}_\lambda < 0$. Then $\tilde{K} < \lambda$ as an equivalent condition from Lemma 2.3(2). In this case, by (vi) and (vii), we can see that $\tau_{\tilde{h}} = (\frac{K}{\alpha\lambda})^p < (\frac{\beta\lambda}{K})^q = \tau_{\tilde{g}}$, and that

$$\tilde{\phi}(\tau) = \begin{cases} nG_\lambda(\tau) \quad (= n\tilde{H}_\lambda(\tau)) & \text{if } 0 < \tau \leqslant \tau_{\tilde{h}} \\ n\tilde{I}_\lambda(\tau) & \text{if } \tau_{\tilde{h}} < \tau < \tau_{\tilde{g}} \\ nH_\lambda(\tau) \quad (= n\tilde{G}_\lambda(\tau)) & \text{if } \tau_{\tilde{g}} \leqslant \tau. \end{cases} \tag{3.22}$$

By an elementary calculation, or by tracing the graphs of $\tilde{G}_\lambda(\tau)$, $\tilde{H}_\lambda(\tau)$ and $\tilde{I}_\lambda(\tau)$, we can see that the minimum $\tilde{\phi}_0$ of $\tilde{\phi}(\tau)/n$ is given as follows:

$$\tilde{\phi}_0 = \begin{cases} \tilde{I}_\lambda(\tau_{\tilde{g}}) & \text{if } \tau_{\tilde{g}} \leqslant \tau_{\tilde{I}} \\ \tilde{I}_\lambda(\tau_{\tilde{I}}) & \text{if } \tau_{\tilde{h}} < \tau_{\tilde{I}} < \tau_{\tilde{g}} \\ \tilde{I}_\lambda(\tau_{\tilde{h}}) & \text{if } \tau_{\tilde{I}} \leqslant \tau_{\tilde{h}}. \end{cases} \tag{3.23}$$

Now we want to express $\tilde{\phi}_0$ in terms of K_α, K_β, K, α, β and λ. First let $(\tau_{\tilde{h}} <)\tau_{\tilde{g}} \leqslant \tau_{\tilde{I}}$. Then we have

$$\frac{\tau_{\tilde{g}}}{\tau_{\tilde{I}}} = \frac{(\beta\lambda/K)^q}{q\beta^q K_\alpha/p\alpha^p K_\beta} = \frac{\alpha^p p^q \lambda^q}{K_\alpha^q} \leqslant 1.$$

Hence as an equivalent condition to $\tau_{\tilde{g}} \leqslant \tau_{\tilde{I}}$, we have $\lambda \leqslant \frac{\tilde{K}_\alpha}{p} (= \frac{K_\alpha}{p\alpha^{p/q}})$, or more precisely

$$\tilde{K} < \lambda \leqslant \frac{\tilde{K}_\alpha}{p}\left(= \max\left\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\right\}\right),$$

because $\tilde{K}$ lies between $\frac{\tilde{K}_\alpha}{p}$ and $\frac{\tilde{K}_\beta}{q}$ as a weighted geometric mean of them (cf. (2.6)). In this case, from (3.20) we have

$$\begin{aligned} \tilde{\phi}_0 &= \tilde{I}_\lambda(\tau_{\tilde{g}}) = K\left(\frac{\alpha^p \tau_{\tilde{g}}^{1/q}}{K_\alpha} + \frac{\beta^q \tau_{\tilde{g}}^{-1/p}}{K_\beta}\right) - \alpha\beta\lambda \\ &= \left\{\frac{\alpha^p}{K_\alpha} + \frac{1}{K_\beta}\left(\frac{K}{\lambda}\right)^q - \alpha\right\}\beta\lambda. \end{aligned} \tag{3.24}$$

Next let $\tau_{\tilde{I}} \leqslant \tau_{\tilde{h}}(< \tau_{\tilde{g}})$. Then similarly as before, we have $\lambda \leqslant \frac{\tilde{K}_\beta}{q} (= \frac{K_\beta}{q\beta^{q/p}})$, or more precisely

$$\tilde{K} < \lambda \leqslant \frac{\tilde{K}_\beta}{q}\left(= \max\left\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\right\}\right)$$

as an equivalent condition to $\tau_{\tilde{I}} \leqslant \tau_{\tilde{h}}$. In this case, from (3.20) we have

$$\begin{aligned} \tilde{\phi}_0 &= \tilde{I}_\lambda(\tau_{\tilde{h}}) = K\left(\frac{\alpha^p \tau_{\tilde{h}}^{1/q}}{K_\alpha} + \frac{\beta^q \tau_{\tilde{h}}^{-1/p}}{K_\beta}\right) - \alpha\beta\lambda \\ &= \left\{\frac{1}{K_\alpha}\left(\frac{K}{\lambda}\right)^p + \frac{\beta^q}{K_\beta} - \beta\right\}\alpha\lambda. \end{aligned} \tag{3.25}$$

Finally let $\tau_{\tilde{h}} < \tau_{\tilde{I}} < \tau_{\tilde{g}}$. Then by the above argument, we observe that

$$\lambda > \max\left\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\right\} \ (\geqslant \tilde{K})$$

as an equivalent condition to $\tau_{\tilde{h}} < \tau_{\tilde{I}} < \tau_{\tilde{g}}$. In this case, from (viii) we have

$$\tilde{\phi}_0 = \tilde{I}_\lambda(\tau_{\tilde{I}}) = \alpha\beta(1-\lambda). \tag{3.26}$$

Putting $F(\lambda) = \tilde{\phi}_0$ in each subcase of Case II, we obtain (3.3). This completes the proof. □

Now applying Theorem S, we prove the following fact which shows that $nM_1M_2F(\lambda)$, the right side of (3.1) in Lemma 3.1 is the best bound in a reasonable sense.

LEMMA 3.2. *With the same notations as in Lemma 3.1 (with the assumption* $M_1 = M_2 = 1$ *added),*

$$nF(\lambda) = \begin{cases} \max_{(x,y)\in\Delta} S[x,y] & (\textit{Case I}) \\ \max_{(x,y)\in\Delta} \tilde{S}[x,y] & (\textit{Case II}). \end{cases} \tag{3.27}$$

Proof. We begin with Case I. By (3.7)

$$F(\lambda) = \min_{\tau>0}\max_{(x,y)\in\Delta} T_\lambda(x,y;\tau).$$

From (3.6) we have

$$S[x,y] \leqslant T_\lambda(x,y;\tau)$$

and from (2.2) we can see that the equality holds for

$$\tau = \tau_{x,y} = \frac{\mu_1}{\nu_1} = \frac{qK_\alpha\{n-(1-\beta^q)y\}}{pK_\beta\{n-(1-\alpha^p)x\}} \ (>0), \tag{3.28}$$

and so

$$S[x,y] = T_\lambda(x,y;\tau_{x,y}) = \min_{\tau>0} T_\lambda(x,y;\tau).$$

Hence what we have to show is

$$\min_{\tau>0}\max_{(x,y)\in\Delta} T_\lambda(x,y;\tau) = \max_{(x,y)\in\Delta}\min_{\tau>0} T_\lambda(x,y;\tau). \tag{3.29}$$

Let $z = (x,y) \in \Delta$ and write $T_\lambda(z,\tau) = T_\lambda(x,y;\tau)$. Then the function $T_\lambda(z,\tau)$ defined on $\Delta \times \mathbb{R}^+$ ($\mathbb{R}^+ = (0,\infty)$) satisfies all assumptions of Theorem S. In fact, first we see that the assumptions (1) and (4) hold, because Δ is a compact convex set in the plane and $T_\lambda(\cdot,\cdot)$ is continuous on $\Delta \times \mathbb{R}^+$. Next we see that the assumptions (2) and (3) hold: For any $\tau \in \mathbb{R}^+$, $T_\lambda(\cdot,\tau)$ is an affine function, hence quasiconcave function on Δ. Furthermore, $T_\lambda(z,\tau)$ can be rewritten as follows:

$$\begin{aligned} T_\lambda(z,\tau) &= A\tau^{1/q} + B\tau^{-1/p} + C, \\ A &= \frac{K}{K_\alpha}\{n-(1-\alpha^p)y\} \ (>0), \\ B &= \frac{K}{K_\beta}\{n-(1-\beta^q)x\} \ (>0), \\ C &= (1-\alpha)\lambda y + (1-\beta)\lambda x - n\lambda \ (>0). \end{aligned}$$

Hence we can check that $T_\lambda(z,\cdot)$ is a quasiconvex function on $\mathbb{R}^+$. Consequently, we obtain the desired identity (3.29).

For Case II, by the similar argument as in Case I we can also obtain the corresponding identity in (3.27). This completes the proof. □

REMARK 3.3. *Say, in Case I, let* τ_0 *and* (x_0,y_0) *be points such that* $\phi(\tau_0) = \min_{\tau>0}\phi(\tau)$ *and* $S[x_0,y_0] = \max_{(x,y)\in\Delta} S[x,y]$ *respectively.* (*Say,* $\tau_0 = \tau_*$ *if* $\lambda \geqslant K, \tau_\beta \leqslant \tau_* \leqslant \tau_\alpha$ *by (3.13).*) *Then the minimax relation (3.29) ensures* (*with the assumption* $M_1 = M_2 = 1$) *that*

$$\begin{aligned} T_\lambda(x,y;\tau_0) &\leqslant T_\lambda(x_0,y_0;\tau_0) = S[x_0,y_0] = \phi(\tau_0) \\ &= nF(\lambda) \leqslant T_\lambda(x_0,y_0;\tau) \quad \textit{for} \quad (x,y)\in\Delta, \tau>0. \end{aligned}$$

The point $((x_0,y_0),\tau_0)$ *is called a saddle point of* $T_\lambda(x,y;\tau)$ *with respect to* $\Delta \times R^+$ *[13, p. 72].*

From the above lemma $nF(\lambda)$ *is the best upper bound of* $S[x,y]$ (*for Case I*) *on the whole set* Δ*. However, the domain of* $S[x,y]$ *is the integer points of* Δ *in itself. Hence if* (x_0,y_0) *is precisely an integer point in* Δ *then* $nF(\lambda)$ *is really the best bound of* $S[x,y]$*. This justifies that we call* $nF(\lambda)$ *the best bound in a reasonable sense.*

In order to obtain a refined form of Lemma 3.1 we prepare

LEMMA 3.4. *Under the same notations as in Lemma 3.1, if* $K \leqslant \lambda \leqslant \tilde{K}$*, then* $\tau_\beta \leqslant \tau_* \leqslant \tau_\alpha$*.*

Proof. Let $L(\lambda,\tau) = L(\tau)$ (cf. (2.14)), that is,

$$L(\lambda,\tau) = (1-\alpha)(\lambda - K\tau^{1/q}) - (1-\beta)(\lambda - K\tau^{-1/p}).$$

Then first, putting $\lambda = K$, we have

$$L(K,1) = 0, \tag{3.30}$$

and moreover

$$\tau_\beta < 1 < \tau_\alpha. \tag{3.31}$$

Indeed, we have by (2.7)

$$\tau_\beta = \frac{qK_\alpha}{pK_\beta}\beta^q = \frac{K_\alpha/p}{K_\beta/q}\beta^q = \theta_\alpha^{p-1}\theta_\beta\left(\frac{\beta}{\theta_\beta}\right)^q < 1,$$

and similarly $1 < \tau_\alpha$. Since $L(K,\tau)$ is a strictly decreasing function in τ, we see, from the facts (3.30) and (3.31), that

$$L(K,\tau_\beta) > 0 \quad \text{and} \quad L(K,\tau_\alpha) < 0. \tag{3.32}$$

Next, putting $\lambda = \tilde{K}$, we have

$$L(\tilde{K}, \frac{\beta}{\alpha}) = 0,$$

and moreover

$$\tau_\beta < \frac{\beta}{\alpha} < \tau_\alpha.$$

Hence similarly as (3.32) we have

$$L(\tilde{K},\tau_\beta) > 0 \quad \text{and} \quad L(\tilde{K},\tau_\alpha) < 0. \tag{3.33}$$

Now note that any $\lambda \in [K,\tilde{K}]$ is written as $\lambda = \mu K + (1-\mu)\tilde{K}$ for some $\mu \in [0,1]$, and that $L(\lambda,\tau)$ is an affine function with respect to λ. Hence we have

$$L(\lambda,\tau) = L(\mu K + (1-\mu)\tilde{K},\tau) = \mu L(K,\tau) + (1-\mu)L(\tilde{K},\tau),$$

so that by (3.32) and (3.33) we have

$$L(\lambda,\tau_\beta) = \mu L(K,\tau_\beta) + (1-\mu)L(\tilde{K},\tau_\beta) > 0.$$

Similarly we have $L(\lambda,\tau_\alpha) < 0$. Hence there exists a unique $\tau_0 \in [\tau_\beta,\tau_\alpha]$ such that

$$L(\lambda,\tau_0) = 0,$$

and this solution τ_0 is nothing but τ_*. $\square$

Now we show our main result as a refined form of Lemma 3.1.

THEOREM 3.5. *Let $a = (a_1,\dots,a_n)$ and $b = (b_1,\dots,b_n)$ be n-tuples of positive numbers satisfying (1.2). Then with the same notations as in Lemma 3.1,*

$$S_{p,\lambda}(a,b) = \left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q} - \lambda\sum a_k b_k \leqslant nM_1M_2F_0(\lambda), \tag{3.34}$$

where $F_0(\lambda) = F_0(\lambda;\alpha,\beta,p)$ *is defined by*

$$F_0(\lambda) = \begin{cases} 1-\lambda & \text{if } 0<\lambda<\min\{\frac{K_\alpha}{p},\frac{K_\beta}{q}\} \\ \{\frac{1}{K_\alpha}(\frac{K}{\lambda})^p+\frac{1}{K_\beta}-1\}\lambda & \text{if } \frac{K_\beta}{q}(=\min\{\frac{K_\alpha}{p},\frac{K_\beta}{q}\}) \leqslant \lambda < K \\ \{\frac{1}{K_\alpha}+\frac{1}{K_\beta}(\frac{K}{\lambda})^q-1\}\lambda & \text{if } \frac{K_\alpha}{p}(=\min\{\frac{K_\alpha}{p},\frac{K_\beta}{q}\}) \leqslant \lambda < K \\ (\frac{1}{K_\alpha}+\frac{1}{K_\beta}-1)\lambda - c_\lambda(\frac{1}{1-\alpha^p}+\frac{1}{1-\beta^q}-1) & \text{if } K\leqslant\lambda\leqslant\tilde{K} \\ \{\frac{\alpha^p}{K_\alpha}+\frac{1}{K_\beta}(\frac{K}{\lambda})^q-\alpha\}\beta\lambda & \text{if } \tilde{K}<\lambda\leqslant\frac{\tilde{K}_\alpha}{p}(=\max\{\frac{\tilde{K}_\alpha}{p},\frac{\tilde{K}_\beta}{q}\}) \\ \{\frac{1}{K_\alpha}(\frac{K}{\lambda})^p+\frac{\beta^q}{K_\beta}-\beta\}\alpha\lambda & \text{if } \tilde{K}<\lambda\leqslant\frac{\tilde{K}_\beta}{q}(=\max\{\frac{\tilde{K}_\alpha}{p},\frac{\tilde{K}_\beta}{q}\}) \\ \alpha\beta(1-\lambda) & \text{if } \max\{\frac{\tilde{K}_\alpha}{p},\frac{\tilde{K}_\beta}{q}\}<\lambda\,. \end{cases} \tag{3.35}$$

Furthermore, the constant $nM_1M_2F_0(\lambda)$ *is the best bound of* $S_{p,\lambda}(a,b)$ *in a reasonable sense as stated in Remark 3.3.*

Proof. First let $K \leqslant \lambda \leqslant \tilde{K}$. Then we have $\tau_\beta \leqslant \tau_* \leqslant \tau_\alpha$ from Lemma 3.4, and $\phi(\tau) = \tilde{\phi}(\tau) = n\max\{G_\lambda(\tau), H_\lambda(\tau)\}$ from (3.12) and (3.21). Hence by the argument in the proof of Lemma 3.1 the upper bound $F_0(\lambda)(=F(\lambda))$ of $S_{p,\lambda}(a,b)$ is given by

$$\phi_0 = \tilde{\phi}_0 = G(\tau_*)\ (= H(\tau_*)).$$

Next let $0<\lambda<K$. Then from (3.14) and (3.21) (since $\lambda < \tilde{K}$)

$$\phi(\tau) = \begin{cases} nG_\lambda(\tau) & \text{if } 0<\tau\leqslant\tau_g \\ nI_\lambda(\tau) & \text{if } \tau_g<\tau<\tau_h \\ nH_\lambda(\tau) & \text{if } \tau_h\leqslant\tau\,, \end{cases} \quad \text{and} \quad \tilde{\phi}(\tau) = \begin{cases} nG_\lambda(\tau) & \text{if } 0<\tau\leqslant\tau_* \\ nH_\lambda(\tau) & \text{if } \tau_*<\tau\,. \end{cases}$$

Note that the function $L(\tau)$ (defined by (2.14)) is strictly decreasing and $L(\tau_h) < L(\tau_*) = 0 < L(\tau_g)$, so that we have $\tau_g < \tau_* < \tau_h$. Now, comparing $\phi(\tau)$ and $\tilde{\phi}(\tau)$, we can see that $\phi(\tau) \geqslant \tilde{\phi}(\tau)$. So by a simple computation (or by tracing the graph of $\phi(\tau)$), we can obtain its minimum $\phi_0 = F_0(\lambda)$ as follows (identical to (3.15)):

$$\phi_0 = \begin{cases} I_\lambda(\tau_g) & \text{if } \tau_I\leqslant\tau_g \\ I_\lambda(\tau_I) & \text{if } \tau_g<\tau_I<\tau_h \\ I_\lambda(\tau_h) & \text{if } \tau_h\leqslant\tau_I\,. \end{cases} \tag{3.36}$$

Finally let $\tilde{K} < \lambda$. Then from (3.22) and (3.12) (since $K<\lambda$)

$$\tilde{\phi}(\tau) = \begin{cases} nG_\lambda(\tau) & \text{if } 0<\tau\leqslant\tau_{\tilde{h}} \\ n\tilde{I}_\lambda(\tau) & \text{if } \tau_{\tilde{h}}<\tau<\tau_{\tilde{g}} \\ nH_\lambda(\tau) & \text{if } \tau_{\tilde{g}}\leqslant\tau \end{cases} \quad \text{and} \quad \phi(\tau) = \begin{cases} nG_\lambda(\tau) & \text{if } 0<\tau\leqslant\tau_* \\ nH_\lambda(\tau) & \text{if } \tau_*<\tau\,. \end{cases}$$

Here note $\tau_{\tilde{h}} < \tau_* < \tau_{\tilde{g}}$. Then, comparing $\tilde{\phi}(\tau)$ and $\phi(\tau)$, we can see that $\tilde{\phi}(\tau) \geqslant \phi(\tau)$. So, similarly as in the previous case we can obtain $\tilde{\phi}_0 = F_0(\lambda)$ as follows (identical to (3.23)):

$$\tilde{\phi}_0 = \begin{cases} \tilde{I}_\lambda(\tau_{\tilde{g}}) & \text{if } \tau_{\tilde{g}}\leqslant\tau_{\tilde{I}} \\ \tilde{I}_\lambda(\tau_{\tilde{I}}) & \text{if } \tau_{\tilde{h}}<\tau_{\tilde{I}}<\tau_{\tilde{g}} \\ \tilde{I}_\lambda(\tau_{\tilde{h}}) & \text{if } \tau_{\tilde{I}}\leqslant\tau_{\tilde{h}}\,. \end{cases} \tag{3.37}$$

Expressing ϕ_0 or $\tilde{\phi}_0$ in each case in terms of α, β, K_α, K_β and λ, we obtain (3.35). This completes the proof. □

THEOREM 3.6. *Let $F_0(\lambda)$ be the constant defined in Theorem 3.5. Then $F_0(\lambda)$ is a strictly decreasing continuous function on $[K,\tilde{K}]$, and the equation $F_0(\lambda)=0$ has a unique solution (denoted by $\lambda=\lambda_0$) in the interval.*

Proof. First if $0<\lambda<K(<1)$, then by (3.36) $F_0(\lambda)$ coincides with one of $I_\lambda(\tau_g)$, $I_\lambda(\tau_h)$ and $I_\lambda(\tau_I)$, and they are all positive because (by (iii))

$$\min\{I_\lambda(\tau_g),I_\lambda(\tau_h),I_\lambda(\tau_I)\}=I_\lambda(\tau_I)=1-\lambda>0,$$

so that $F_0(\lambda)>0$. Next if $(1<)\tilde{K}<\lambda$, then by (3.37) $F_0(\lambda)$ coincides with one of $\tilde{I}_\lambda(\tau_{\tilde{g}})$, $\tilde{I}_\lambda(\tau_{\tilde{h}})$ and $\tilde{I}_\lambda(\tau_{\tilde{I}})$, and they are all negative. In fact, from (2.7) and (3.24)

$$\begin{aligned}\tilde{I}_\lambda(\tau_{\tilde{g}})&=\left\{\frac{\alpha^p}{K_\alpha}+\frac{1}{K_\beta}\left(\frac{K}{\lambda}\right)^q-\alpha\right\}\beta\lambda\leqslant\left\{\frac{\alpha^p}{K_\alpha}+\frac{1}{K_\beta}\left(\frac{K}{\tilde{K}}\right)^q-\alpha\right\}\beta\lambda\\&=\left\{\frac{1}{p}\left(\frac{\alpha}{\theta_\alpha}\right)^{p-1}+\frac{1}{q}\left(\frac{\beta}{\theta_\beta}\right)^{q-1}-1\right\}\alpha\beta\lambda<0.\end{aligned}$$

Similarly, from (2.7) and (3.25)

$$\tilde{I}_\lambda(\tau_{\tilde{h}})=\left\{\frac{1}{K_\alpha}\left(\frac{K}{\lambda}\right)^p+\frac{\beta^q}{K_\beta}-\beta\right\}\alpha\lambda<0.$$

Furthermore, from (3.26) $\tilde{I}_\lambda(\tau_{\tilde{I}})=\alpha\beta(1-\lambda)<0$. Hence $F_0(\lambda)<0$. Now it suffices to see that $F_0(\lambda)$ is strictly decreasing or $F_0'(\lambda)<0$ for $\lambda\in(K,\tilde{K})$. By (3.9) or (3.10) the function $F_0(\lambda)$ is expressed on $[K,\tilde{K}]$ as follows:

$$\begin{aligned}F_0(\lambda)&=\phi_0=G_\lambda(\tau_*)\ (=\ H_\lambda(\tau_*))\\&=K\left(\frac{\alpha^p\tau_*^{1/q}}{K_\alpha}+\frac{\tau_*^{-1/p}}{K_\beta}\right)-\alpha\lambda.\end{aligned}$$

Here τ_* is the unique solution of the equation (2.10) depending on $\lambda\in[K,\tilde{K}]$. Write $\tau_*=\tau(\lambda)$ and $F_0(\lambda)=F^*(\tau(\lambda))=F^*(\lambda,\tau)$. Then we have by (2.10)

$$(1-\alpha)\left(1-\frac{K}{q}\tau^{\frac{1}{q}-1}\frac{d\tau}{d\lambda}\right)=(1-\beta)\left(1+\frac{K}{p}\tau^{-\frac{1}{p}-1}\frac{d\tau}{d\lambda}\right),$$

so that

$$\frac{d\tau}{d\lambda}=\frac{\tau^{\frac{1}{p}+1}(\beta-\alpha)}{K(\frac{1-\alpha}{q}\tau+\frac{1-\beta}{p})}.$$

Hence we have

$$\begin{aligned}\frac{d}{d\lambda}F_0(\lambda)&=\frac{\partial F^*(\lambda,\tau)}{\partial\tau}\frac{d\tau}{d\lambda}+\frac{\partial F^*(\lambda,\tau)}{\partial\lambda}\\&=K\left\{\frac{\alpha^p}{K_\alpha}\frac{1}{q}\tau^{\frac{1}{q}-1}+\frac{1}{K_\beta}\left(-\frac{1}{p}\right)\tau^{-\frac{1}{p}-1}\right\}\frac{d\tau}{d\lambda}-\alpha\\&=-\frac{\frac{\alpha(1-\alpha^{p-1}\beta)}{qK_\alpha}\tau+\frac{\beta(1-\alpha\beta^{q-1})}{pK_\beta}}{\frac{1-\alpha}{q}\tau+\frac{1-\beta}{p}}<0.\end{aligned}$$

This completes the proof. $\square$

REMARK 3.7. *(1) By a more precise calculation we can see that* $F_0(\lambda)$ *is a strictly decreasing continuous function on* $(0,\infty)$.
(2) For the unique solution $\lambda = \lambda_0$ *of the equation* $F_0(\lambda) = 0$, *the corresponding solution* $\tau = \tau_* = \tau(\lambda_0)$ *of (2.10) is in* $[\tau_\beta, \tau_\alpha]$ *from Lemma 3.4.*

4. Applications to difference inequalities

Recently the following theorem was given in [5].

THEOREM I. *Under the same assumption as in Theorem 3.5*

$$A_0 := \left(\sum a_k^p\right)^{1/p} \left(\sum b_k^q\right)^{1/q} - \sum a_k b_k \leqslant nM_1M_2F_0, \tag{4.1}$$

where $F_0 = \frac{1}{K_\alpha} + \frac{1}{K_\beta} - 1 - c_1\left(\frac{1}{1-\alpha^p} + \frac{1}{1-\beta^q} - 1\right)$, *and* c_1 *is given by* $\lambda = 1$ *in (2.11).*

In particular, the following inequalities hold.
(i) If $\beta \to 1$, *then*

$$\left(\frac{\sum a_k^p}{n}\right)^{1/p} - \frac{\sum a_k}{n} \leqslant M_1\left[\frac{1}{q}\left\{\frac{1-\alpha^p}{p(1-\alpha)}\right\}^{q-1} - \frac{\alpha-\alpha^p}{1-\alpha^p}\right]. \tag{4.2}$$

(ii) If $p = q = 2$, *then*

$$\left(\sum a_k^2\right)^{1/2} \left(\sum b_k^2\right)^{1/2} - \sum a_k b_k \leqslant nM_1M_2\frac{(1-\alpha\beta)^2}{2(1+\alpha)(1+\beta)}. \tag{4.3}$$

(iii) If $p = q = 2$ *and* $\beta \to 1$, *then*

$$\left(\frac{\sum a_k^2}{n}\right)^{1/2} - \frac{\sum a_k}{n} \leqslant \frac{(M_1-m_1)^2}{4(M_1+m_1)}. \tag{4.4}$$

Furthermore, it was shown in [6] that the constant $nM_1M_2F_0$ in (4.1) is best possible in a reasonable sense as stated in Remark 3.3. If we put $\lambda = 1$ in Theorem 3.5, then since $K < 1 < \tilde{K}$ we obtain (4.1) (with $F_0 = F_0(1)$) at once.

Now as an extension of (4.2), concerning the difference of the p-th power mean and the positive scalar multiple of the usual arithmetic mean of an n-tuple, we have the following theorem:

THEOREM 4.1. *Let* $a = (a_1, \dots, a_n)$ *be an* n*-tuple satisfying the same assumption as in Theorem 3.5. Then for any* $\lambda > 0$

$$A_1(\lambda) := \left(\frac{\sum a_k^p}{n}\right)^{1/p} - \lambda\frac{\sum a_k}{n} \leqslant M_1F_1(\lambda), \tag{4.5}$$

where $F_1(\lambda)$ *is defined by*

$$F_1(\lambda) = \begin{cases} 1-\lambda & \text{if } 0 < \lambda < \frac{K_\alpha}{p} \\ \frac{1}{q}\left\{\frac{1-\alpha^p}{p(1-\alpha)\lambda}\right\}^{q-1} - \frac{\alpha-\alpha^p}{1-\alpha^p}\lambda & \text{if } \frac{K_\alpha}{p} \leqslant \lambda \leqslant \frac{K_\alpha}{\alpha^{p-1}p} \\ \alpha(1-\lambda) & \text{if } \frac{K_\alpha}{\alpha^{p-1}p} < \lambda\,. \end{cases} \tag{4.6}$$

Proof. We shall obtain all facts, putting $M_2 = 1$ and $m_2 = b_1 = b_2 = \cdots = b_n = \beta \to 1$ in Theorem 3.5. (Instead we could give a direct proof, starting from (3.4) with $\beta = 1$.) From the inequality (3.34) or the one devided by n in the both sides, we can obtain (4.5), letting $\beta \to 1$ and writing $F_1(\lambda) = \lim_{\beta\to 1} F_0(\lambda,\alpha,\beta,p)$. To see that $F_1(\lambda)$ is given by (4.6), first note that $K_\beta/q, \tilde{K}_\beta/q \to 1$ (cf. (2.8), (2.9)), so that we have

$$\min\left\{\frac{K_\alpha}{p},\frac{K_\beta}{q}\right\} \to \frac{K_\alpha}{p} \quad \text{and} \quad \max\left\{\frac{\tilde{K}_\alpha}{p},\frac{\tilde{K}_\beta}{q}\right\} \to \frac{\tilde{K}_\alpha}{p}.$$

Furthermore,

$$K \to \left(\frac{K_\alpha}{p}\right)^{1/p} \quad \text{and} \quad \tilde{K} \to \left(\frac{K_\alpha}{\alpha^{p-1}p}\right)^{1/p}.$$

Hence we see that the seven cases with respect to λ in (3.35) are reduced to the following five cases:

(i) $0 < \lambda < \frac{K_\alpha}{p}$, (ii) $\frac{K_\alpha}{p} \leqslant \lambda < \left(\frac{K_\alpha}{p}\right)^{1/p}$, (iii) $\left(\frac{K_\alpha}{p}\right)^{1/p} \leqslant \lambda < \left(\frac{K_\alpha}{\alpha^{p-1}p}\right)^{1/p}$,
(iv) $\left(\frac{K_\alpha}{\alpha^{p-1}p}\right)^{1/p} \leqslant \lambda < \frac{K_\alpha}{\alpha^{p-1}p}$ and (v) $\frac{K_\alpha}{\alpha^{p-1}p} \leqslant \lambda$.

For each of the cases (i) and (v) we obtain the desired $F_1(\lambda)$ immediately. For the remaining cases (ii), (iii) and (iv) we obtain a common

$$F_1(\lambda) = \frac{1}{q}\left\{\frac{1-\alpha^p}{p(1-\alpha)\lambda}\right\}^{q-1} - \frac{\alpha-\alpha^p}{1-\alpha^p}\lambda.$$

In fact, write ψ the right-hand side of the above identity for brevity, then for both (ii) and (iv) we can obtain the same ψ as the limit of the corresponding $F_0(\lambda,\alpha,\beta,p)$ in (3.35). For (iii), since the equation (2.10) becomes $(1-\alpha)(\lambda - \left(\frac{K_\alpha}{p}\right)^{1/p}\tau^{1/q}) = 0$, so that

$$\tau_* \to \left\{\frac{\lambda}{(K_\alpha/p)^{1/p}}\right\}^q = \frac{\lambda^q}{(K_\alpha/p)^{q-1}}. \tag{4.7}$$

by continuity of the solution and $c_\lambda \to 0$. Hence we have

$$\frac{c_\lambda}{1-\beta^q} = \frac{(1-\beta)(\lambda - K\tau_*^{-1/p})}{1-\beta^q} \to \frac{1}{q}\left\{\lambda - \left(\frac{K_\alpha}{p\lambda}\right)^{q-1}\right\}.$$

Now it follows that

$$\begin{aligned}&\left(\frac{1}{K_\alpha}+\frac{1}{K_\beta}-1\right)\lambda - c_\lambda\left(\frac{1}{1-\alpha^p}+\frac{1}{1-\beta^q}-1\right)\\ &\to \left(\frac{1}{K_\alpha}+\frac{1}{q}-1\right)\lambda - \frac{1}{q}\left\{\lambda - \left(\frac{K_\alpha}{p\lambda}\right)^{q-1}\right\} = \psi.\end{aligned}$$

This completes the proof. □

REMARK 4.2. *The constant $M_1F_1(\lambda)$ defined in Theorem 4.1 is the best bound of the difference $A_1(\lambda)$ in a reasonable sense as stated in Remark 3.3, that is, for the case $\frac{K_\alpha}{p} \leqslant \lambda \leqslant \frac{K_\alpha}{\alpha^{p-1}p}$, if $x_0 = \frac{1-\left(\frac{K_\alpha}{p\lambda}\right)^p}{1-\alpha^p}n$ is an integer then $A_1(\lambda) = A_1(\lambda, a)$ attains $M_1F_1(\lambda)$ at*

$$a = (\overbrace{M_1, \ldots, M_1}^{n-x_0}, \overbrace{m_1, \ldots, m_1}^{x_0}).$$

Here the integer $x = x_0$ is obtained from the relation (cf. (3.28), (4.7))

$$\frac{qK_\alpha\{n-(1-\beta^q)y\}}{pK_\beta\{n-(1-\alpha^p)x\}} \to \frac{\lambda^q}{(K_\alpha/p)^{q-1}},$$

that is, from

$$\frac{nK_\alpha}{p\{n-(1-\alpha^p)x\}} = \frac{\lambda^q}{(K_\alpha/p)^{q-1}}.$$

For the other cases $0 < \lambda < \frac{K_\alpha}{p}$ and $\frac{K_\alpha}{\alpha^{p-1}p} < \lambda$, $A_1(\lambda)$ attains $M_1F_1(\lambda)$ at $a = (M_1, \ldots, M_1)$ and $(m_1, \ldots, m_1)$, respectively.

Next for $p = q = 2$, Theorem 3.5 implies the following theorem:

THEOREM 4.3. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be n-tuples satisfying (1.2). Put $\alpha = \min\{\frac{m_1}{M_1}, \frac{m_2}{M_2}\}, \beta = \max\{\frac{m_1}{M_1}, \frac{m_2}{M_2}\}, \gamma = \frac{(1+\alpha)^{1/2}(1+\beta)^{1/2}}{2}$ and $\tilde{\gamma} = \frac{\gamma}{\alpha^{1/2}\beta^{1/2}}$. Write c'_λ the constant of (2.11) with respect to $p = q = 2$. Then for any $\lambda > 0$*

$$A_2(\lambda) := \left(\sum a_k^2\right)^{1/2}\left(\sum b_k^2\right)^{1/2} - \lambda \sum a_k b_k \leqslant nM_1M_2F_2(\lambda), \tag{4.8}$$

where $F_2(\lambda)$ is a constant defined by

$$F_2(\lambda) = \begin{cases} 1-\lambda & \text{if } 0 < \lambda < \frac{1+\alpha}{2} \\ \left(\frac{1+\alpha}{4\lambda^2} - \frac{\alpha}{1+\alpha}\right)\lambda & \text{if } \frac{1+\alpha}{2} \leqslant \lambda < \gamma \\ \frac{(1-\alpha\beta)\lambda}{(1+\alpha)(1+\beta)} - c'_\lambda\left\{\frac{1-\alpha^2\beta^2}{(1-\alpha^2)(1-\beta^2)}\right\} & \text{if } \gamma \leqslant \lambda \leqslant \tilde{\gamma} \\ \left(\frac{1+\alpha}{4\lambda^2} - \frac{\alpha}{1+\alpha}\right)\beta\lambda & \text{if } \tilde{\gamma} < \lambda \leqslant \frac{1+\alpha}{2\alpha} \\ \alpha\beta(1-\lambda) & \text{if } \frac{1+\alpha}{2\alpha} < \lambda. \end{cases} \tag{4.9}$$

In particular, if $\lambda = 1$ then we have the inequality (4.3) in Theorem I.

Proof. Let $p = q = 2$ in Theorem 3.5 (3.34). Then the condition $\alpha < \beta$ implies $\frac{K_\alpha}{p} = \frac{1+\alpha}{2} < \frac{1+\beta}{2} = \frac{K_\beta}{q}$, so that we have $\min\left\{\frac{K_\alpha}{p}, \frac{K_\beta}{q}\right\} = \frac{1+\alpha}{2}$. Similarly, we have $\max\left\{\frac{\tilde{K}_\alpha}{p}, \frac{\tilde{K}_\beta}{q}\right\} = \frac{1+\alpha}{2\alpha}$. Moreover, $K = \gamma$ and $\tilde{K} = \tilde{\gamma}$. Now using these facts, we can obtain (4.8) and (4.9) by an elementary calculation. In particular, if $\lambda = 1$ then $\gamma \leqslant 1 \leqslant \tilde{\gamma}$, so that

$$\left(\sum a_k^2\right)^{1/2}\left(\sum b_k^2\right)^{1/2} - \sum a_k b_k \leqslant nM_1M_2\frac{1-\alpha\beta}{(1+\alpha)(1+\beta)} - c_1'\left\{\frac{1-\alpha^2\beta^2}{(1-\alpha^2)(1-\beta^2)}\right\}.$$

Since the equation (2.10) becomes

$$(1-\alpha)(1-\gamma\tau^{1/2}) = (1-\beta)(1-\gamma\tau^{-1/2}),$$

we have the solution $\tau = \tau_* = (1+\beta)/(1+\alpha)$. Hence we have

$$c_1' = (1-\alpha)(1-\gamma\tau_*^{1/2}) = \frac{1}{2}(1-\alpha)(1-\beta),$$

from which we obtain the inequality (4.3) in Theorem I. □

In the above theorem, if $\lambda = 1$, then a saddle point (Remark 3.3) of $T_\lambda(x,y;\tau)$ (and also $\tilde{T}_\lambda(x,y;\tau)$) is given as follows (see. [5, Theorem 3.1]):

$$((x_0,y_0),\tau_0) = \left(\left(\frac{1+2\beta+\alpha\beta}{2(1+\alpha)(1+\beta)}n, \frac{1+2\alpha+\alpha\beta}{2(1+\alpha)(1+\beta)}n\right), \frac{1+\beta}{1+\alpha}\right).$$

Putting $p = 2$ in Theorem 4.1, or letting $\beta \to 1$ in Theorem 4.3, we have

COROLLARY 4.4. *Under the same assumption as in Theorem 4.1,*

$$A_3(\lambda) := \left(\frac{\sum a_k^2}{n}\right)^{1/2} - \lambda\frac{\sum a_k}{n} \leqslant M_1F_3(\lambda), \tag{4.10}$$

where $F_3(\lambda)$ *is a constant defined by*

$$F_3(\lambda) = \begin{cases} 1-\lambda & \text{if } 0 < \lambda < \frac{1+\alpha}{2} \\ \frac{1+\alpha}{4\lambda} - \frac{\alpha}{1+\alpha}\lambda & \text{if } \frac{1+\alpha}{2} \leqslant \lambda \leqslant \frac{1+\alpha}{2\alpha} \\ \alpha(1-\lambda) & \text{if } \frac{1+\alpha}{2\alpha} < \lambda\,. \end{cases}$$

In particular, if $\lambda = 1$ *then we obtain the inequality (4.4) in Theorem I.*

5. Applications to ratio inequalities

In this section we shall show some ratio inequalities which are induced by putting $F_0(\lambda) = 0$. It was shown in Theorem 3.6 and Remark 3.7 that the equation $F_0(\lambda) = 0$ has a unique solution $\lambda = \lambda_0 \in [K, \tilde{K}]$, and that the corresponding $\tau = \tau_* = \tau(\lambda_0)$ lies in $[\tau_\beta, \tau_\alpha]$. First of all, we shall give the explicit expressions of the constants λ_0 and $\tau(\lambda_0)$ in terms of p, q, α and β. In preparation, we cite the following result due to Gheorghiu [3] (or reverse Hölder's inequality [9, p.685]):

THEOREM G. *Under the same assumption as in Theorem 3.5,*

$$\left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q} \leqslant \lambda_\sharp \sum a_k b_k, \tag{5.1}$$

where

$$\lambda_\sharp = \frac{1-\alpha^p\beta^q}{p^{1/p}q^{1/q}(\beta-\alpha\beta^q)^{1/p}(\alpha-\alpha^p\beta)^{1/q}}.$$

Put

$$\tau_\sharp = \frac{(1-\beta)(1-\alpha^p)(\beta-\alpha\beta^q)}{(1-\alpha)(1-\beta^q)(\alpha-\alpha^p\beta)}.$$

Then concerning $\lambda_\sharp$ and $\tau_\sharp$, we have the following lemma:

LEMMA 5.1. *Under the same assumption as before, the following properties hold:*
(i) $K \leqslant \lambda_\sharp \leqslant \tilde{K}$.
(ii) $\tau = \tau_\sharp$ *is the (unique) solution of (2.10) for* $\lambda = \lambda_\sharp$, *and* $\tau_\beta \leqslant \tau_\sharp \leqslant \tau_\alpha$.
(iii) $F_0(\lambda_\sharp) = 0$, *that is,* $\lambda = \lambda_\sharp$ *is the (unique) solution of the equation* $F_0(\lambda) = 0$.

Hence we see that $\lambda_\sharp$ and λ_0 are identical from Theorem 3.6, and that $\tau_\sharp = \tau(\lambda_0)$ from Lemma 2.3. Consequently we would have proved Theorem G.

Proof of Lemma 5.1. For (i), first to see $\lambda_\sharp \leqslant \tilde{K}$, putting $\gamma_1 \equiv \beta^{q-1}$, we have from convexity of the function $f(t) = t^p \quad (t > 0)$,

$$\frac{1-\alpha^p\beta^q}{1-\alpha\beta^{q-1}} = \frac{1-(\alpha\gamma_1)^p}{1-\alpha\gamma_1} \leqslant \frac{1-\alpha^p}{1-\alpha},$$

because $0 < \alpha\gamma_1 < \alpha < 1$. Similarly, we have

$$\frac{1-\alpha^p\beta^q}{1-\alpha^{p-1}\beta} \leqslant \frac{1-\beta^q}{1-\beta}.$$

Hence

$$\lambda_\sharp = \frac{1}{\alpha^{1/q}\beta^{1/p}}\left\{\frac{1-\alpha^p\beta^q}{p(1-\alpha\beta^{q-1})}\right\}^{1/p}\left\{\frac{1-\alpha^p\beta^q}{q(1-\alpha^{p-1}\beta)}\right\}^{1/q}$$
$$\leqslant \frac{1}{\alpha^{1/q}\beta^{1/p}}\left\{\frac{1-\alpha^p}{p(1-\alpha)}\right\}^{1/p}\left\{\frac{1-\beta^q}{q(1-\beta)}\right\}^{1/q} = \tilde{K}.$$

To see $K \leqslant \lambda_\sharp$, putting $\gamma_2 = \frac{1}{\beta^{q-1}}$, we have similarly as before,

$$\frac{1-\alpha^p\beta^q}{\beta-\alpha\beta^q} = \frac{\frac{1}{\beta^q}-\alpha^p}{\frac{1}{\beta^{q-1}}-\alpha} = \frac{\gamma_2^p-\alpha^p}{\gamma_2-\alpha} \geqslant \frac{1-\alpha^p}{1-\alpha},$$

because $\gamma_2 > 1 > \alpha > 0$. We also have

$$\frac{1-\alpha^p\beta^q}{\alpha-\alpha^p\beta} \geqslant \frac{1-\beta^q}{1-\beta}.$$

Hence

$$\lambda_\sharp = \left\{\frac{1-\alpha^p\beta^q}{p(\beta-\alpha\beta^q)}\right\}^{1/p}\left\{\frac{1-\alpha^p\beta^q}{q(\alpha-\alpha^p\beta)}\right\}^{1/q} \geqslant \left\{\frac{1-\alpha^p}{p(1-\alpha)}\right\}^{1/p}\left\{\frac{1-\beta^q}{q(1-\beta)}\right\}^{1/q} = K.$$

For (ii), computing the both sides of (2.10) for $\lambda = \lambda_\sharp$ and $\tau = \tau_\sharp$ directly, we obtain

$$(1-\alpha)(\lambda_\sharp - K\tau_\sharp^{1/q}) = c_\sharp = (1-\beta)(\lambda_\sharp - K\tau_\sharp^{-1/p}),$$

where

$$c_\sharp = \frac{(1-\alpha)(1-\beta)-(\alpha-\alpha^p)(\beta-\beta^q)}{p^{1/p}q^{1/q}(\beta-\alpha\beta^q)^{1/p}(\alpha-\alpha^p\beta)^{1/q}}.$$

Another property $\tau_\beta \leqslant \tau_\sharp \leqslant \tau_\alpha$ is now clear from Lemma 3.4.

For (iii), by Theorem 3.5 and (i), we can show

$$F_0(\lambda_\sharp) = \left(\frac{1}{K_\alpha}+\frac{1}{K_\beta}-1\right)\lambda_\sharp - c_\sharp\left(\frac{1}{1-\alpha^p}+\frac{1}{1-\beta^q}-1\right) = 0, \tag{5.2}$$

which implies the assertion (iii) from Theorem 3.6. □

Now we define $B_0(a,b,p)$ by

$$B_0(a,b,p) = \frac{\left(\sum a_k^p\right)^{1/p}\left(\sum b_k^q\right)^{1/q}}{\sum a_k b_k}.$$

Then fom Lemma 5.1 and Theorem 3.5 we have the following fact with respect to Gheorghiu's inequality (5.1).

THEOREM 5.2. *Under the same assumption as in Theorem 3.5, $\lambda_0(=\lambda_\sharp)$ is the best bound of the ratio $B_0(a,b,p)$ in a reasonable sense as stated in Remark 3.3. More precisely if there exists an integer point (x,y) in Δ such that*

$$\frac{qK_\alpha\{n-(1-\beta^q)y\}}{pK_\beta\{n-(1-\alpha^p)x\}} = \tau_\sharp \quad \textit{and} \quad x+y=n,$$

then $B_0(a,b,p)$ attains $\lambda_\sharp$.

In fact (say, for Case I), we have by Remark 3.3

$$\begin{aligned} T_{\lambda_\sharp}(x,y;\tau_\sharp) &= nK\left(\frac{\tau_\sharp^{1/q}}{K_\alpha}+\frac{\tau_\sharp^{-1/p}}{K_\beta}\right) - n\lambda_\sharp \\ &\quad + (1-\alpha)(\lambda_\sharp - K\tau_\sharp^{1/q})x + (1-\beta)(\lambda_\sharp - K\tau_\sharp^{-1/p})y \\ &= \frac{(1-\alpha)(1-\beta)-(\alpha-\alpha^p)(\beta-\beta^q)}{p^{1/p}q^{1/q}(\beta-\alpha\beta^q)^{1/p}(\alpha-\alpha^p\beta)^{1/q}}(x+y-n) \quad (\leqslant 0), \end{aligned}$$

so that the maximum of $T_{\lambda_\sharp}(x,y;\tau_\sharp)$ is obtained whenever $x+y=n$.

Now for the ratio of the p-th power mean by the usual arithmetic mean of an n-tuple, we have the following corollary:

COROLLARY 5.3. *Under the same assumption as in Theorem 4.1,*

$$B_1(a) := \frac{\left(\sum a_k^p\right)^{1/p} n^{1/q}}{\sum a_k} \leqslant \frac{1-\alpha^p}{p^{1/p}q^{1/q}(1-\alpha)^{1/p}(\alpha-\alpha^p)^{1/q}} (= \lambda_1). \tag{5.3}$$

Proof. In Theorem 5.2 or Theorem G, we have only to put $M_2 = 1$ and $m_2 = \beta$, and let $\beta \to 1$. □

Here we note that the constant λ_1 coincides with the p-th root of the constant given by Ky Fan [2].

Next, putting $p = q = 2$ in Theorem 5.2 or Theorem G, we obtain the following Pólya-Szegö inequality ([11], [9, p. 684]):

COROLLARY 5.4. *Under the same assumption as in Theorem 3.5,*

$$B_2(a,b) := \frac{\left(\sum a_k^2\right)^{1/2}\left(\sum b_k^2\right)^{1/2}}{\sum a_k b_k} \leqslant \frac{1+\alpha\beta}{2\alpha^{1/2}\beta^{1/2}}. \tag{5.4}$$

In particular, we obtain the following result due to Kantorovich from the above corollary, or from Corollary 5.3 for $p = 2$.

COROLLARY 5.5. *Under the same assumption as in Corollary 5.3,*

$$B_3(a) := \frac{\left(n\sum a_k^2\right)^{1/2}}{\sum a_k} \leqslant \frac{1+\alpha}{2\alpha^{1/2}}. \tag{5.5}$$

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