Mathematical
Inequalities
& Applications
Volume 4, Number 2 (2001), 239–246

# BASIC RESULTS FOR FUZZY IMPULSIVE DIFFERENTIAL EQUATIONS

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(*communicated by D. Bainov*)

*Abstract.* The basic theory of fuzzy impulsive differential equations is initiated by combining suitably the theories of impulsive differential equations and fuzzy differential equations.

## 1. Introduction

In mathematical modeling of real world problems, we encounter two inconveniences. The first one is caused by the complexity of the situation that is being modeled. The second inconvenience consists of indeterminacy resulting by our inability to differentiate events exactly. The main property of indeterminacy is vagueness of its sematics. Since the classical mathematics could not cope with such vagueness, a new mathematical apparatus which enables us to describe vagueness is very useful. Such an apparatus is known as the fuzzy set theory. The theory of fuzzy sets, fuzzy valued functions and necessary calculus of fuzzy function has been investigated [1, 2, 3, 4, 12]. Recently, the framework for the study of fuzzy differential equations has also been developed and the basic properties of solutions of fuzzy differential equations is available [5, 6, 7, 8, 10, 11].

In this paper, we shall attempt to initiate the theory of fuzzy impulsive differential equations by combining the theories of impulsive differential equations [9] and fuzzy differential equations. Since the fuzziness and impulsiveness occurs in several real world problems, the proposed union would be of immense value. For example, the interest rate models in bond pricing, where the interest rates are unpredictable and vague could be modeled by means of fuzzy impulsive differential equations.

## 2. Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact, convex subsets of $R^n$. If $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

*Mathematics subject classification* (2000): 34A47, 34D20, 34L99.
*Key words and phrases*: Fuzzy impulsive differential equations, basic theory.

and if $\alpha,\beta \geqslant 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. Let $I = [t_0, t_0+a]$, $t_0 \geqslant 0$ and $a > 0$ and denote by $E^n = [u: R^n \to [0,1]$ such that $u$ satisfies $(i)$ to $(iv)$ mentioned below]:

$(i)$ $u$ is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

$(ii)$ $u$ is fuzzy convex, that is, for $x,y \in R^n$ and $0 \leqslant \lambda \leqslant 1$,

$$u(\lambda x + (1-\lambda)y) \geqslant \min[u(x), u(y)];$$

$(iii)$ $u$ is upper semicontinuous;

$(iv)$ $[u]^0 = \overline{[x \in R^n: u(x) > 0]}$ is compact.

For $0 < \alpha \leqslant 1$, we denote $[u]^\alpha = [x \in R^n: u(x) \geqslant \alpha]$. Then from $(i)$ to $(iv)$, it follows that the $\alpha$-level sets $[u]^\alpha \in P_k(R^n)$ for $0 \leqslant \alpha \leqslant 1$. For later purposes, we define $\widehat{o} \in E^n$ as $\widehat{o}(x) = 1$ if $x = 0$ and $\widehat{o}(x) = 0$ if $x \neq 0$.

Let $d_H(A,B)$ be the Hausdorff distance between the sets $A, B \in P_k(R^n)$. Then we define

$$d[u,v] = \sup_{0 \leqslant \alpha \leqslant 1} d_H[[u]^\alpha, [v]^\alpha],$$

which defines a metric in $E^n$ and $(E^n, d)$ is a complete metric space. We list the following properties of $d[u,v]$:

$$d[u+w, v+w] = d[u,v] \quad \text{and} \quad d[u,v] = d[v,u],$$

$$d[\lambda u, \lambda v] = |\lambda| \, d[u,v],$$

$$d[u,v] \leqslant d[u,w] + d[w,v],$$

for all $u, v, w \in E^n$ and $\lambda \in R$.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then $z$ is called the $H$-difference of $x$ and $y$ and is denoted by $x - y$. A mapping $F: I \to E^n$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in the metric space $(E^n, d)$.

Moreover, if $F: I \to E^n$ is continuous, then it is integrable and

$$\int_a^b F = \int_a^c F + \int_c^b F.$$

Also, the following properties of the integral are valid. If $F, G\colon I \to E^n$ are integrable, $\lambda \in R$, then the following hold:

$$\int (F+G) = \int F + \int G;$$

$$\int \lambda F = \lambda \int F, \quad \lambda \in R;$$

$$d[F,G] \quad \text{is integrable;}$$

$$d\left[\int F, \int G\right] \leqslant \int d[F,G].$$

Finally, let $F: I \to E^n$ be continuous. Then the integral $G(t) = \int_{t_0}^{t} F$ is differentiable and $G'(t) = F(t)$. Furthermore,

$$F(t) - F(t_0) = \int_a^t F'(t).$$

(See [2-6] for details.)

Consider the fuzzy differential system

$$u' = f(t, u), \quad u(t_0) = u_0, \tag{2.1}$$

where $f \in C[I \times E^n, E^n]$ and $I = [t_0, t_0 + a]$, $t_0 \geqslant 0$, $a > 0$. Before proceeding any further, we note that a mapping $u: I \to E^n$ is a solution of the initial value problem (2.1) if and only if it is continuous and satisfies the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds \quad \text{for} \quad t \in I.$$

An application of contraction mapping principle yields the following existence and uniqueness result [5].

THEOREM 2.1. *Assume that* $f \in C[I \times E^n, E^n]$ *and satisfies*

$$d[f(t, u), f(t, v)] \leqslant Ld[u, v], \quad L > 0,$$

*for* $(t, u), (t, v) \in I \times E^n$. *Then the initial value problem* (2.1) *has a unique solution* $u(t) = u(t, t_0, u_0)$ *on* $I$.

We also need the following known [9] impulsive differential inequalities result. For this purpose, we let $PC$ denote the class of piecewise continuous functions from $R_+$ to $R$ with discontinuities of the first kind only at $t = t_k, k = 1, 2, \ldots$. We can now state the needed results.

THEOREM 2.2. *Assume that*

($A_0$) *the sequence* $\{t_k\}$ *satisfies* $0 \leqslant t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ *with* $t_k \to \infty$ *as* $k \to \infty$;

($A_1$) $m \in PC'[R_+, R]$ *and* $m(t)$ *is left-continuous at* $t_k, k = 1, 2, \ldots$;

($A_2$) $\forall k = 1, 2, \ldots$ *and* $t \geqslant t_0$,

$$\left[\begin{array}{ll} D^+m(t) \leqslant g(t, m(t)), & t \neq t_k, \\ m(t_k^+) \leqslant \psi_k(m(t_k)), & \\ \quad m(t_0) \leqslant w_0, & \end{array}\right. \tag{2.2}$$

*where* $g: R_+^2 \to R$ *is continuous in* $(t_{k-1}, t_k] \times R_+$ *and for each* $w \in R_+$, $\lim_{(t,z) \to (t_k^+, w)} g(t, z) = g(t_k^+, w)$ *exists and* $\psi_k: R_+ \to R$ *is nondecreasing;*

($A_3$) $r(t) = r(t, t_0, w_0)$ *is the maximal solution of*

$$\left[\begin{array}{ll} w' = g(t, w), & t \neq t_k, \\ w(t_k^+) = \psi_k(w(t_k)), & \\ w(t_0) = w_0 \geqslant 0, & \end{array}\right. \tag{2.3}$$

*existing on* $[t_0, \infty)$. *Then*

$$m(t) \leqslant r(t), t \geqslant t_0. \tag{2.4}$$

We recall that the maximal solution $r(t)$ of (2.3) means the following

$$r(t) = \begin{bmatrix} r_0(t, t_0, w_0), & t \in [t_0, t_1], \\ r_1(t, t_1, r_0(t_1^+)), & t \in (t_1, t_2], \\ \vdots & \\ r_k(t, t_k, r_{k-1}(t_k^+)), & t \in (t_k, t_{k+1}], \\ \vdots & \\ \vdots & \end{bmatrix} \tag{2.5}$$

where each $r_i(t, t_i, r_{i-1}(t_i^+))$ is the maximal solution of (2.3) on the interval $(t_i, t_{i+1}]$ for each $i = 1, 2, \ldots,$ and $r_{i-1}(t_i^+) = \psi_i(r_{i-1}(t_i, t_{i-1}, r_{i-2}(t_{i-1}^+)))$.

## 3. Fuzzy impulsive differential equations

Let us consider the initial value problem for fuzzy impulsive differential equation

$$\begin{bmatrix} u' = f(t, u), \quad t \neq t_k, \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), \\ u(t_0) = u_0 \end{bmatrix} \tag{3.1}$$

where $(A_0)$ of Theorem 2.2 holds and $f: R_+ \times E^n \to E^n$, $f$ is continuous in $(t_{k-1}, t_k] \times E^n$ and for each $u \in E^n$, $\lim f(t, v) = f(t_k^+, u)$ exists as $(t, v) \to (t_k^+, u)$. Also $I_k: E^n \to E^n$ and $u_0 \in E^n$. If the assumptions of Theorem 2.1 hold on each set $[t_{k-1}, t_k] \times E^n$, then clearly there exists a unique solution $u_i(t) = u(t, t_i, u_{i-1}(t_i^+))$ on each interval $[t_{i-1}, t_i]$. As a result, employing the impulsive condition in (3.1) at each $t = t_i$, we can define the solution $u(t)$ of (3.1) on the interval $[t_0, \infty)$ as we did in (2.5).

We shall next extend a typical result in Lyapunov-like theory. Let $V: R_+ \times E^n \to R_+$. Then $V$ is said to belong to the class $V_0$ if

(*i*) $V(t, u)$ is continuous in $(t_{k-1}, t_k] \times E^n$ and for each $u \in E^n$, $k = 1, 2, \ldots,$ $\lim_{(t,v) \to (t_k^+, u)} V(t, v) = V(t_k^+, u)$ exists;

(*ii*) $V(t, u)$ satisfies $| V(t, u) - V(t, v) | \leqslant Ld[u, v], L > 0$ for $(t, u), (t, v) \in (t_{k-1}, t_k] \times E^n$;

For $(t, u) \in (t_{k-1}, t_k] \times E^n$, we define

$$D^+ V(t, u) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, u + hf(t, u)) - V(t, u)]. \tag{3.2}$$

We can then prove the following comparison result.

THEOREM 3.1. *Let* $V: R_+ \times E^n \to R_+$ *and* $V \in V_0$. *Suppose further that*

$$D^+ V(t,u) \leqslant g(t, V(t,u)), \quad t \neq t_k, \tag{3.3}$$

$$V(t, u + I_k(u)) \leqslant \psi_k(V(t,u)), \quad t = t_k, \quad 3.4$$

*where* $g: R_+^2 \to R$ *is continuous in* $(t_{k-1}, t_k] \times R_+$ *and for each* $w \in R_+$, $\lim_{(t,z)\to(t_k^+,w)} g(t,z) = g(t_k^+, w)$ *exists and* $\psi_k: R_+ \to R_+$ *is nondecreasing. Let* $r(t) = r(t, t_0, w_0)$ *be the maximal solution of the scalar impulsive differential equation* (2.3) *existing on* $[t_0, \infty)$. *Then* $V(t_0^+, u_0) \leqslant w_0$ *implies*

$$V(t, u(t)) \leqslant r(t), \quad t \geqslant t_0, \tag{3.5}$$

*where* $u(t) = u(t, t_0, u_0)$ *is any solution of fuzzy impulsive differential equation* (3.1) *existing on* $[t_0, \infty)$.

*Proof.* Let $u(t) = u(t, t_0, w_0)$ be any solution of (3.1) existing on $[t_0, \infty)$. Define $m(t) = V(t, u(t))$ so that $m(t_0^+) = V(t_0^+, u_0)$ and suppose that $m(t_0^+) \leqslant w_0$. Now for small $h > 0$ and $t \in (t_{k-1}, t_k]$, $k = 1, 2, \ldots$,

$$\begin{aligned}
m(t+h) - m(t) &= V(t+h, u(t+h)) - V(t, u(t)) \\
&= V(t+h, u(t+h)) - V(t+h, u(t) + hf(t, u(t))) \\
&\quad + V(t+h, u(t) + hf(t, u(t))) - V(t, u(t)), \\
&\leqslant Ld[u(t+h, u(t) + hf(t, u(t))] \\
&\quad + V(t+h, u(t) + hf(t, u(t))) - V(t, u(t)),
\end{aligned}$$

using the Lipschitz condition assumed in $(ii)$ of the definition of $V_0$. Thus

$$D^+ m(t) = \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \leqslant D^+ V(t, u(t))$$

$$+ L \limsup_{h\to 0^+} \frac{1}{h}[d[u(t+h), u(t) + hf(t, u(t))]].$$

Let $u(t+h) = u(t) + z(t)$, where $z(t)$ is the $H$-difference for small $h > 0$ which is assumed to exist. Hence employing the properties of $d[u,v]$, we see that

$$\begin{aligned}
d[u(t+h), u(t) + hf(t, u(t))] &= d[u(t) + z(t), u(t) + hf(t, u(t))] \\
&= d[z(t), hf(t, u(t))] \\
&= d[u(t+h) - u(t), hf(t, u(t))].
\end{aligned}$$

Consequently,

$$\frac{1}{h} d[u(t+h), u(t) + hf(t, u(t))] = d[\frac{u(t+h) - u(t)}{h}, f(t, u(t))]$$

and therefore,

$$\limsup_{h\to 0^+} \frac{1}{h} d[u(t+h), u(t) + hf(t, u(t))]$$

$$= \limsup_{h \to 0^+} \frac{1}{h} d[\frac{u(t+h)-u(t)}{h}, f(t,u(t))]$$
$$= d[u'(t), f(t,u(t))] = 0,$$

since $u(t)$ is the solution of (3.1). We therefore have the scalar differential inequality

$$D^+ m(t) \leqslant g(t,m(t)), \quad t \neq t_k.$$

From (3.4), we get for $t = t_k$,

$$m(t_k^+) = V(t_k^+, u(t_k^+)) = V(t_k^+, u(t_k) + I_k(u(t_k)))$$
$$\leqslant \psi_k(V(t_k, u(t_k)) = \psi_k(m(t_k)).$$

Hence by Theorem 2.2, we arrive at

$$m(t) \leqslant r(t), \quad t \geqslant t_0,$$

proving (3.5).

Some special cases of $g(t,w)$ and $\psi_k(w)$ which are instructive are given below.

COROLLARY 3.1. *In Theorem* 3.1*, suppose that*

(1) $g(t,w) \equiv 0$, $\psi_k(w) = w$ *for all* $k$*, then* $V(t,u(t))$ *is nondecreasing in* $t$ *and* $V(t,u(t)) \leqslant V(t_0^+, u_0)$, $t \geqslant t_0$*;*

(2) $g(t,w) \equiv 0$, $\psi_k(w) = d_k w$, $d_k \geqslant 0$ *for all* $k$*, then*

$$V(t,u(t)) \leqslant V(t_0^+, u_0) \prod_{t_0 < t_k < t} d_k, \quad t \geqslant t_0;$$

(3) $g(t,w) = -\alpha w$, $\alpha > 0$ *and* $\psi_k(w) = d_k w$, $d_k \geqslant 0$ *for all* $k$*, then*

$$V(t,u(t)) \leqslant [V(t_0^+, u_0) \prod_{t_0 < t_k < t} d_k] e^{-\alpha(t-t_0)}, \quad t \geqslant t_0.$$

(4) $g(t,w) = \lambda'(t) w$, $\psi_k(w) = d_k w$, $d_k \geqslant 0$ *for all* $k$, $\lambda \in C^1[R_+, R_+]$*, then*

$$V(t,u(t)) \leqslant [V(t_0^+, u_0) \prod_{t_0 < t_k < t} d_k] exp(\lambda(t) - \lambda(t_0)), \quad t \geqslant t_0.$$

We shall prove a typical result on stability criteria.

THEOREM 3.2. *Assume that*

(*i*) $V: R_+ \times S(\rho) \to R_+$, $V \in V_0$ *and* $D^+ V(t,u) \leqslant g(t, V(t,u))$, $t \neq t_k$*, where* $S(\rho) = [u \in E^n : d[u, \hat{o}] < \rho]$, $g: R_+^2 \to R$, $g(t,0) \equiv 0$ *and* $g$ *satisfies assumptions given in Theorem 3.1;*

(*ii*) *there exists a* $\rho_0 > 0$ *such that* $u \in S(\rho_0)$ *implies* $u + I_k(u) \in S(\rho)$ *for all* $k$ *and*

$$V(t, u + I_k(u)) \leqslant \psi_k(V(t,u)), \quad t = t_k, \quad u \in S(\rho_0),$$

*where* $\psi_k: R_+ \to R_+$ *is nondecreasing;*

(*iii*) $b(d[u,\hat{o}]) \leqslant V(t,u) \leqslant a(d[u,\hat{o}])$, $(t,u) \in R_+ \times S(\rho)$*, where* $a, b \in \mathscr{K} = [\sigma \in C[R_+, R_+] : \sigma(0) = 0$ *and* $\sigma(w)$ *is increasing in* $w]$.

*Then the stability properties of the trivial solution of* (2.3) *implies the corresponding stability properties of the trivial solution of* (3.1).

*Proof.* We shall give the proof of stability only since the proofs of other stability concepts can be proved based on this and the standard proofs. See [8, 9]. Let $0 < \epsilon < \min(\rho, \rho_0)$, $t_0 \in R_+$ be given. Suppose that the trivial solution of (2.3) is stable. Then given $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$0 \leqslant w_0 < \delta_1 \quad \text{implies} \quad w(t, t_0, w_0) < b(\epsilon), \quad t \geqslant t_0,$$

where $w(t, t_0, w_0)$ is any solution of (2.3). Let $w_0 = a(d[u_0, \widehat{o}])$ and choose a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(\delta) < \delta_1.$$

We claim that with this $\delta$, we have

$$d[u_0, \widehat{o}] < \delta \quad \text{implies} \quad d[u(t), \widehat{o}] < \epsilon, \quad t \geqslant t_0,$$

for any solution $u(t) = u(t, t_0, u_0)$ of (3.1). If this is not true, there would exist a solution $u(t) = u(t, t_0, u_0)$ of (3.1) with $d[u_0, \widehat{o}] < \delta$ and a $t^* > t_0$ satisfying $t_k < t^* \leqslant t_{k+1}$, for some $k$

$$\epsilon \leqslant d[u(t^*), \widehat{o}] \quad \text{and} \quad d[u(t), \widehat{o}] < \epsilon, \quad t_0 \leqslant t \leqslant t_k.$$

Since $0 < \epsilon < \rho_0$, condition $(ii)$ shows that

$$d[u(t_k^+), \widehat{o}] = d[u(t_k) + I_k(u(t_k)), \widehat{o}] < \rho \quad \text{and} \quad d[u(t_k), \widehat{o}] < \epsilon.$$

Hence we can find a $t^0$ such that $t_k < t^0 \leqslant t^*$ satisfying

$$\epsilon \leqslant d[u(t^0), \widehat{o}] < \rho.$$

Now setting $m(t) = V(t, u(t))$ for $t_0 \leqslant t \leqslant t^0$ and using $(i)$ and $(ii)$ we get by Theorem 3.1, the estimate

$$V(t, u(t)) \leqslant r(t, t_0, a(d[u_0, \widehat{o}])), \quad t_0 \leqslant t \leqslant t^0,$$

where $r(t, t_0, w_0)$ is the maximal solution of (2.3). We are then led to the contradiction, because of $(iii)$,

$$b(\epsilon) \leqslant b(d[u(t^0), \widehat{o}]) \leqslant V(t^0, u(t^0)) \leqslant r(t^0, t_0, a(d[u_0, \widehat{o}]))$$
$$< r(t^0, t_0, a(\delta)) < r(t^0, t_0, \delta_1) < b(\epsilon),$$

which proves the claim. The proof is complete.

A simple example of $V(t, u)$ is $d[u, \widehat{o}]$ so that

$$D^+V(t, u) = \limsup_{h \to 0^+} \frac{1}{h}[d[u + hf(t, u), \widehat{o}] - d[u, \widehat{o}]].$$

Consider the special case (4) given in Corollary 3.1, namely $g(t, w) = \lambda'(t)w$, $\psi_k(w) = d_k w$, $d_k \geqslant 0$ for all $k$ and $\lambda \in C^1[R_+, R_+]$ with $\lambda'(t) \geqslant 0$. If $\lambda(t)$ satisfies

$$\lambda(t_{k+1}) + \ln d_k \leqslant \lambda(t_k) \quad \text{for all} \quad k, \tag{3.6}$$

then the trivial solution of (2.3) is stable. This follows because the solution of (2.3) in this case is

$$w(t,t_0,w_0) = w_0 \prod_{t_0<t_k<t} d_k \exp[\lambda(t) - \lambda(t_0)], \quad t \geqslant t_0.$$

Since $\lambda(t)$ is nondecreasing in $t$, it follows that, using (3.6),

$$0 \leqslant w(t,t_0,w_0) \leqslant w_0 \exp[\lambda(t_1) - \lambda(t_0)], \quad t \geqslant t_0,$$

provided $0 < t_0 < t_1$. Hence choosing $\delta = \frac{\epsilon}{2}\exp[\lambda(t_0) - \lambda(t_1)]$ stability follows. For details on impulsive differential equations, see [9].

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(Received April 21, 2000)

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