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ON KY FAN'S INEQUALITY

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Abstract. In this paper we prove several Ky Fan type inequalities involving certain Stolarsky-Tobey means.

1. Introduction and notation

Let $n \geqslant 2$ be a given integer, let

$$A_{n-1} = \{ (\lambda_1, \dots, \lambda_{n-1}) \mid \lambda_i \geqslant 0,\ i = 1, \dots, n-1,\ \lambda_1 + \dots + \lambda_{n-1} \leqslant 1 \}$$

be the Euclidean simplex, and let μ be a probability measure on A_{n-1}. For each $i \in \{ 1, \dots, n \}$, the ith weight w_i associated to μ is defined by

$$w_i = \int_{A_{n-1}} \lambda_i d\mu(\lambda) \qquad \text{if} \quad 1 \leqslant i \leqslant n-1,$$

$$w_n = \int_{A_{n-1}} (1 - \lambda_1 - \dots - \lambda_{n-1}) d\mu(\lambda),$$

where $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in A_{n-1}$. Obviously, $w_i > 0$ for all $i \in \{ 1, \dots, n \}$, and $w_1 + \dots + w_n = 1$. Moreover, if $\mu = (n-1)!$ (i. e. $d\mu(\lambda) = (n-1)!\, d\lambda_1 \cdots d\lambda_{n-1}$), then $w_i = 1/n$ for all $i \in \{ 1, \dots, n \}$.

Given $X = (x_1, \dots, x_n) \in]0, \infty[^n$, the weighted harmonic, geometric, and arithmetic mean, respectively, of $x_1, \dots, x_n$ are defined by

$$H(X; w) = \frac{1}{\sum_{i=1}^n \dfrac{w_i}{x_i}}, \qquad G(X; w) = \prod_{i=1}^n x_i^{w_i}, \qquad A(X; w) = \sum_{i=1}^n w_i x_i.$$

For $\mu = (n-1)!$ the usual unweighted harmonic, geometric, and arithmetic mean, respectively, of $x_1, \dots, x_n$ are obtained:

$$H(X) = \frac{n}{\sum_{i=1}^n \dfrac{1}{x_i}}, \qquad G(X) = \left(\prod_{i=1}^n x_i \right)^{1/n}, \qquad A(X) = \frac{\sum_{i=1}^n x_i}{n}.$$

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Next, recall that the logarithmic mean of the positive real numbers x_1 and x_2 is defined by

$$L(x_1,x_2)=\frac{x_1-x_2}{\log x_1-\log x_2}\qquad \text{if}\quad x_1\neq x_2,$$
$$L(x_1,x_1)=x_1,$$

while the identric mean of x_1 and x_2 is defined by

$$I(x_1,x_2)=\frac{1}{e}\left(\frac{x_2^{x_2}}{x_1^{x_1}}\right)^{1/(x_2-x_1)}\qquad \text{if}\quad x_1\neq x_2,$$
$$I(x_1,x_1)=x_1.$$

These two means were generalized for n variables, too. Thus, starting from the integral representation

$$L(x_1,x_2)=\left(\int_0^1\frac{dt}{tx_1+(1-t)x_2}\right)^{-1},$$

A. O. Pittenger [8] introduced the weighted logarithmic mean of $x_1,\ldots,x_n$. It is defined by

$$L(X;\mu)=\left(\int_{A_{n-1}}\frac{1}{\lambda\cdot X}d\mu(\lambda)\right)^{-1},$$

where

$$\lambda\cdot X=\lambda_1x_1+\cdots+\lambda_{n-1}x_{n-1}+(1-\lambda_1-\cdots-\lambda_{n-1})x_n$$

for all $\lambda=(\lambda_1,\ldots,\lambda_{n-1})\in A_{n-1}$. On the other hand, starting from the integral representation

$$I(x_1,x_2)=\exp\left(\int_0^1\log(tx_1+(1-t)x_2)dt\right),$$

in [10] it was pointed out that

$$I(X;\mu)=\exp\left(\int_{A_{n-1}}\log(\lambda\cdot X)d\mu(\lambda)\right)$$

can be considered as the weighted identric mean of $x_1,\ldots,x_n$. For $\mu=(n-1)!$ we get the unweighted and symmetric logarithmic and identric mean, respectively, of $x_1,\ldots,x_n$:

$$L(X)=\left((n-1)!\int_{A_{n-1}}\frac{1}{\lambda\cdot X}d\lambda_1\cdots d\lambda_{n-1}\right)^{-1},$$
$$I(X)=\exp\left((n-1)!\int_{A_{n-1}}\log(\lambda\cdot X)d\lambda_1\cdots d\lambda_{n-1}\right).$$

For properties and explicit forms of these means, the reader is referred to [7] and [8].

However, it should be noted that all the above weighted means are special cases of the so-called Stolarsky-Tobey means, and they verify the following chain of inequalities (see [7]):

$$H(X;w) \leqslant G(X;w) \leqslant L(X;\mu) \leqslant I(X;\mu) \leqslant A(X;w). \tag{1.1}$$

A remarkable counterpart of the arithmetic-geometric mean inequality $G(X) \leqslant A(X)$ was obtained by Ky Fan: if $0 < x_i \leqslant 1/2$ for all $i \in \{1,\dots,n\}$, then

$$\frac{G(X)}{G(\mathbf{1}-X)} \leqslant \frac{A(X)}{A(\mathbf{1}-X)}, \tag{1.2}$$

where $\mathbf{1}-X := (1-x_1,\dots,1-x_n)$. This inequality evoked the interest of many mathematicians, and numerous proofs, generalizations, or sharpenings were published (see, for instance, [1]–[6], [9]). The following weighted refinement of (1.2) has been recently obtained in [10]:

$$\frac{G(X;w)}{G(\mathbf{1}-X;w)} \leqslant \frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)} \leqslant \frac{A(X;w)}{A(\mathbf{1}-X;w)}. \tag{1.3}$$

On the other hand, W.-L. Wang and P.-F. Wang [13] established the following counterpart of (1.2):

$$\frac{H(X)}{H(\mathbf{1}-X)} \leqslant \frac{G(X)}{G(\mathbf{1}-X)}. \tag{1.4}$$

Another proof of (1.4) can be found in [4].

2. Main results

It is the main purpose of this paper to prove a weighted version of (1.4) as well as a counterpart of the right inequality in (1.3). They are contained in the following theorem.

THEOREM 1. *If* $X = (x_1,\dots,x_n) \in \,]0,1/2]^n$ *then*

$$\frac{H(X;w)}{H(\mathbf{1}-X;w)} \leqslant \frac{G(X;w)}{G(\mathbf{1}-X;w)} \tag{2.1}$$

and

$$\frac{L(X;\mu)}{L(\mathbf{1}-X;\mu)} \leqslant \frac{I(X;\mu)}{I(\mathbf{1}-X;\mu)}. \tag{2.2}$$

The inequalities (2.1) and (2.2) are strict unless $x_1 = \cdots = x_n$.

In the proof we shall use

LEMMA 2. *Let* $J \subseteq \mathbf{R}$ *be a nonempty interval, let* $X = (x_1,\dots,x_n) \in J^n$, *and let* $f \in C^1(J)$ *be a strictly monotone convex function. Then the following inequalities hold:*

$$\sum_{i=1}^{n} w_i f(x_i) \leqslant f\left(\frac{\sum_{i=1}^{n} w_i x_i f'(x_i)}{\sum_{i=1}^{n} w_i f'(x_i)}\right), \tag{2.3}$$

$$\int_{A_{n-1}} f(\lambda\cdot X)d\mu(\lambda) \leqslant f\left(\frac{\int_{A_{n-1}}(\lambda\cdot X)f'(\lambda\cdot X)d\mu(\lambda)}{\int_{A_{n-1}} f'(\lambda\cdot X)d\mu(\lambda)}\right). \tag{2.4}$$

Moreover, if f is strictly convex, then the inequalities (2.3) and (2.4) are strict unless $x_1 = \cdots = x_n$.

Proof. Let $\bar{x}$ be a point in J which will be chosen later. The convexity of f ensures that

$$f(x_i) + f'(x_i)(\bar{x} - x_i) \leqslant f(\bar{x}) \qquad \text{for all } i \in \{1,\ldots,n\}.$$

Multiplying both sides by w_i and then summing the obtained inequalities yields

$$\sum_{i=1}^{n} w_i f(x_i) + \bar{x}\sum_{i=1}^{n} w_i f'(x_i) - \sum_{i=1}^{n} w_i x_i f'(x_i) \leqslant f(\bar{x}).$$

Set

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i f'(x_i)}{\sum_{i=1}^{n} w_i f'(x_i)}.$$

This implies (2.3) because $\bar{x} \in J$ in virtue of the strictly monotonicity of f.

Let $\tilde{x}$ be another point in J which will be chosen later, too. The convexity of f ensures that

$$f(\lambda\cdot X) + f'(\lambda\cdot X)(\tilde{x} - \lambda\cdot X) \leqslant f(\tilde{x}) \qquad \text{for all } \lambda \in A_{n-1}.$$

Integrating over A_{n-1} with respect to μ yields

$$\int_{A_{n-1}} f(\lambda\cdot X)d\mu(\lambda) + \tilde{x}\int_{A_{n-1}} f'(\lambda\cdot X)d\mu(\lambda) - \int_{A_{n-1}}(\lambda\cdot X)f'(\lambda\cdot X)d\mu(\lambda) \leqslant f(\tilde{x}).$$

Set

$$\tilde{x} = \frac{\int_{A_{n-1}}(\lambda\cdot X)f'(\lambda\cdot X)d\mu(\lambda)}{\int_{A_{n-1}} f'(\lambda\cdot X)d\mu(\lambda)}.$$

This implies (2.4) because $\tilde{x} \in J$ in virtue of the strictly monotonicity of f. □

Proof of Theorem 1. The inequalities (2.1) and (2.2) follow at once from (2.3) and (2.4), respectively, if we choose $J =]0,1/2]$ and $f : J \to \mathbf{R}$ to be the function $f(x) = \log(1-x) - \log x$. Indeed, it is immediately seen that f is strictly decreasing and strictly convex. □

Having in mind the chain (1.1), it is naturally to ask whether the inequalities (1.3), (2.1), and (2.2) can be completed by

$$\frac{G(X;w)}{G(\mathbf{1}-X;w)} \leqslant \frac{L(X;\mu)}{L(\mathbf{1}-X;\mu)}? \tag{2.5}$$

Unfortunately, the inequality (2.5) cannot be true for an arbitrary measure μ, even in the special case $n = 2$. To prove this, let $x_1 \neq x_2$ be positive real numbers lying in $]0,1/2]$. For each $\varepsilon \in]0,1/2[$, let $F_\varepsilon : [0,1] \to \mathbf{R}$ be the absolutely continuous function defined by

$$F_\varepsilon(t) := \begin{cases} \dfrac{t}{2\varepsilon} & \text{if } 0 \leqslant t \leqslant \varepsilon \\ \dfrac{1}{2} & \text{if } \varepsilon < t < 1-\varepsilon \\ \dfrac{t-1+2\varepsilon}{2\varepsilon} & \text{if } 1-\varepsilon \leqslant t \leqslant 1, \end{cases}$$

and let μ_ε be the Lebesgue-Stieltjes measure on $[0,1]$ generated by F_ε (i. e. $d\mu_\varepsilon(t) = F'_\varepsilon(t)\,dt$). Since

$$\int_0^1 d\mu_\varepsilon(t) = \int_0^\varepsilon \frac{1}{2\varepsilon}dt + \int_{1-\varepsilon}^1 \frac{1}{2\varepsilon}dt = 1,$$

it follows that μ_ε is a probability measure on $[0,1]$. The weights of μ_ε are

$$w_1 = \int_0^1 t d\mu_\varepsilon(t) = \int_0^1 tF'_\varepsilon(t)dt = \frac{1}{2}$$

and $w_2 = 1 - w_1 = \dfrac{1}{2}$.

On the other hand, we have

$$\begin{aligned} \frac{1}{L(x_1,x_2;\mu_\varepsilon)} &= \int_0^1 \frac{1}{tx_1+(1-t)x_2}d\mu_\varepsilon(t) = \int_0^1 \frac{F'_\varepsilon(t)}{tx_1+(1-t)x_2}dt \\ &= \frac{1}{2\varepsilon(x_1-x_2)} \log \frac{x_1(\varepsilon x_1+(1-\varepsilon)x_2)}{x_2(\varepsilon x_2+(1-\varepsilon)x_1)}, \end{aligned}$$

and, analogously,

$$\frac{1}{L(1-x_1,1-x_2;\mu_\varepsilon)} = \frac{1}{2\varepsilon(x_2-x_1)} \log \frac{(1-x_1)\left(\varepsilon(1-x_1)+(1-\varepsilon)(1-x_2)\right)}{(1-x_2)\left(\varepsilon(1-x_2)+(1-\varepsilon)(1-x_1)\right)}.$$

Now, a simple computation shows that

$$\lim_{\varepsilon\to 0+} \frac{L(x_1,x_2;\mu_\varepsilon)}{L(1-x_1,1-x_2;\mu_\varepsilon)} = \frac{H(x_1,x_2)}{H(1-x_1,1-x_2)} < \frac{G(x_1,x_2)}{G(1-x_1,1-x_2)}.$$

So, for sufficiently small ε, we have

$$\frac{G(x_1,x_2)}{G(1-x_1,1-x_2)} > \frac{L(x_1,x_2;\mu_\varepsilon)}{L(1-x_1,1-x_2;\mu_\varepsilon)}.$$

Another interesting result concerning the inequality (2.5) is related to the so-called Dirichlet measures. For each $\alpha > 0$, let μ_α be the Dirichlet measure on $[0,1]$ defined by

$$d\mu_\alpha(t) = \frac{t^{\alpha-1}(1-t)^{\alpha-1}}{B(\alpha,\alpha)}dt.$$

Clearly, μ_α is a probability measure on $[0,1]$. Its weights are

$$w_1 = \int_0^1 t d\mu_\alpha(t) = \frac{1}{B(\alpha,\alpha)} \int_0^1 t^\alpha (1-t)^{\alpha-1} dt = \frac{B(\alpha+1,\alpha)}{B(\alpha,\alpha)} = \frac{1}{2}$$

and $w_2 = 1 - w_1 = \frac{1}{2}$.

Further, let $g_\alpha :]0,1] \to \mathbf{R}$ be the function defined by

$$g_\alpha(x) := \sqrt{x} \int_0^1 \frac{1}{tx+1-t} d\mu_\alpha(t) = \frac{\sqrt{x}}{B(\alpha,\alpha)} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\alpha-1}}{tx+1-t} dt.$$

LEMMA 3. *The following assertions are true:*
1° *If* $\alpha \in]0,1/2[$, *then* g_α *is strictly decreasing.*
2° $g_{1/2}(x) = 1$ *for all* $x \in]0,1]$.
3° *If* $\alpha \in]1/2,\infty[$, *then* g_α *is strictly increasing.*

Proof. For each $x \in]0,1]$ we have

$$g_\alpha(x) = \sqrt{x} \int_0^1 \frac{1}{1-t(1-x)} d\mu_\alpha(t) = \sqrt{x} \sum_{k=0}^\infty \mu_k (1-x)^k, \tag{2.6}$$

where

$$\mu_k = \int_0^1 t^k d\mu_\alpha(t) = \frac{B(k+\alpha,\alpha)}{B(\alpha,\alpha)}, \qquad k = 0,1,2,\dots. \tag{2.7}$$

From (2.7), it follows that

$$\mu_{k+1} = \frac{k+\alpha}{k+2\alpha} \mu_k \qquad \text{for all } k \geqslant 0. \tag{2.8}$$

By virtue of (2.6) and (2.8), it is easily seen that

$$\begin{aligned} g'_\alpha(x) &= \frac{1}{2\sqrt{x}} \sum_{k=1}^\infty \left[(2k+1)\mu_k - 2(k+1)\mu_{k+1}\right](1-x)^k \\ &= \frac{2\alpha-1}{2\sqrt{x}} \sum_{k=1}^\infty \frac{k\mu_k}{k+2\alpha} (1-x)^k \end{aligned}$$

for all $x \in]0,1]$. This equality ensures the validity of the assertions 1° and 3°.

On the other hand, for each $x \in]0,1]$ we have

$$g_{1/2}(x) = \frac{\sqrt{x}}{\pi} \int_0^1 \frac{1}{\sqrt{t(1-t)}} \cdot \frac{1}{tx+1-t} dt.$$

Substituting $t = \sin^2\theta$ yields

$$g_{1/2}(x) = \frac{2\sqrt{x}}{\pi} \int_0^{\pi/2} \frac{d\theta}{x\sin^2\theta + \cos^2\theta} = \frac{2\sqrt{x}}{\pi} \cdot \frac{1}{\sqrt{x}} \arctan(\sqrt{x}\tan\theta)\Big|_0^{\pi/2} = 1. \quad \square$$

THEOREM 4. *Let* $x_1, x_2 \in]0, 1/2]$, *and let* $L_\alpha(x_1, x_2) := L(x_1, x_2; \mu_\alpha)$. *Then the following assertions are true:*

1° *If* $\alpha \in]0, 1/2[$, *then* $\dfrac{L_\alpha(x_1,x_2)}{L_\alpha(1-x_1,1-x_2)} \leqslant \dfrac{G(x_1,x_2)}{G(1-x_1,1-x_2)}$.

2° $\dfrac{L_{1/2}(x_1,x_2)}{L_{1/2}(1-x_1,1-x_2)} = \dfrac{G(x_1,x_2)}{G(1-x_1,1-x_2)}$.

3° *If* $\alpha \in]1/2, \infty[$, *then* $\dfrac{L_\alpha(x_1,x_2)}{L_\alpha(1-x_1,1-x_2)} \geqslant \dfrac{G(x_1,x_2)}{G(1-x_1,1-x_2)}$.

Moreover, the inequalities in the assertions 1° *and* 3° *are strict unless* $x_1 = x_2$.

Proof. 1° Suppose that $0 < x_1 < x_2 \leqslant \dfrac{1}{2}$. Then $\dfrac{x_1}{x_2} < \dfrac{1-x_2}{1-x_1}$. According to the assertion 1° in Lemma 3, we have $g_\alpha\left(\dfrac{x_1}{x_2}\right) > g_\alpha\left(\dfrac{1-x_2}{1-x_1}\right)$. But this inequality is equivalent to $\dfrac{L_\alpha(x_1,x_2)}{L_\alpha(1-x_1,1-x_2)} < \dfrac{G(x_1,x_2)}{G(1-x_1,1-x_2)}$.

The assertions 2° and 3° follow in the same way from the corresponding assertions in Lemma 3. □

Returning to the inequality (2.5), it still remains open the question whether the unweighted inequality

$$\frac{G(X)}{G(\mathbf{1}-X)} \leqslant \frac{L(X)}{L(\mathbf{1}-X)}$$

is valid ? A partial answer, in the special case $n = 2$, is given in Proposition 5. It would be interesting to investigate whether this proposition can be generalized for n variables (the authors do not know the answer).

PROPOSITION 5. *If* $x_1, x_2 \in]0, 1/2]$, *then*

$$\frac{G(x_1,x_2)}{G(1-x_1,1-x_2)} \leqslant \frac{L(x_1,x_2)}{L(1-x_1,1-x_2)},$$

with equality if and only if $x_1 = x_2$.

Proof. Obviously, it suffices to prove that if $x_1 \neq x_2$, then

$$\frac{\sqrt{x_1x_2}}{\sqrt{(1-x_1)(1-x_2)}} < \frac{\log(1-x_2)-\log(1-x_1)}{\log x_1 - \log x_2}. \tag{2.9}$$

Suppose, for instance, that $0 < x_2 < x_1 \leqslant 1/2$. Then (2.9) is equivalent to

$$\sqrt{\frac{x_1}{1-x_1}}\sqrt{\frac{x_2}{1-x_2}}(\log x_1 - \log x_2) + \log(1-x_1) - \log(1-x_2) < 0. \tag{2.10}$$

Set $u = \sqrt{x_1/(1-x_1)}$ and $v = \sqrt{x_2/(1-x_2)}$. Then we have $0 < v < u \leqslant 1$, and (2.10) can be written as

$$uv\left(\log\frac{u^2}{1+u^2} - \log\frac{v^2}{1+v^2}\right) - \log(1+u^2) + \log(1+v^2) < 0. \tag{2.11}$$

Let $v \in]0,1[$ be fixed, and let $f : [v,1] \to \mathbf{R}$ be the function defined by

$$f(u) = uv\left(\log\frac{u^2}{1+u^2} - \log\frac{v^2}{1+v^2}\right) - \log(1+u^2) + \log(1+v^2).$$

We have

$$f'(u) = v\left(\log\frac{u^2}{1+u^2} - \log\frac{v^2}{1+v^2}\right) + \frac{2(v-u)}{1+u^2}$$

and

$$f''(u) = \frac{2(v-u)(1-u^2)}{u(1+u^2)^2}.$$

Since $f''(u) < 0$ for all $u \in]v,1[$, it follows that f' is strictly decreasing on $[v,1]$. Consequently, $f'(u) < 0$ for all $u \in]v,1]$, because $f'(v) = 0$. Therefore f is strictly decreasing on $[v,1]$, hence $f(u) < 0$ for all $u \in]v,1]$, because $f(v) = 0$. This proves the validity of (2.11). $\square$

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