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EXISTENCE OF SOLUTIONS FOR $2n^{th}$ ORDER NONLINEAR GENERALIZED STURM–LIOUVILLE BOUNDARY VALUE PROBLEMS

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Abstract. Higher order upper and lower solutions are used to establish the existence of solutions to $y^{(2n)} = f(t, y, y'', \dots, y^{(2n-2)})$, satisfying nonlinear boundary conditions, either of the form $g_i(y^{(2i-2)}(0), y^{(2i-1)}(0)) = 0$, $h_i(y^{(2i-2)}(1), y^{(2i-1)}(1)) = 0$, $1 \leqslant i \leqslant n$, or of the form $k_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) = 0$, $\ell_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) = 0$, $1 \leqslant i \leqslant n$.

1. Introduction

Investigations of relationships between the existence of functions satisfying differential inequalities and the existence of a solution of a boundary value problem for an ordinary differential equation have quite a history. Historical work along those lines can be found in the papers by Ako [4, 5, 6], Gaines [13], Jackson [17], Mawhin [20], and Nagumo [22]. The methods remain yet fruitful and appear in more recent works by authors such as Eloe and Henderson [12] and Thompson [25, 26].

In this paper, higher order upper and lower solutions methods are used to establish the existence of solutions of the even order ordinary differential equation,

$$y^{(2n)} = f(t, y, y'', \dots, y^{(2n-2)}), \quad 0 \leqslant t \leqslant 1, \tag{1}$$

satisfying the nonlinear boundary conditions, either of the form

$$\begin{aligned} g_i(y^{(2i-2)}(0), y^{(2i-1)}(0)) = 0, \ \ 1 \leqslant i \leqslant n, \\ h_i(y^{(2i-2)}(1), y^{(2i-1)}(1)) = 0, \ \ 1 \leqslant i \leqslant n, \end{aligned} \tag{2}$$

or of the form

$$\begin{aligned} k_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) = 0, \ \ 1 \leqslant i \leqslant n, \\ \ell_i(y^{(2i-2)}(0), y^{(2i-2)}(1)) = 0, \ \ 1 \leqslant i \leqslant n, \end{aligned} \tag{3}$$

where $f(t, x_1, \dots, x_n) : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and $g_i, h_i, k_i, \ell_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, $1 \leqslant i \leqslant n$.

We note that the boundary conditions 2 generalize the classical Sturm-Liouville linear conditions, and each of 2 or 3 generalize Lidstone linear conditions. Borrowing from terminology introduced by Thompson [25, 26], we will refer to conditions 2 or

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3 as *fully nonlinear* boundary conditions. It will be necessary to make various growth assumptions concerning the boundary conditions. However, it is surprising that it is not necessary to make assumptions concerning the smoothness of the boundary conditions.

Some attention has been devoted to higher order upper and lower solutions methods yielding solutions of higher order boundary value problems. The papers by Klaasen [19] and Kelley [18] would be considered classics in that sense, and Schmitt [24] used upper and lower solutions methods to obtain comparison theorems for higher order conjugate boundary value problems. More recently, Šeda [23], Eloe and Grimm [11] and Hong and Hu [16] made use of monotone methods in conjunction with upper and lower solutions to obtain solutions of higher order boundary value problems.

We also remark that even order differential equations such as 1 have very much been in the recent literature, but usually when f depends at most on t and y, but not on higher order derivatives, and when the boundary conditions involve linear g_i and h_i or linear k_i and ℓ_i; see, for example, [1, 2, 3, 9, 10, 14, 15]. When $n = 1$, much is known about the applications from which equation 1 arises, and when $n = 2$, applications of fourth order differential equations arise in modeling axially loaded beams fastend together [7] or in modeling the effects of soil settlement on elastically bedded building girders [8]. For $n > 2$, Meirovitch [21] used higher even order boundary value problems (involving a differential stiffness operator of order $2n$ along with boundary operators of maximum order $2n-1$), in studying the open-loop control of a distributed structure.

The existence of solutions of 1, 2 and 1, 3 will be dealt with in separate sections. In Section 3, boundedness conditions are imposed on g_i and h_i, whereas in Section 4, monotonicity conditions are imposed on k_i and ℓ_i. In both cases, it is assumed that appropriate upper and lower solutions exist from which solutions of the respective boundary value problems are obtained.

2. Two background representation lemmas

In this section, we present a couple of integral representation lemmas. The first will be useful to represent solutions of 1, 2, and the second will be indispensable in passing sign information from higher order derivatives to lower order derivatives. While their proofs are standard, we include them for completeness.

LEMMA 2.1. *Suppose $x(t)$ is a solution to the integral equation*

$$\begin{aligned} x(t) = &\sum_{i=1}^{n}(g_i(x^{(2i-2)}(0), x^{(2i-1)}(0)) + x^{(2i-2)}(0))p_i(t) \\ &+ \sum_{i=1}^{n}(h_i(x^{(2i-2)}(1), x^{(2i-1)}(1)) + x^{(2i-2)}(1))q_i(t) \\ &+ \int_0^1 G(t,s)f(s, x(s), x''(s), \ldots, x^{(2n-2)}(s))ds \end{aligned}$$

where $G(t,s)$ is the Green's function for $x^{(2n)} = 0, x^{(2i-2)}(0) = x^{(2i-2)}(1) = 0$, $1 \leqslant i \leqslant n$. Here the functions p_i and q_i satisfy $p_i^{(2j-2)}(0) = \delta_{ij}$, $p_i^{(2j-2)}(1) = 0$,

$q_i^{(2j-2)}(0) = 0$, $q_i^{(2j-2)}(1) = \delta_{ij}, 1 \leqslant i,j \leqslant n$, *with* p_i *and* q_i *solutions to* $x^{(2n)} = 0$. *Then* x *is a solution to* 1,2.

Proof. Suppose x is a solution to the integral equation above. Then using the boundary conditions that the Green's function and the p_i and q_i satisfy at $t = 0$, we obtain $x^{(2j-2)}(0) = (g_j(x^{(2j-2)}(0), x^{(2j-1)}(0)) + x^{(2j-2)}(0))p_j^{(2j-2)}(0)$. However, $p_j^{(2j-2)}(0) = 1$ implies $g_j(x^{(2j-2)}(0),\ x^{(2j-1)}(0)) = 0$. A similar argument at $t = 1$ shows $h_i(x^{(2i-2)}(1), x^{(2i-1)}(1)) = 0$. This shows x satisfies the boundary conditions 2. The right hand side of the integral equation is $2n$ times differentiable. Differentiating the integral equation $2n$ times yields x satisfies 1.

LEMMA 2.2. *If* $x(t) \in C^{(2)}[0,1]$, *then* $x(t) = x(0)(1-t)+x(1)t+\int_0^1 H(t,s)x''(s)ds$ *where* $H(t,s)$ *is the Green's function for* $x'' = 0, x(0) = x(1) = 0$.

Proof. Let $u(t) = x(0)(1-t)+x(1)t+\int_0^1 H(t,s)x''(s)ds$. Then $u(0) = x(0), u(1) = x(1)$, and $u''(t) = x''(t)$. Since the only solution of $x'' = 0, x(0) = 0, x(1) = 0$, is the trivial solution, it follows that $u(t) = x(t)$ for all t.

3. An existence theorem

In this section, we define what we mean by an upper solution and a lower solution of 1, 2. We also impose certain boundedness conditions on the x-variable for which $g_i(x,y) = 0$ and $h_i(x,y) = 0$. We then use upper and lower solutions methods in conjunction with Lemmas 2.1 and 2.2 to obtain a solution of 1, 2.

We make the following assumption concerning the boundary conditions.

Assumption (A): For each i, the set of x-values such that $g_i(x,y) = 0, h_i(x,y) = 0$ are bounded. More specifically, we assume that, for $1 \leqslant i \leqslant n$, there exist real numbers $a_i < b_i$ and $c_i < d_i$, such that $g_i(x,y) = 0$ implies $a_i \leqslant x \leqslant b_i$ and $h_i(x,y) = 0$ implies $c_i \leqslant x \leqslant d_i$. We also assume the boundary conditions are such that $g_i(x,y) + x$ and $h_i(x,y) + x$ are bounded functions.

An *upper solution* to 1, 2 is a function $q \in C^{(2n)}[0,1]$ satisfying

$$\begin{aligned}
q^{(2n)}(t) &\leqslant f(t, q(t), , q''(t), \ldots, q^{(2n-2)}(t)),\ t \in [0,1],\\
q^{(2n-2)}(0) &\geqslant b_n,\ q^{(2n-2)}(1) \geqslant d_n,\\
q^{(2n-4)}(0) &\leqslant a_{n-1},\ q^{(2n-4)}(1) \leqslant c_{n-1},\\
q^{(2n-6)}(0) &\geqslant b_{n-2},\ q^{(2n-6)}(1) \geqslant d_{n-2},\\
&\vdots
\end{aligned}$$

Similarly, a *lower solution* to 1, 2 is a function $p \in C^{(2n)}[0,1]$ satisfying

$$\begin{aligned} p^{(2n)}(t) &\geqslant f(t,p(t),p''(t),\dots,p^{(2n-2)}(t)),\ t \in [0,1],\\ p^{(2n-2)}(0) &\leqslant a_n,\ p^{(2n-2)}(1) \leqslant c_n,\\ p^{(2n-4)}(0) &\geqslant b_{n-1},\ p^{(2n-4)}(1) \geqslant d_{n-1},\\ p^{(2n-6)}(0) &\leqslant a_{n-2},\ p^{(2n-6)}(1) \leqslant c_{n-2},\\ &\vdots \end{aligned}$$

We remark that the upper and lower solutions thus defined have the advantage that they do not involve the nonlinear boundary conditions directly.

We now present the main result of the section which establishes the existence of a solution of 1, 2 which lies between a pair of upper and lower solutions.

THEOREM 3.1. *Assume*

1. $f(t,x_1,x_2,\dots,x_n) : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ *is continuous;*
2. $f(t,x_1,x_2,\dots,x_n)$ *is increasing in the* $x_{n-1},x_{n-3},x_{n-5},\dots$ *variables;*
3. $f(t,x_1,x_2,\dots,x_n)$ *is decreasing in the* $x_{n-2},x_{n-4},x_{n-6},\dots$ *variables;*
4. *The boundary conditions satisfy assumption (A).*

If, in addition, there exist an upper solution q *and a lower solution* p *for* 1, 2 *such that* $(-1)^{i+1}q^{(2n-2i)}(t) \geqslant (-1)^{i+1}p^{(2n-2i)}(t)$ *for all* $t \in [0,1]$ *and* $i = 1,2,\dots,n$, *then there exists a solution* $x(t)$ *to* 1, 2. *Moreover,* $(-1)^{i+1}p^{(2n-2i)}(t) \leqslant (-1)^{i+1}x^{(2n-2i)}(t) \leqslant (-1)^{i+1}q^{(2n-2i)}(t)$, *for* $t\in[0,1]$ *and* $i = 1,2,\dots,n$.

Proof. For $1 \leqslant j \leqslant n$, define $\alpha_{2n-2j}(y^{(2n-2j)}(t)) =$

$$\begin{cases} \max\{p^{(2n-2j)}(t),\ \min\{y^{(2n-2j)}(t), q^{(2n-2j)}(t)\}\} & \text{if } j \text{ is odd},\\ \max\{q^{(2n-2j)}(t),\ \min\{y^{(2n-2j)}(t), p^{(2n-2j)}(t)\}\} & \text{if } j \text{ is even},\end{cases}$$

where y is a function defined on $[0,1]$. If $y^{(2n-2j)}$ is continuous, then α_{2n-2j} is continuous. Moreover, $(-1)^{i+1}p^{(2n-2i)}(t) \leqslant (-1)^{i+1}\alpha_{2n-2i}(y^{(2n-2i)}(t)) \leqslant (-1)^{i+1}q^{(2n-2i)}(t)$, for $i = 1,2,\dots,n$. Define $F : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ by $F(t,y,y'',\dots,y^{(2n-2)}) =$

$$\begin{cases} f(t,\alpha_0(y(t)),\dots,\alpha_{2n-4}(y^{(2n-4)}(t)),q^{(2n-2)}(t)) + \dfrac{y^{(2n-2)}(t)-q^{(2n-2)}(t)}{1+(y^{(2n-2)}(t))^2},\\ \qquad \text{if } y^{(2n-2)}(t) \geqslant q^{(2n-2)}(t),\\ f(t,\alpha_0(y(t)),\dots,\alpha_{2n-4}(y^{(2n-4)}(t)),y^{(2n-2)}(t)),\\ \qquad \text{if } p^{(2n-2)}(t) \leqslant y^{(2n-2)}(t) \leqslant q^{(2n-2)}(t),\\ f(t,\alpha_0(y(t)),\dots,\alpha_{2n-4}(y^{(2n-4)}(t)),p^{(2n-2)}(t)) - \dfrac{p^{(2n-2)}(t)-y^{(2n-2)}(t)}{1+(y^{(2n-2)}(t))^2},\\ \qquad \text{if } y^{(2n-2)}(t) \leqslant p^{(2n-2)}(t).\end{cases}$$

Then F is continuous and bounded for any continuous $y \in C^{(2n-2)}$. Moreover, $F(t,x_1,\dots,x_n)$ has the same increasing/decreasing properties as does f. Define

$T: C^{(2n-2)}[0,1] \to C^{(2n-2)}[0,1]$ by

$$\begin{aligned} Ty(t) = &\sum_{i=1}^{n}(g_i(y^{(2i-2)}(0), y^{(2i-1)}(0)) + y^{(2i-2)}(0))p_i(t) \\ &+ \sum_{i=1}^{n}(h_i(y^{(2i-2)}(1), y^{(2i-1)}(1)) + y^{(2i-2)}(1))q_i(t) \\ &+ \int_0^1 G(t,s)F(s, y(s), y''(s), \ldots, y^{(2n-2)}(s))ds. \end{aligned}$$

For each $0 \leqslant k \leqslant 2n-2$, consider

$$\begin{aligned} |(Ty)^{(k)}(t)| \leqslant &\sum_{i=1}^{n}|g_i(y^{(2i-2)}(0), y^{(2i-1)}(0)) + y^{(2i-2)}(0)| \cdot |p_i^{(k)}(t)| \\ &+ \sum_{i=1}^{n}|h_i(y^{(2i-2)}(1), y^{(2i-1)}(1)) + y^{(2i-2)}(1)| \cdot |q_i^{(k)}(t)| \\ &+ \int_0^1 |\frac{\partial^k G}{\partial t^k}(t,s)| \cdot |F(s, y(s), y''(s), \ldots, y^{(2n-2)}(s))|ds. \end{aligned}$$

By assumption, $\sum_{i=1}^{n}|g_i(y^{(2i-2)}(0), y^{(2i-1)}(0)) + y^{(2i-2)}(0)|$ and $\sum_{i=1}^{n}|h_i(y^{(2i-2)}(1),$ $y^{(2i-1)}(1)) + y^{(2i-2)}(1)|$ are bounded. As $\dfrac{\partial^k G}{\partial t^k}$ and F are also bounded, it follows that the $|(Ty)^{(k)}(t)|$ are uniformly bounded for all $y \in C^{(2n-2)}$. By Schauder's Theorem, there exists a function x satisfying $Tx(t) = x(t)$. To complete the proof using Lemma 2.1, it must be demonstrated that $f(t, x(t), \ldots, x^{(2n-2)}(t)) = F(t, x(t), \ldots, x^{(2n-2)}(t))$.

Suppose there exists t_0 such that $x^{(2n-2)}(t_0) > q^{(2n-2)}(t_0)$. Without loss of generality, assume $x^{(2n-2)}(t) - q^{(2n-2)}(t)$ is maximized at t_0. If $t_0 = 0$, then $b_n \geqslant x^{(2n-2)}(t_0) > q^{(2n-2)}(t_0) > b_n$. A similar contradiction occurs at $t_0 = 1$. Assume $t_0 \in (0,1)$. Then $x^{(2n)}(t_0) \leqslant q^{(2n)}(t_0)$ and hence

$$\begin{aligned} 0 &\geqslant x^{(2n)}(t_0) - q^{(2n)}(t_0) \geqslant F(t_0, x(t_0), \ldots, x^{(2n-2)}(t_0)) - f(t_0, q(t_0), \ldots, q^{(2n-2)}(t_0)) \\ &= f(t_0, \alpha_0(x(t_0)), \ldots, \alpha_{2n-4}(x^{(2n-4)}(t_0)), q^{(2n-2)}(t_0)) + \frac{x^{(2n-2)}(t_0) - q^{(2n-2)}(t_0)}{1 + (x^{(2n-2)}(t_0))^2} \\ &\quad - f(t_0, q(t_0), \ldots, q^{(2n-2)}(t_0)). \end{aligned}$$

Using the increasing/decreasing properties we obtain the above expression is greater than or equal to

$$f(t_0, q(t_0), \ldots, q^{(2n-2)}(t_0)) + \frac{x^{(2n-2)}(t_0) - q^{(2n-2)}(t_0)}{1 + x^{(2n-2)}(t_0)^2} - f(t_0, q(t_0), \ldots, q^{(2n-2)}(t_0)) > 0.$$

This contradiction implies $x^{(2n-2)}(t) \leqslant q^{(2n-2)}(t)$ for all $t \in [0,1]$. A similar argument shows $p^{(2n-2)}(t) \leqslant x^{(2n-2)}(t)$ for all $t \in [0,1]$.

Lemma 2.2 implies

$$x^{(2n-4)}(t)-q^{(2n-4)}(t)=\left(x^{(2n-4)}(0)-q^{(2n-4)}(0)\right)(1-t)+\left(x^{(2n-4)}(1)-q^{(2n-4)}(1)\right)t$$
$$+\int_0^1 H(t,s)\left(x^{(2n-2)}(s)-q^{(2n-2)}(s)\right)ds.$$

The assumptions on the boundary conditions and the definition of the upper solution implies the first two terms are positive. It is well known that the Green's function $H(t,s)\leqslant 0$. Moreover, the previous paragraph demonstrated that $x^{(2n-2)}(s)-q^{(2n-2)}(s)\leqslant 0$. The combined effect is $x^{(2n-4)}(t)-q^{(2n-4)}(t)\geqslant 0$ for all $t\in[0,1]$. Similar analysis demonstrates $p^{(2n-4)}(t)-x^{(2n-4)}(t)\geqslant 0$ for all $t\in[0,1]$.

Repeated application of the above argument yields

$$(-1)^{i+1}p^{(2n-2i)}(t)\leqslant(-1)^{i+1}x^{(2n-2i)}(t)\leqslant(-1)^{i+1}q^{(2n-2i)}(t),$$
$$\text{for } t\in[0,1] \text{ and } i=1,2,\ldots,n,$$

which in turn implies $f\left(t,x(t),\ldots,x^{(2n-2)}(t)\right)=F(t,x(t),\ldots,x^{(2n-2)}(t))$. Thus x is a solution to 1, 2.

4. Second existence theorem

In this section we will apply the techniques developed in the previous section to prove a second existence theorem with regard to the boundary value problem 1, 3. For this boundary value problem, there is a representation of solutions analogous to the one in Lemma 2.1. We will not state that representation. In the previous section, we assumed a boundedness condition on our boundary conditions. In this section, that condition is removed. Instead we make the following monotonicity assumption concerning our boundary conditions.

Assumption (B): Assume each $k_i(x,y),\ell_i(x,y)$ is an *increasing* function of each of its variables.

It will be convenient to write our boundary conditions in an alternate algebraic form. Set $\widehat{k}_i(x,y)=k_i(x,y)+x$ and $\widehat{\ell}_i(x,y)=\ell_i(x,y)+y$. Then $\widehat{k}_i$ and $\widehat{\ell}_i$ are increasing and 3 has the form $\widehat{k}_i(y^{(2i-2)}(0),y^{(2i-2)}(1))=y^{(2i-2)}(0)$, $\widehat{\ell}_i(y^{(2i-2)}(0),y^{(2i-2)}(1))=y^{(2i-2)}(1)$.

An *upper solution* to 1, 3 is a function $q\in C^{(2n)}[0,1]$ satisfying

$$\begin{aligned}
&q^{(2n)}\leqslant f\left(t,q,q'',\ldots,q^{(2n-2)}\right)\\
&\widehat{k}_i(q^{(2i-2)}(0),q^{(2i-2)}(1))\leqslant q^{(2i-2)}(0),\\
&\widehat{\ell}_i(q^{(2i-2)}(0),q^{(2i-2)}(1))\leqslant q^{(2i-2)}(1),\ i=n-2k,\\
&\widehat{k}_i(q^{(2i-2)}(0),q^{(2i-2)}(1))\geqslant q^{(2i-2)}(0),\\
&\widehat{\ell}_i(q^{(2i-2)}(0),q^{(2i-2)}(1))\geqslant q^{(2i-2)}(1),\ i=n-2k-1,
\end{aligned}$$

where $k\geqslant 1$.

A *lower solution* to 1, 3 is a function $p \in C^{(2n)}[0,1]$ satisfying

$$p^{(2n)} \geqslant f(t,p,p'',\dots,p^{(2n-2)})$$
$$\widehat{k}_i(p^{(2i-2)}(0),p^{(2i-2)}(1)) \geqslant p^{(2i-2)}(0),$$
$$\widehat{\ell}_i(p^{(2i-2)}(0),p^{(2i-2)}(1)) \geqslant p^{(2i-2)}(1),\ i=n-2k$$
$$\widehat{k}_i(p^{(2i-2)}(0),p^{(2i-2)}(1)) \leqslant p^{(2i-2)}(0),$$
$$\widehat{\ell}_i(p^{(2i-2)}(0),p^{(2i-2)}(1)) \leqslant p^{(2i-2)}(1),\ i=n-2k-1,$$

where $k \geqslant 1$.

We now present the main result of this section which establishes the existence of a solution to 1, 3 that again lies between an upper solution and a lower solution.

THEOREM 4.1. *Assume*

1. $f(t,x_1,x_2,\dots,x_n) : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ *is continuous;*
2. $f(t,x_1,x_2,\dots,x_n)$ *is increasing in the* x_{n-2k+1} *variables,* $k \geqslant 1$*;*
3. $f(t,x_1,x_2,\dots,x_n)$ *is decreasing in the* x_{n-2k} *variables,* $k \geqslant 1$*;*
4. *The boundary conditions satisfy assumption (B).*

If, in addition, there exist an upper solution q *and a lower solution* p *for* 1, 3 *such that* $(-1)^{i+1}q^{(2n-2i)}(t) \geqslant (-1)^{i+1}p^{(2n-2i)}(t)$ *for all* $t\in[0,1]$ *and* $i = 1,2,\dots,n,$ *then there exists a solution* $x(t)$ *to* 1, 3. *Moreover,* $(-1)^{i+1}p^{(2n-2i)}(t) \leqslant (-1)^{i+1}x^{(2n-2i)}(t) \leqslant (-1)^{i+1}q^{(2n-2i)}(t)$, *for* $t\in[0,1]$ *and* $i=1,2,\dots,n$.

Proof. For $1\leqslant j\leqslant n$, let α_{2n-2j} and F be defined as in Theorem 3.1. Define

$$\widetilde{k}_i(y^{(2i-2)}(0),y^{(2i-2)}(1)) = \widehat{k}_i(\alpha_{2i-2}(y^{(2i-2)}(0)),\alpha_{2i-2}(y^{(2i-2)}(1))),$$
$$\widetilde{\ell}_i(y^{(2i-2)}(0),y^{(2i-2)}(1)) = \widehat{\ell}_i(\alpha_{2i-2}(y^{(2i-2)}(0)),\alpha_{2i-2}(y^{(2i-2)}(1))).$$

Then each $\widetilde{k}_i$ and $\widetilde{\ell}_i$ is bounded and increasing in both of its variables. Define $T : C^{(2n-2)}[0,1] \to C^{(2n-2)}[0,1]$ by

$$\begin{aligned} Ty(t) = &\sum_{i=1}^{n}\widetilde{k}_i(y^{(2i-2)}(0),y^{(2i-2)}(1))p_i(t) \\ &+\sum_{i=1}^{n}\widetilde{\ell}_i(y^{(2i-2)}(0),y^{(2i-2)}(1))q_i(t) \\ &+\int_0^1 G(t,s)F(s,y(s),y''(s),\dots,y^{(2n-2)}(s))ds. \end{aligned}$$

Using the same type of reasoning as in Theorem 3.1, we obtain T has a fixed point $x(t)$ via Schauder's Theorem. The definition of T ensures $x^{(2i-2)}(0) = \widetilde{k}_i(x^{(2i-2)}(0),x^{(2i-2)}(1))$ and $x^{(2i-2)}(1) = \widetilde{\ell}_i(x^{(2i-2)}(0),x^{(2i-2)}(1))$. It remains to demonstrate

$$\widetilde{k}_i(x^{(2i-2)}(0),x^{(2i-2)}(1)) = \widehat{k}_i(x^{(2i-2)}(0),x^{(2i-2)}(1)),$$
$$\widetilde{\ell}_i(x^{(2i-2)}(0),x^{(2i-2)}(1)) = \widehat{\ell}_i(x^{(2i-2)}(0),x^{(2i-2)}(1)),\ \text{and}$$
$$f(t,x(t),\dots,x^{(2n-2)}(t)) = F(t,x(t),\dots,x^{(2n-2)}(t)).$$

Suppose there exists t_0 such that $x^{(2n-2)}(t_0) > q^{(2n-2)}(t_0)$. Without loss of generality, assume $x^{(2n-2)}(t) - q^{(2n-2)}(t)$ is maximized at t_0. If $t_0 = 0$, then the fact that $\widetilde{k}_n$ is increasing and $q(t)$ is an upper solution imply $x^{(2n-2)}(0) = \widetilde{k}_n(x^{(2n-2)}(0), x^{(2n-2)}(1)) \leqslant \widetilde{k}_n(q^{(2n-2)}(0), q^{(2n-2)}(1)) = \widehat{k}_n(q^{(2n-2)}(0), q^{(2n-2)}(1)) \leqslant q^{(2n-2)}(0)$. Thus $t_0 \neq 0$. A similar argument involving $\widehat{\ell_n}$ shows $t_0 \neq 1$. Thus $t_0 \in (0,1)$. Applying the same argument involving F as in Theorem 3.1, we obtain $x^{(2n-2)}(t) \leqslant q^{(2n-2)}(t), t \in [0,1]$. In a like manner, it can be established that $p^{(2n-2)}(t) \leqslant x^{(2n-2)}(t) \leqslant q^{(2n-2)}(t)$.

Now suppose there exists t_0 such that $x^{(2n-4)}(t_0) < q^{(2n-4)}(t_0)$. If $t_0 = 0$, then

$$\begin{aligned} x^{(2n-4)}(0) &= \widetilde{k}_{n-1}(x^{(2n-4)}(0), x^{(2n-4)}(1)) \\ &\geqslant \widetilde{k}_{n-1}(q^{(2n-4)}(0), q^{(2n-4)}(1)) \\ &= \widehat{k}_{n-1}(q^{(2n-4)}(0), q^{(2n-4)}(1)) \\ &\geqslant q^{(2n-4)}(0). \end{aligned}$$

A similar contradiction occurs if $t_0 = 1$. Hence $x^{(2n-4)}(0) - q^{(2n-4)}(0) \geqslant 0$ and $x^{(2n-4)}(1) - q^{(2n-4)}(1) \geqslant 0$. The representation Lemma 2.2 yields

$$\begin{aligned} x^{(2n-4)}(t) - q^{(2n-4)}(t) = \left(x^{(2n-4)}(0) - q^{(2n-4)}(0)\right)(1-t) + \left(x^{(2n-4)}(1) - q^{(2n-4)}(1)\right)t \\ + \int_0^1 H(t,s)(x^{(2n-2)}(s) - q^{(2n-2)}(s))ds \geqslant 0,\ t \in [0,1]. \end{aligned}$$

Similarly, $x^{(2n-4)}(t) - p^{(2n-4)}(t) \leqslant 0,\ t \in [0,1]$. Continuing in this manner we obtain

$$(-1)^{i+1} p^{(2n-2i)}(t) \leqslant (-1)^{i+1} x(t)^{(2n-2i)} \leqslant (-1)^{i+1} q^{(2n-2i)}(t),$$

for $t \in [0,1]$ and $i = 1, 2, \ldots, n$, and hence $x(t)$ is a solution to 1, 3.

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