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REVERSED JENSEN TYPE INTEGRAL INEQUALITIES FOR MONOTONE FUNCTIONS

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Abstract. Reversed Jensen type integral inequalities for monotone functions are deduced from a more general inequality. Special cases are of interest in the study of Volterra integral operators.

1. Introduction and statement of results

The integral version of the classical Jensen inequality is

$$g\left(\frac{1}{\sigma(I)}\int f\,d\sigma\right) \leqslant \frac{1}{\sigma(I)}\int_I g(f)\,d\sigma \tag{1.1}$$

where I is an interval in $\mathbb{R}$, g is convex in $\mathbb{R}$, σ is a positive measure in I and $f, g(f) \in L^1(I,\sigma)$.

This paper concerns inequalities similar to (1.1), with g convex and f monotone, but with the inequality reversed, specifically inequalities of the type

$$\int_0^x g(f(s))k(s)d\sigma(s) \leqslant \left(\int_0^x k(s)d\sigma(s)\right) g\left(\frac{1}{x}\int_0^x f(s)ds\right) \tag{1.2}$$

and

$$\int_0^x g(f(s))k(x-s)ds \leqslant \left(\int_0^x k(s)ds\right) g\left(\frac{1}{x}\int_0^x f(s)ds\right), \tag{1.3}$$

for $0 < x \leqslant b$.

There is an extensive literature on inequalities of reverse Hölder and Jensen type. Recent work is cited in papers by Bergh [3], Heinig and Maligranda [5], Barza, Pečarić and Persson [1] and Pečarić, Perić and Persson [7].

The special case of (1.3),

$$\alpha\int_0^x (x-s)^{\alpha-1} f(s)^\alpha ds \leqslant \left(\int_0^x f(s)ds\right)^\alpha, \quad 0 \leqslant x \leqslant 1, \tag{1.4}$$

with f increasing and $\alpha \geqslant 1$, has attracted some attention with proofs by Walter and Weckesser [8], Egorov [4] and Jagers [6].

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Inequalities (1.2) and (1.3) are consequences of the general inequality (1.6) of Theorem 1, proved in §2. A second proof of (1.3) is given in §3, allowing an analysis of the inequality

$$\beta \int_0^x (x-s)^{\beta-1} f(s)^\alpha ds \leqslant x^{\beta-\alpha} \left(\int_0^x f(s) ds \right)^\alpha, \quad 0 < x \leqslant 1, \tag{1.5}$$

for increasing or decreasing f and different choices of α and β.

Throughout the paper increasing and decreasing will mean respectively non-decreasing and non-increasing.

THEOREM 1.

(i) Let f be increasing and non-negative in $[0,b]$ and let g be convex in $[0,\infty)$ with $g(0) = 0$.

(ii) Let $k \in L^1[0,b]$ be non-negative in $[0,b]$ and let σ be a positive measure in $[0,b]$.

Let

$$F(x) = \int_0^x f(s) ds, \quad 0 \leqslant x \leqslant b.$$

Then

$$\int_0^x g(f(s)) k(s) d\sigma(s) \leqslant \sup_{0<c\leqslant 1} \left\{ g(F(x)/cx) \int_{(1-c)x}^x k(s) d\sigma(s) \right\} \tag{1.6}$$

for $0 \leqslant x \leqslant b$.

In the statement of the corollaries

$$R = \{(x,y) : 0 \leqslant x \leqslant b,\ y > 0\}.$$

COROLLARY 1. *Assuming (i) and (ii) of the Theorem, if*

$$g(y/c) \int_{(1-c)x}^x k(s) d\sigma(s) \leqslant g(y) \int_0^x k(s) d\sigma(s) \tag{1.7}$$

for $(x,y) \in R$ and $0 < c \leqslant 1$, then inequality (1.2) is valid.

COROLLARY 2.

(i) Let g be convex in $[0,\infty)$ with $g(0) = 0$, let $k \in L^1[0,b]$ be non-negative in $[0,b]$ and let

$$K(x) = \int_0^x k(s) ds, \quad 0 \leqslant x \leqslant b.$$

(ii) Suppose that

$$g(y/c) K(cx) \leqslant g(y) K(x), \tag{1.8}$$

for $(x,y) \in R$ and $0 \leqslant c < 1$.

Then, if f is increasing and non-negative in $[0,b]$,

$$\int_0^x k(x-s) g(f(s)) ds \leqslant K(x) g\left(\frac{1}{x} F(x)\right), \quad 0 < x \leqslant b, \tag{1.9}$$

and if f is decreasing and non-negative in $[0,b]$,

$$\int_0^x k(s)g(f(s))ds \leqslant K(x)g\left(\frac{1}{x}F(x)\right), \quad 0 < x \leqslant b. \tag{1.10}$$

Remarks. (i) The restriction $g(0) = 0$ can be removed by applying the theorem to the function $G(x) = g(x) - g(0)$ and making the appropriate changes to the stated results.

(ii) If g is concave, applying the theorem to $-g$ gives (1.2) – (1.10) with the inequalities reversed.

(iii) Walter and Wecknesser [8] proved that if f is increasing and non-negative in $[0,b]$, g is convex in $[0,\infty)$, $k \in L^1[0,b]$ and the function

$$h_c(y) = g(cy) - g(y)K(c)$$

is increasing for $y \geqslant 0$ and $0 \leqslant c \leqslant b$, then

$$\int_0^x k(x-s)g(f(s))ds \leqslant g\left(\int_0^x f(s)ds\right), \quad 0 \leqslant x \leqslant b. \tag{1.11}$$

Taking $g(y) = y^\alpha$, $K(x) = x^\beta$ and $1 \leqslant \alpha \leqslant \beta$ in (1.11) gives

$$\beta\int_0^x (x-s)^{\beta-1}f(s)^\alpha ds \leqslant \left(\int_0^x f(s)ds\right)^\alpha, \quad 0 \leqslant x \leqslant 1,$$

an apparently weaker inequality than (1.5).

2. Proof of Theorem 1

Take $x \in (0,b]$. We can suppose that $F(x) > 0$; otherwise the result is trivial. Suppose at first that $f(0) = 0$, so that

$$F(x) = \int_0^x (x-t)df(t),$$

and $d\mu(t) = \{(x-t)/F(x)\}\, df(t)$ is a probability measure in $[0,x]$. Then

$$f(s) = \int_0^x \frac{F(x)}{(x-t)}\chi_{(0,s)}(t)d\mu(t),$$

where χ_I is the characteristic function of the interval I, and by Jensen's inequality

$$g(f(s)) \leqslant \int_0^x g\left(\frac{F(x)}{(x-t)}\chi_{(0,s)}(t)\right)d\mu(t).$$

Thus, since $g(0) = 0$,

$$g(f(s)) \leqslant \int_0^s g\left(\frac{F(x)}{(x-t)}\right)d\mu(t), \qquad \text{for } 0 \leqslant s \leqslant x,$$

and hence

$$\int_0^x g(f(s))k(s)d\sigma(s) \leqslant \int_0^x d\mu(t) \int_t^x g\left(\frac{F(x)}{(x-t)}\right) k(s)d\sigma(s),$$

after reversing the order of integration. Hence

$$\int_0^x g(f(s))k(s)d\sigma(s) \leqslant \sup_{0\leqslant t<x} \int_t^x g\left(\frac{F(x)}{(x-t)}\right) k(s)d\sigma(s)$$

and (1.3) follows on taking $c = 1-(t/x)$.

The restriction $f(0) = 0$ can be removed by standard density arguments using dominated convergence of a sequence (f_n) of increasing functions with $f_n(0) = 0$ for $n \in \mathbb{N}$ and such that $f_n \to f(x)$ for $x \in (0,b]$.

The proofs of the corollaries are straightforward. For Corollary 2, fix x in $(0,b]$ and observe that $f(x-s)$ is a decreasing function of s in $[0,x]$ if and only if $f(s)$ is an increasing function of s in $[0,x]$, and hence (1.9) and (1.10) are equivalent. □

3. An alternative proof of Corollary 2

In this section we give an alternative proof of Corollary 2, making use of the observations that g is differentiable a.e. and $g(y)/y$ is increasing in $(0,\infty)$.

Proof. Let $\delta, y > 0$. Taking $c = y/(y+\delta)$ in (1.8),

$$g(y+\delta)K(xy/(y+\delta)) \leqslant g(y)K(x).$$

Hence

$$[g(y+\delta)-g(y)]\,K\left(\frac{xy}{y+\delta}\right) \leqslant g(y)\left[K(x) - K\left(x - \frac{x}{y+\delta}\delta\right)\right],$$

and dividing by δ and letting δ tend to zero, we obtain

$$yg'(y)K(x) \leqslant xg(y)k(x), \tag{3.1}$$

for $(x,y) \in R$ and a. e. $y > 0$.

To prove (1.10), assume that f is decreasing and non-negative in $[0,b]$, that is, for some b_0 in $(0,b]$,

$$f > 0 \text{ in } [0,b_0), \quad f = 0 \quad \text{in} \quad [b_0,b). \tag{3.2}$$

and so

$$h(x) := F(x)/x \geqslant f(x) > 0 \quad \text{in} \quad I_0 = [0,b_0), \tag{3.3}$$

while $h(0) = \lim_{x\to 0} F(x)/x = f(0)$.

Let

$$\Delta(x) = xg(h(x))k(x) - h(x)g'(h(x))K(x)$$

and

$$\phi(x) = K(x)g(h(x)) - \int_0^x k(s)g(f(s))ds,$$

for $0 \leqslant x \leqslant b$. Then $\phi(0) = \phi'(0) = 0$,

$$\phi'(x) = k(x)f(x)\left\{\frac{g(h(x))}{h(x)} - \frac{g(f(x))}{f(x)}\right\} + \frac{\Delta(x)}{F(x)}\{h(x) - f(x)\}, \quad 0 < x < b_0,$$

and

$$\phi'(x) = k(x)g(A/x) - K(x)g'(A/x)(A/x^2), \quad b_0 \leqslant x \leqslant b.$$

From (3.1) and (3.3), $\phi'(x) \geqslant 0$ a. e. in $[0,b]$ and inequality (1.10) follows immediately.

To establish (1.9), fix x in $(0,b]$ and replace $f(s)$ in (1.10) by $f(x-s)$. □

4. An application

In this section we consider the inequalities

$$\beta \int_0^x s^{\beta-1} f(s)^\alpha ds \leqslant x^{\beta-\alpha}\left(\int_0^x f(s)ds\right)^\alpha, \quad 0 \leqslant x \leqslant b, \tag{4.1}$$

and

$$\beta \int_0^x (x-s)^{\beta-1} f(s)^\alpha ds \geqslant x^{\beta-\alpha}\left(\int_0^x f(s)ds\right)^\alpha, \quad 0 \leqslant x \leqslant b. \tag{4.2}$$

Inequality (4.1) is (1.10) with $g(y) = y^\alpha$ and $K(x) = x^\beta$, which gives, in the notation of the previous section,

$$\phi'(x) = \beta x^{\beta-1} f(x)\left\{h(x)^{\alpha-1} - f(x)^{\alpha-1}\right\} + (\beta-\alpha)x^{\beta-\alpha}F(x)^{\alpha-1}\{h(x) - f(x)\}.$$

Using this expression for ϕ' it is easy to verify the following:

1. Let $\beta \geqslant \alpha \geqslant 1$: then, if f is decreasing and non-negative (4.1) and (4.2) are valid, and if f is increasing and non-negative then inequalities (4.1) and (4.2) are reversed.

2. Let f be decreasing and non-negative and not identically zero. There is equality in (4.1) when $\alpha = \beta > 1$ only if

$$f(x) = \begin{cases} \mu, & 0 < x \leqslant b_0, \\ 0, & b_0 < x \leqslant b, \end{cases}$$

and when $\beta > \alpha > 1$ only if

$$f(x) = \mu \quad (0 < x \leqslant b)$$

where μ is a constant.

3. Let f be increasing and non-negative and not identically zero. There is equality in (4.2) when $\alpha = \beta > 1$ only if

$$f(x) = \begin{cases} 0, & 0 < x \leqslant b_0, \\ \mu, & b_0 < x \leqslant b, \end{cases}$$

and when $\beta > \alpha > 1$ only if

$$f(x) = \mu \quad (0 < x \leqslant b),$$

where μ is a constant.

REFERENCES

[1] S. BARZA, J. PEČARIĆ AND L. E. PERSSON,, *Reversed Hölder type inequalities for monotone functions of several variables*, Math. Nachr. **186** (1997), 67–80.

[2] G. BENNETT, *Factorising the classical inequalities*, Mem. Amer. Math. Soc. **120** (1996).

[3] J. BERGH, *A converse inequality of Hölder type,*, Math. Z. **215** (1994), 205–208.

[4] YU. V. EGOROV, *On an integral inequality, Operator Theory: Advances and Applications, to appear.*

[5] H. HEINIG AND L. MALIGRANDA, *Weighted inequalities for monotone and concave functions*, Studia Math. **116 (2)** (1995), 133–165.

[6] A. A. JAGERS, *An integral inequality of Bushell and Okrasinski, to appear.*

[7] J. PEČARIĆ, I. PERIĆ AND L. E. PERSSON, *Integral inequalities for monotone functions*, J. Math. Anal. and Appl. **215** (1997), 235–251.

[8] W. WALTER AND V. WECKESSER, *An integral inequality of convolution type*, Aequationes Math. **46** (1993), 212–219.

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