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TWO–POINT OSTROWSKI INEQUALITY

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Abstract. A generalization of a classical Ostrowski inequality is proved. As a consequence, an improvement of a recent result of Barnett and Dragomir is given.

1. Introduction

In 1938, A. Ostrowski proved the following integral inequality [3]:

THEOREM 1. *Let* $f : [a,b] \to \mathbb{R}$ *is continuous on* $[a,b]$ *and differentiable on* (a,b). *If the derivative* f' *is bounded on* (a,b), *that is*

$$\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty,$$

then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqslant \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1}$$

for all $x \in [a,b]$.

Note that (1) can be given in the equivalent form

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \leqslant \frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty. \tag{2}$$

For a further details on Ostrowski inequality and its generalizations see [2, Chapter XV].

Recently, N. S. Barnett and S. S. Dragomir [1] proved the following result:

THEOREM 2. *If* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$ *and if* $[c,d] \subset [a,b]$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{d-c}\int_c^d f(s)\mathrm{d}s\right| \tag{3}$$
$$\leqslant \left\{\frac{b-a}{4} + \frac{d-c}{2} + \frac{1}{b-a}\left[\left|\frac{c+d}{2} - \frac{a+b}{2}\right| - \frac{d-c}{2}\right]^2\right\}\|f'\|_\infty.$$

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Note that for $c=d=x$ we can assume $\frac{1}{d-c}\int_c^d f(s)\mathrm{d}s = f(x)$, as a limit case, so that (3) reduces to the Ostrowski inequality (1).

As a main result in this paper we prove a new inequality which gives estimate for $\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{d-c}\int_c^d f(s)\mathrm{d}s\right|$ tighter than the estimate given by (3). This new inequality can be regarded as a natural generalization of Ostrowski inequality (1) and we call it two-point Ostrowski inequality. We show that this two-point Ostrowski inequality is sharp in the sense that there exists a function for which the equality is attained. This is not true for the inequality (3) in which $\leqslant$ should be replaced by $<$. Also, we establish our main result for a class of Lipschitzian functions rather than for a class of functions with bounded first derivatives.

2. Main result

First we note that the assumption on differentiability of f and boundedness of f' on (a,b) can be replaced by a weaker assumption that f is M-Lipschitzian on $[a,b]$, that is

$$|f(t)-f(s)| \leqslant M|t-s|, \quad \forall t,s \in [a,b],$$

and (1) (equivalently (2)) with $\|f'\|_\infty$ replaced by M remains valid (see [2, p.470]). Consequently the inequality (3) with $\|f'\|_\infty$ replaced by a constant $M>0$ is valid for any function $f:[a,b]\to\mathbb{R}$ which is M-Lipschitzian on $[a,b]$. This is easily seen by reviewing the proof of (3) in [1] since (3) has been obtained there by a direct application of Ostrowski inequality (1).

Now we state and prove our main result.

THEOREM 3. *Let $a,b,c,d\in\mathbb{R}$, are such that*

$$a\leqslant c<d\leqslant b, \quad c-a+b-d>0.$$

(i) *If $f:[a,b]\to\mathbb{R}$ is M-Lipschitzian on $[a,b]$, with some constant $M>0$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{d-c}\int_c^d f(s)\mathrm{d}s\right| \leqslant \frac{(c-a)^2+(b-d)^2}{2(c-a+b-d)}M. \tag{4}$$

(ii) *If $f_0:[a,b]\to\mathbb{R}$ is defined as*

$$f_0(t)=|t-s_0|, \quad t\in[a,b],$$

where

$$s_0=\frac{bc-ad}{c-a+b-d}, \tag{5}$$

then f_0 is 1-Lipschitzian on $[a,b]$ and we have

$$\left|\frac{1}{b-a}\int_a^b f_0(t)\mathrm{d}t - \frac{1}{d-c}\int_c^d f_0(s)\mathrm{d}s\right| = \frac{(c-a)^2+(b-d)^2}{2(c-a+b-d)}. \tag{6}$$

Proof. By the substitution

$$t = \frac{b-a}{d-c}s - \frac{bc-ad}{d-c}, \quad s \in [c,d]$$

we get

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = \frac{1}{d-c}\int_c^d f\left(\frac{b-a}{d-c}s - \frac{bc-ad}{d-c}\right)\mathrm{d}s.$$

Therefore,

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{d-c}\int_c^d f(s)\mathrm{d}s\right| \\
&= \frac{1}{d-c}\left|\int_c^d \left[f\left(\frac{b-a}{d-c}s - \frac{bc-ad}{d-c}\right) - f(s)\right]\mathrm{d}s\right| \\
&\leqslant \frac{1}{d-c}\int_c^d \left|f\left(\frac{b-a}{d-c}s - \frac{bc-ad}{d-c}\right) - f(s)\right|\mathrm{d}s \\
&\leqslant \frac{M}{d-c}\int_c^d \left|\frac{b-a}{d-c}s - \frac{bc-ad}{d-c} - s\right|\mathrm{d}s \\
&= \frac{M}{d-c}\int_c^d \left|\frac{c-a+b-d}{d-c}s - \frac{bc-ad}{d-c}\right|\mathrm{d}s \\
&= \frac{M(c-a+b-d)}{(d-c)^2}\int_c^d |s-s_0|\,\mathrm{d}s, \qquad (7)
\end{aligned}$$

where s_0 is given by (5). Further, we have

$$s_0 - c = \frac{d-c}{c-a+b-d}(c-a) \geqslant 0$$

and

$$d - s_0 = \frac{d-c}{c-a+b-d}(b-d) \geqslant 0,$$

which implies that $s_0 \in [c,d]$ and

$$\begin{aligned}
\int_c^d |s-s_0|\,\mathrm{d}s &= \int_c^{s_0}(s_0-s)\mathrm{d}s + \int_{s_0}^d (s-s_0)\mathrm{d}s \\
&= \frac{1}{2}\left[(s_0-c)^2 + (d-s_0)^2\right] \\
&= \frac{(d-c)^2}{2(c-a+b-d)^2}\left[(c-a)^2 + (b-d)^2\right]. \qquad (8)
\end{aligned}$$

Substituting this in (7) we get (4).

Obviously we have

$$|f_0(t) - f_0(s)| = |t-s|$$

when $t,s \in [a,s_0]$ or $t,s \in [s_0,b]$, while for $t \in [a,s_0]$ and $s \in [s_0,b]$ we have

$$|f_0(t) - f_0(s)| = |2s_0 - t - s| \leqslant |t-s|.$$

This shows rhat f_0 is 1-Lipschitzian on $[a,b]$. Further, from (8) we get

$$\frac{1}{d-c}\int_c^d f_0(s)\mathrm{d}s = \frac{d-c}{2(c-a+b-d)^2}\left[(c-a)^2+(b-d)^2\right]. \tag{9}$$

By a similar calculation we have

$$s_0-a=\frac{b-a}{c-a+b-d}(c-a), \quad b-s_0=\frac{b-a}{c-a+b-d}(b-d)$$

and

$$\begin{aligned}\frac{1}{b-a}\int_a^b f_0(t)\mathrm{d}t &= \frac{(s_0-a)^2+(b-s_0)^2}{2(b-a)} \\ &= \frac{b-a}{2(c-a+b-d)^2}\left[(c-a)^2+(b-d)^2\right].\end{aligned} \tag{10}$$

Now it is evident from (9) and (10) that

$$\frac{1}{b-a}\int_a^b f_0(t)\mathrm{d}t - \frac{1}{d-c}\int_c^d f_0(s)\mathrm{d}s = \frac{(c-a)^2+(b-d)^2}{2(c-a+b-d)},$$

which gives (6). $\square$

REMARK 1. As we already noted, for $c=d=x$ we can assume $\frac{1}{d-c}\int_c^d f(s)\mathrm{d}s = f(x)$ so that (4) reduces to the Ostrowski inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f(x)\right| \leqslant \frac{(x-a)^2+(b-x)^2}{2(b-a)}M$$

for M-Lipschitzian function f on $[a,b]$. On the other side, the condition

$$c-a+b-d>0$$

is equivalent to the requirement that $c \neq a$ or $d \neq b$. If $c=a$ and $d=b$, then the left-hand side of (4) is equal to zero, while the right-hand side of (4) can be regarded as zero in the case $c=a$ and $d=b$, since

$$\frac{(c-a)^2+(b-d)^2}{c-a+b-d} \leqslant c-a+b-d.$$

Note that this is not the case with (3) whose right-hand side is equal to $(b-a)\|f'\|_\infty$ when $c=a$ and $d=b$.

As we noted above, both inequalities (3) and (4) can be regarded as generalizations of Ostrowski inequality (1). The following Corollary shows that our two-point Ostrowski inequality (4) gives much tighter estimate than the estimate given by (3).

COROLLARY 1. *Let $a,b,c,d \in \mathbb{R}$, are such that*

$$a \leqslant c < d \leqslant b, \quad c-a+b-d>0.$$

Define β_1, β_2, β_3 *and* β_4 *as*

$$\begin{aligned}
\beta_1 &= \frac{b-a}{4} + \frac{d-c}{2} + \frac{1}{b-a}\left[\left|\frac{c+d}{2} - \frac{a+b}{2}\right| - \frac{d-c}{2}\right]^2, \\
\beta_2 &= \frac{(b-c)^2 + (d-a)^2}{2(b-c+d-a)}, \\
\beta_3 &= \frac{(d-c)^2}{3(b-a)} + \frac{(d-a)(c-a) + (b-d)(b-c)}{2(b-a)}, \\
\beta_4 &= \frac{(c-a)^2 + (b-d)^2}{2(c-a+b-d)}.
\end{aligned}$$

If $f : [a,b] \to \mathbb{R}$ *is* M*-Lipschitzian on* $[a,b]$, *with some constant* $M > 0$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - \frac{1}{d-c}\int_c^d f(s)\mathrm{d}s\right| \leqslant \beta_4 M < \beta_3 M < \beta_2 M < \beta_1 M. \qquad (11)$$

Proof. The first inequality in (11) was proved in Theorem 3. To prove the rest it is sufficint to prove that $\beta_3 - \beta_4 > 0$, $\beta_2 - \beta_3 > 0$ and $\beta_1 - \beta_2 > 0$.

By an easy calculation we get

$$\beta_3 - \beta_4 = \frac{d-c}{b-a}\left[\frac{d-c}{3} + \frac{(c-a)(b-d)}{c-a+b-d}\right]$$

and evidently $\beta_3 - \beta_4 > 0$, under given assumptions. Similarly, by suitable rearrangement of the expressions for β_2 and β_3, we easily get

$$\beta_2 - \beta_3 = \frac{(d-c)(b-c)(d-a)}{3(b-a)(b-a+d-c)}\left[3 - \left(\frac{d-c}{d-a} + \frac{d-c}{b-c}\right)\right].$$

Since $0 < \frac{d-c}{d-a} < 1$ and $0 < \frac{d-c}{b-c} < 1$, we conclude that $\beta_2 - \beta_3 > 0$. Finally, it is easy to see that β_1 can be rewritten as

$$\beta_1 = \frac{1}{2(b-a)}\left[(b-c)^2 + (d-a)^2 - |c+d-a-b|\right],$$

so that

$$\begin{aligned}
\beta_1 - \beta_2 &= \frac{d-c}{2(b-a)(b-a+d-c)}\left[(d-a)^2 + (b-c)^2 - \left|(d-a)^2 - (b-c)^2\right|\right] \\
&= \frac{d-c}{2(b-a)(b-a+d-c)}\left[\min\{d-a, b-c\}\right]^2.
\end{aligned}$$

From the above expression it is obvious that $\beta_1 - \beta_2 > 0$, under given assumptions. □

3. Concluding remarks

Let $a, b, c, d \in \mathbb{R}$, are such that

$$a \leqslant c < d \leqslant b, \quad c - a + b - d > 0$$

and let β_1, β_2, β_3 and β_4 be defined as in Corollary 1. If f is differentiable and f' is bounded on (a, b), then the inequalities given by (11) hold with

$$M = \|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| .$$

Especially we have

$$\left| \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - \frac{1}{d-c} \int_c^d f(s) \mathrm{d}s \right| \leqslant \beta_4 \|f'\|_\infty < \beta_1 \|f'\|_\infty ,$$

which shows that the estimate given by (4) is actually much tighter than the estimate given by (3).

Further, as a consequence of Corollary 1 we have the following two inequalities:

$$\left| \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - \frac{1}{d-c} \int_c^d f(s) \mathrm{d}s \right| < \beta_2 \|f'\|_\infty \tag{12}$$

and

$$\left| \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - \frac{1}{d-c} \int_c^d f(s) \mathrm{d}s \right| < \beta_3 \|f'\|_\infty . \tag{13}$$

The inequality (12) can be proved independently by a slight modification of the original proof for (3) from [1].

Similarly, the inequality (13) can be proved by a direct application of the Ostrowski inequality, but in a slight different way than that of Barnett and Dragomir from [1]. Namely we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - \frac{1}{d-c} \int_c^d f(s) \mathrm{d}s \right| \\ = & \left| \frac{1}{d-c} \int_c^d \left[\frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - f(x) \right] \mathrm{d}x \right| \\ \leqslant & \frac{1}{d-c} \int_c^d \left| \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - f(x) \right| \mathrm{d}x. \end{aligned} \tag{14}$$

Now, applying Ostrowski inequality in the form given by (2) to the expressin

$$\left| \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t - f(x) \right|$$

and then integrating, it is easy to get (13) from (14). Moreover, using Ostrowski inequality in the form given by (1) we can get the inequality (13) in the equivalent form

$$\left|\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t-\frac{1}{d-c}\int_c^d f(s)\mathrm{d}s\right| < \left[\frac{1}{4}+\frac{(d-c)^2}{3(b-a)^2}+\frac{\left(c-\frac{a+b}{2}\right)\left(d-\frac{a+b}{2}\right)}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty .$$

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