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SOME INEQUALITIES FOR COSINE SUMS

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(*communicated by Th. M. Rassias*)

Abstract. We establish sharp lower and upper estimates for the cosine sums $\sum_{k=1}^{n} \frac{\cos k\theta}{k+1}$. We also discuss the possibility of extending these results to other cosine sums of this type.

1. Introduction

In 1928, Rogosinski and Szegő [8], showed that for every positive integer n and for $0 < \theta < \pi$,

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geqslant 0. \tag{1}$$

Over the years, several extensions and generalizations of this inequality have been obtained by different authors (see for example [1], [2], [4]). In [9], Tomić improved the Rogosinski-Szegő inequality above, when $n \geqslant 2$, by showing that there is a constant $K > 0$, independent of n and θ such that the inequality

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geqslant K > \frac{1}{168}, \tag{2}$$

holds (see also [6], ch. 4).

Here, we are able to prove that this result of Tomić admits considerable improvement by determining the best constant K in (1.2). In this paper, we develop a method of estimating cosine sums of this type which enables us to find also an n-independent functional upper bound for these sums.

Our main results are the following:

THEOREM 1. *For any positive integer $n > 1$ and $0 < \theta < \pi$ we have the inequality*

$$\frac{41}{96} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geqslant 0. \tag{3}$$

The leading constant $\frac{41}{96}$ in the above inequality is best possible and corresponds to the case $n = 2$.

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THEOREM 2. *For all positive integers* n *and for* $0<\theta<\pi$*, we have*

$$\sum_{k=1}^{n}\frac{\cos k\theta}{k+1}<Ci\left(\frac{\pi}{2}\right)+\sum_{k=1}^{\infty}\frac{\cos k\theta}{k+1}. \tag{4}$$

Note that

$$Ci\left(\frac{\pi}{2}\right)=-\int_{\frac{\pi}{2}}^{\infty}\frac{\cos t}{t}\,dt=0.472\ldots$$

and this is the best constant in the inequality (1.4).

Of course, it is well-known and easy to see that

$$\sum_{k=1}^{\infty}\frac{\cos k\theta}{k+1}=-1-\cos\theta\,\log(2\sin\frac{\theta}{2})+\left(\frac{\pi-\theta}{2}\right)\sin\theta. \tag{5}$$

(cf. [7]).

In [3], we established a corresponding result of (1.3) for a simpler case of cosine sums. More specifically, it is shown in [3] that the inequality

$$1+\sum_{k=1}^{n}\frac{\cos k\theta}{k}\geqslant\frac{1}{6}, \tag{6}$$

holds for all $n\geqslant 2$ and $0<\theta<\pi$, the constant $\frac{1}{6}$ being the best possible.

In this paper we are able to prove an analogous result of (1.4) for the sums of (1.6). Namely, we have

THEOREM 3. *For all positive integers* n *and for* $0<\theta<\pi$*, we have*

$$\sum_{k=1}^{n}\frac{\cos k\theta}{k}\leqslant\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\cos k\theta}{k}. \tag{7}$$

The only case of equality in (1.7) occurs when $n=1$ *and* $\theta=\frac{\pi}{3}$.

Note that the constant $1/2$ in the above inequality is best possible. This result improves an old result of W. H. Young [10], *viz.*

$$\sum_{k=1}^{n}\frac{\cos k\theta}{k}\leqslant 5-\log(2\sin\frac{\theta}{2}),\quad 0<\theta<\pi.$$

As it is well known $\sum_{k=1}^{\infty}\frac{\cos k\theta}{k}=-\log(2\sin\frac{\theta}{2})$.

A different functional upper bound for the cosine sums of (1.7) has been obtained by Hyltén-Cavallius [5]. This is

$$\sum_{k=1}^{n}\frac{\cos k\theta}{k}\leqslant-\log(\sin\frac{\theta}{2})+\frac{\pi-\theta}{2},\quad 0<\theta<\pi. \tag{8}$$

(See also [6], ch. 4). Our inequality (1.7) is sharper than (1.8) for the same range of θ.

In the following sections we prove Theorem 1, Theorem 2 and Theorem 3. In the final section we discuss the more general problem of extending (1.4) to the case of cosine sums

$$\sum_{k=1}^{n} \frac{\cos k\theta}{k+p} .$$

2. Proof of Theorem 1

Let

$$S_n(\theta) = \sum_{k=1}^{n} \frac{\cos k\theta}{k+1}.$$

Summing twice by parts and using the familiar formulae

$$\frac{1}{2} + \sum_{k=1}^{n} \cos k\theta \;=\; \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$$

and

$$\sum_{k=0}^{n} \frac{\sin(k+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} \;=\; \frac{1}{2}\left(\frac{\sin(n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^2$$

we get $(n \geqslant 2)$

$$\begin{aligned} S_n(\theta) \;&=\; -\frac{1}{3} + \sum_{k=1}^{n-2} \frac{1}{(k+1)(k+2)(k+3)}\left(\frac{\sin(k+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^2 \\ &+\; \frac{1}{2n(n+1)}\left(\frac{\sin n\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right)^2 + \frac{\sin(n+\frac{1}{2})\theta}{2(n+1)\sin\frac{\theta}{2}}. \end{aligned} \tag{1}$$

It follows from this that, for $0 < \theta \leqslant \frac{2\pi}{2n+1}$,

$$\frac{41}{96} + S_n(\theta) \geqslant \frac{3}{32} .$$

This settles (1.3) for this range of θ.

Suppose that $\theta > \frac{2\pi}{2n+1}$. Using again (2.1) we obtain

$$\frac{41}{96} + S_n(\theta) \geqslant \frac{25}{96} - \frac{1}{6}\sin^2\frac{\theta}{2} - \frac{1}{2(n+1)\sin\frac{\theta}{2}}.$$

We first consider the case where $\frac{2\pi}{n+1/2} \leqslant \theta < \pi$. We set $t = \sin\frac{\theta}{2}$ and define

$$g_n(t) = \frac{25}{48} - \frac{1}{3}t^2 - \frac{1}{(n+1)t}.$$

It is not hard to see that, for $\sin\frac{\pi}{n+1/2} \leqslant t < 1$, $g_n(t)$ is a concave function ($g_n''(t) < 0$). Hence

$$g_n(t) \geqslant \min\left\{g_n\left(\sin\frac{\pi}{n+1/2}\right), \, g_n(1)\right\} > 0, \tag{2}$$

for $n \geqslant 5$. Therefore in this case (1.3) is true.

Now for the remaining interval $\frac{\pi}{n+1/2} \leqslant \theta < \frac{2\pi}{n+1/2}$, we use once more (2.1) to get

$$\frac{41}{96} + S_n(\theta) \geqslant \frac{25}{96} - \frac{1}{6}\sin^2\frac{\theta}{2} - \frac{1}{2(n+1)\sin\frac{\theta}{2}} + \frac{1}{60}(3 - 4\sin^2\frac{\theta}{2})^2.$$

As in the previous case setting $t = \sin\frac{\theta}{2}$, it suffices to prove that the function

$$G_n(t) = g_n(t) + \frac{1}{30}(3 - 4t^2)^2,$$

is positive for $\sin\frac{\pi}{2n+1} < t < \sin\frac{\pi}{n+1/2}$. Indeed, we can easily verify that the function $G_n(t)$ is concave for this range of t, so that

$$\begin{aligned} G_n(t) &\geqslant \min\left\{G_n\left(\sin\frac{\pi}{2n+1}\right), \, G_n\left(\sin\frac{\pi}{n+1/2}\right)\right\} \\ &= G_n\left(\sin\frac{\pi}{2n+1}\right) > 0, \end{aligned}$$

for $n \geqslant 5$.

Finally, by a straightforward computation we can establish (1.3) for the cases $n = 2, 3, 4$. Note that the minimum of the cosine polynomial $S_2(\theta)$ is $-\frac{41}{96}$ which specifies the best possible leading constant in (1.3). The proof of Theorem 1 is complete.

3. Proof of Theorem 2

Throughout this section we shall employ the notations

$$R_n(\theta) = S_n(\theta) - \sum_{k=1}^{\infty}\frac{\cos k\theta}{k+1}, \quad 0 < \theta < \pi.$$

and

$$\theta_n = \frac{\pi}{2n+1}.$$

To see that inequality (1.4) is sharp we first show that

$$\lim_{n\to\infty} R_n(\theta_n) = \mathrm{Ci}\left(\frac{\pi}{2}\right). \tag{1}$$

We observe that

$$S_n(\theta_n) = \sum_{k=1}^{n}\frac{1}{k+1} - 2\sum_{k=1}^{n}\frac{\sin^2 k\frac{\pi}{4n+2}}{k+1}.$$

Then using (1.5) we see that

$$\begin{aligned} R_n(\theta_n) &= \sum_{k=1}^{n+1} \frac{1}{k} - 2\sum_{k=1}^{n} \frac{\sin^2 k\frac{\pi}{4n+2}}{k+1} \\ &+ \cos\theta_n \log(2\sin\frac{\theta_n}{2}) - \left(\frac{\pi-\theta_n}{2}\right)\sin\theta_n. \end{aligned} \tag{2}$$

It is easy to verify that

$$\lim_{n\to\infty} 2\sum_{k=1}^{n} \frac{\sin^2 k\frac{\pi}{4n+2}}{k+1} = 2\int_0^{\frac{\pi}{4}} \frac{\sin^2 t}{t}\,dt = \log\frac{\pi}{2} - \mathrm{Ci}\left(\frac{\pi}{2}\right) + \gamma,$$

where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577215\ldots$$

is Euler's constant. From this and (3.2) we obtain (3.1).

As the cases of (1.4) for $n = 1, 2, \ldots, 5$ can be directly checked, in order to establish Theorem 2 we consider the following cases:

Case A. The interval $0 < \theta \leqslant \theta_n,\ \ n \geqslant 4$.

This interval is the most difficult one to handle. We first show that $R_n(\theta)$ is strictly increasing on the subinterval $0 < \theta \leqslant \dfrac{3}{2n+1}$. Differentiating and using (1.5) we obtain

$$\begin{aligned} R_n'(\theta) &= \frac{\cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} - \sin\theta\log(2\sin\frac{\theta}{2}) - \left(\frac{\pi-\theta}{2}\right)\cos\theta \\ &+ \sum_{k=1}^{n}\frac{\sin k\theta}{k+1}. \end{aligned} \tag{3}$$

We write

$$\sum_{k=1}^{n}\frac{\sin k\theta}{k+1} = \sum_{k=1}^{n+1}\frac{\sin k\theta}{k} - \sum_{k=1}^{\infty}\frac{\sin k\theta}{k(k+1)} + \sum_{k=n+2}^{\infty}\frac{\sin k\theta}{k(k+1)} - \frac{\sin(n+1)\theta}{n+2}. \tag{4}$$

One can easily verify that

$$\sum_{k=1}^{\infty}\frac{\sin k\theta}{k(k+1)} = \frac{\pi-\theta}{2}(1-\cos\theta) - \sin\theta\,\log(2\sin\frac{\theta}{2})\,. \tag{5}$$

Then a summation by parts gives

$$\left|\sum_{k=n+2}^{\infty}\frac{\sin k\theta}{k(k+1)}\right| \leqslant \frac{1}{(n+2)(n+3)\sin\frac{\theta}{2}}\,.$$

It follows from this, (3.3), (3.4) and (3.5) that

$$R_n'(\theta) \geqslant \frac{\cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} + \sum_{k=1}^{n+1}\frac{\sin k\theta}{k} - \frac{1}{(n+2)(n+3)\sin\frac{\theta}{2}} - \frac{\pi}{2} + \frac{\theta}{2} - \frac{1}{n+2}. \quad (6)$$

Also, it can be easily seen that the function on the right hand side of (1.5) is convex for $0<\theta<\pi$. Now, for $0<\theta\leqslant\frac{3}{2n+1}$, we have

$$S_n''(\theta) = -\sum_{k=1}^{n}\frac{k^2}{k+1}\cos k\theta < 0,$$

hence $R_n''(\theta) < 0$ for the same range of θ, which is to say that $R_n'(\theta)$ is strictly decreasing on this interval. On the other hand, it is clear that

$$\sum_{k=1}^{n+1}\frac{\sin\frac{3k}{2n+1}}{k} > \int_0^{\frac{3}{2}}\frac{\sin t}{t}\,dt = \mathrm{Si}\left(\frac{3}{2}\right) = 1.3246\ldots .$$

Therefore by (3.6) we get

$$\begin{aligned} R_n'(\theta) &\geqslant \frac{\cos(\frac{3}{2})}{2\sin\frac{3}{4n+2}} - \frac{1}{(n+2)(n+3)\sin\frac{3}{4n+2}} \\ &+ \frac{3}{4n+2} - \frac{1}{n+2} + \mathrm{Si}\left(\frac{3}{2}\right) - \frac{\pi}{2} > 0, \end{aligned}$$

for $n\geqslant 8$. Using (3.3) it can be directly checked that $R_n'(\frac{3}{2n+1})>0$ for $4\leqslant n\leqslant 7$. We deduce from the above that $R_n'(\theta)>0$ for $0<\theta\leqslant\frac{3}{2n+1}$, $n\geqslant 4$.

Next, we prove that $R_n(\theta) < \mathrm{Ci}\left(\frac{\pi}{2}\right)$ for $\frac{3}{2n+1}\leqslant\theta\leqslant\frac{\pi}{2n+1}$.
We write $\theta=\frac{t}{2n+1}$, $3\leqslant t\leqslant\pi$. Then using (1.5) we obtain

$$\begin{aligned} R_n(\theta) &= \sum_{k=1}^{n+1}\frac{1}{k} - 2\sum_{k=1}^{n+1}\frac{\sin^2 k\frac{t}{4n+2}}{k+1} \\ &+ \cos\frac{t}{2n+1}\log\left(2\sin\frac{t}{4n+2}\right) - \left(\frac{\pi}{2}-\frac{t}{4n+2}\right)\sin\frac{t}{2n+1} \\ &+ 2\,\frac{\sin^2\frac{(n+1)t}{4n+2}}{n+2}. \end{aligned} \quad (7)$$

It is not hard to see that $(3\leqslant t\leqslant\pi)$

$$\sum_{k=1}^{n+1}\frac{\sin^2 k\frac{t}{4n+2}}{k+1} > \int_0^{\frac{t}{4}}\frac{\sin^2 u}{u}\,du = \frac{1}{2}\left[\log\left(\frac{t}{2}\right) - \mathrm{Ci}\left(\frac{t}{2}\right) + \gamma\right].$$

We let

$$A_n(t) = \sum_{k=1}^{n+1} \frac{1}{k} + \cos \frac{t}{2n+1} \log\left(2 \sin \frac{t}{4n+2}\right) - \left(\frac{\pi}{2} - \frac{t}{4n+2}\right) \sin \frac{t}{2n+1} + 2 \frac{\sin^2 \frac{(n+1)t}{4n+2}}{n+2}.$$

Then by a routine calculation we verify that

$$A_n(t) < \lim_{n\to\infty} A_n(t) = \gamma + \log\left(\frac{t}{2}\right).$$

Combining the above estimates with (3.7) we obtain

$$R_n(\theta) < \text{Ci}\left(\frac{t}{2}\right). \tag{8}$$

Since the function $\text{Ci}(x)$ attains its absolute maximum in $(0, \pi/2]$ at $x_0 = \frac{\pi}{2}$, by (3.8) the desired result follows.

Case B. The interval $\theta_n < \theta \leqslant 2\theta_n,\ \ n \geqslant 4$.

For this range of θ we shall show that $R_n'(\theta) < 0$. In view of the result of Case A, this establishes (1.4) for the interval under consideration. We observe that

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k+1} = \sum_{k=1}^{n} \frac{\sin k\theta}{k} - \sum_{k=1}^{\infty} \frac{\sin k\theta}{k(k+1)} + \sum_{k=n+1}^{\infty} \frac{\sin k\theta}{k(k+1)}$$

and recall that

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} = \int_0^{\theta} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt - \frac{\theta}{2}$$

then use (3.5) to obtain

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k+1} = \int_0^{\theta} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt - \frac{\pi}{2} + \frac{\pi-\theta}{2}\cos\theta + \sin\theta\,\log(2\sin\frac{\theta}{2}) + \sum_{k=n+1}^{\infty} \frac{\sin k\theta}{k(k+1)}.$$

By virtue of (3.3) we get

$$-R_n'(\theta) = -\frac{\cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} - \int_0^{\theta} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt + \frac{\pi}{2} - \sum_{k=n+1}^{\infty} \frac{\sin k\theta}{k(k+1)}. \tag{9}$$

We recall also that

$$\sum_{k=n+1}^{\infty} \frac{\sin k\theta}{k(k+1)} \leqslant \frac{1}{(n+1)(n+2)\sin\frac{\theta}{2}}. \tag{10}$$

Next we show that $-R_n'(\theta)$ is a concave function in the interval in question. Indeed, writing

$$f(\theta)=\sum_{k=1}^{\infty}\frac{\cos k\theta}{k+1}$$

using (1.5) and a straightforward calculation we have $f'''(\theta)<0$. On the other hand, for the same range of θ

$$\frac{d^3}{d\theta^3}S_n(\theta)=\sum_{k=1}^{n}\frac{k^3}{k+1}\ \sin k\theta>0.$$

In view of the above we obtain

$$-R_n'(\theta)\geqslant \min\left\{-R_n'\left(\frac{\pi}{2n+1}\right),\ -R_n'\left(\frac{2\pi}{2n+1}\right)\right\}. \tag{11}$$

Now using (3.9) and (3.10) we have

$$\begin{aligned}
-\ & R_n'\left(\frac{\pi}{2n+1}\right)\\
\geqslant\ & -\int_0^{\frac{\pi}{2n+1}}\frac{\sin(n+\frac12)t}{2\sin\frac t2}\,dt+\frac{\pi}{2}-\frac{1}{(n+1)(n+2)\sin\frac{\pi}{4n+2}}\\
\geqslant\ & \frac{\pi}{2}-\mathrm{Si}\left(\frac{\pi}{2}\right)\ \frac{\pi}{(4n+2)\sin\frac{\pi}{4n+2}}-\frac{1}{(n+1)(n+2)\sin\frac{\pi}{4n+2}}\\
>\ & 0.0011,\quad \text{for } n\geqslant 4.
\end{aligned}\tag{12}$$

Similarly

$$\begin{aligned}
-\ & R_n'\left(\frac{2\pi}{2n+1}\right)\\
\geqslant\ & \frac{1}{2\sin\frac{\pi}{2n+1}}-\int_0^{\frac{2\pi}{2n+1}}\frac{\sin(n+\frac12)t}{2\sin\frac t2}\,dt+\frac{\pi}{2}-\frac{1}{(n+1)(n+2)\sin\frac{\pi}{2n+1}}\\
\geqslant\ & \frac{1}{2\sin\frac{\pi}{2n+1}}+\frac{\pi}{2}-\mathrm{Si}(\pi)\ \frac{\pi}{(2n+1)\sin\frac{\pi}{2n+1}}-\frac{1}{(n+1)(n+2)\sin\frac{\pi}{2n+1}}\\
>\ & 1\quad \text{for } n\geqslant 4.
\end{aligned}\tag{13}$$

Combining (3.12) and (3.13) with (3.11) we obtain the desired result.

Case C. The interval $2\theta_n<\theta\leqslant 3\theta_n,\ \ n\geqslant 6$.

In this case we write

$$R_n(\theta)=\sum_{k=1}^{n}\frac{\cos k\theta}{k}-\sum_{k=1}^{\infty}\frac{\cos k\theta}{k}+\sum_{k=n+1}^{\infty}\frac{\cos k\theta}{k(k+1)} \tag{14}$$

Then we see that

$$\begin{aligned} & \sum_{k=1}^{n} \frac{\cos k\theta}{k} - \sum_{k=1}^{\infty} \frac{\cos k\theta}{k} \\ = & \sum_{k=1}^{n} \frac{\cos \frac{2k\pi}{2n+1}}{k} + \log(2\sin\frac{\pi}{2n+1}) + \int_{\frac{2\pi}{2n+1}}^{\theta} \frac{\cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt \\ \leqslant & \sum_{k=1}^{n} \frac{\cos \frac{2k\pi}{2n+1}}{k} + \log(2\sin\frac{\pi}{2n+1}). \end{aligned} \tag{15}$$

One can easily check that

$$\begin{aligned} & \sum_{k=1}^{n} \frac{\cos \frac{2k\pi}{2n+1}}{k} = \sum_{k=1}^{n} \frac{1}{k} - 2\sum_{k=1}^{n+1} \frac{\sin^2 \frac{k\pi}{2n+1}}{k} + 2\frac{\sin^2 \frac{(n+1)\pi}{2n+1}}{n+1} \\ < & \sum_{k=1}^{n} \frac{1}{k} - 2\int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{t}\,dt + 2\frac{\sin^2 \frac{(n+1)\pi}{2n+1}}{n+1}. \end{aligned}$$

It follows from this that

$$\begin{aligned} & \sum_{k=1}^{n} \frac{\cos \frac{2k\pi}{2n+1}}{k} + \log(2\sin\frac{\pi}{2n+1}) \\ \leqslant & \ \mathrm{Ci}(\pi) + \frac{2}{n+1} + a_n, \end{aligned} \tag{16}$$

where

$$a_n = \sum_{k=1}^{n} \frac{1}{k} - \log\left(n + \frac{1}{2}\right) - \gamma.$$

An elementary calculation shows that a_n is a strictly decreasing sequence. On the other hand, for the same range of θ

$$\sum_{k=n+1}^{\infty} \frac{\cos k\theta}{k(k+1)} < \frac{1}{(n+1)(n+2)\sin\frac{\pi}{2n+1}}.$$

A combination of this with (3.14), (3.15) and (3.16) yields

$$R_n(\theta) < 0.435 < \mathrm{Ci}\left(\frac{\pi}{2}\right), \quad \text{for } n \geqslant 6,$$

which settles Case C.

Case D. The interval $3\theta_n < \theta \leqslant \pi, \ \ n \geqslant 6$.

This case can be handled in a similar way by writing

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\cos k\theta}{k} - \sum_{k=1}^{\infty} \frac{\cos k\theta}{k} \\
&= \sum_{k=1}^{n} \frac{\cos \frac{3k\pi}{2n+1}}{k} + \log(2\sin\frac{3\pi}{4n+2}) + \int_{\frac{3\pi}{2n+1}}^{\theta} \frac{\cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt. \qquad (17)
\end{aligned}$$

For θ in this interval we have

$$\begin{aligned}
&\int_{\frac{3\pi}{2n+1}}^{\theta} \frac{\cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt \leqslant \int_{\frac{3\pi}{2n+1}}^{\frac{5\pi}{2n+1}} \frac{\cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt \\
&\leqslant \frac{5\pi}{(4n+2)\sin\frac{5\pi}{4n+2}} \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{\cos t}{t}\,dt < 0.343 \ \text{ for } \ n \geqslant 6. \qquad (18)
\end{aligned}$$

Moreover

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\cos \frac{3k\pi}{2n+1}}{k} = \sum_{k=1}^{n} \frac{1}{k} - 2\sum_{k=1}^{n+1} \frac{\sin^2 \frac{3k\pi}{4n+2}}{k} + 2\frac{\sin^2 \frac{(n+1)3\pi}{4n+2}}{n+1} \\
&< \sum_{k=1}^{n} \frac{1}{k} - 2\int_0^{\frac{3\pi}{4}} \frac{\sin^2 t}{t}\,dt + \frac{1}{n+1},
\end{aligned}$$

whence

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\cos \frac{3k\pi}{2n+1}}{k} + \log(2\sin\frac{3\pi}{4n+2}) \\
&\leqslant \operatorname{Ci}\left(\frac{3\pi}{2}\right) + \frac{1}{n+1} + a_n, \qquad (19)
\end{aligned}$$

where a_n as above and $\operatorname{Ci}(\frac{3\pi}{2}) = -0.1984\ldots$. Also, in this case we have

$$\sum_{k=n+1}^{\infty} \frac{\cos k\theta}{k(k+1)} < \frac{1}{(n+1)(n+2)\sin\frac{3\pi}{4n+2}}.$$

Taking into account (3.14) and (3.15) and using the estimates of (3.18), (3.19) and the above, we finally deduce that

$$R_n(\theta) < 0.34 \quad \text{for } \ n \geqslant 6.$$

Hence (1.4) is true for the case D as well.

The proof of Theorem 2 is now complete.

4. Proof of Theorem 3

For $n = 1$ we observe that the function $\frac{1}{2} - \log(2\sin\frac{\theta}{2}) - \cos\theta$ attains its absolute minimum in $[0, \pi]$ at $\theta_0 = \frac{\pi}{3}$ and this yields directly (1.7).

Suppose that $n \geqslant 2$. Let

$$Q_n(\theta) = \sum_{k=1}^{n} \frac{\cos k\theta}{k} + \log(2\sin\frac{\theta}{2}).$$

A straightforward differentiation gives

$$Q_n'(\theta) = \frac{\cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

It follows from this that, for $0 < \theta \leqslant \frac{\pi}{2n+1}$, the function $Q_n(\theta)$ is strictly increasing, so that

$$Q_n(\theta) \leqslant \sum_{k=1}^{n} \frac{\cos\frac{k\pi}{2n+1}}{k} + \log(2\sin\frac{\pi}{4n+2}) = M_n, \text{ say.}$$

A direct computation yields

$$0.472 < M_n < 0.483 \quad \text{for } 2 \leqslant n \leqslant 35.$$

Next we see that

$$M_n < \sum_{k=1}^{n} \frac{1}{k} - 2\int_0^{\frac{\pi}{4}} \frac{\sin^2 t}{t}\,dt + 2\frac{\sin^2\frac{(n+1)\pi}{4n+2}}{n+1} + \log(2\sin\frac{\pi}{4n+2})\,. \tag{1}$$

For $n \geqslant 36$ we note that

$$2\sin^2\frac{(n+1)\pi}{4n+2} < 1.022.$$

It follows from this and (4.1) that

$$M_n < \mathrm{Ci}\left(\frac{\pi}{2}\right) + a_n + \frac{1.022}{n+1}, \tag{2}$$

where the sequence a_n as in the proof of Theorem 2. Recalling that a_n is strictly decreasing we find that the right hand side of (4.2) does not exceed 0.4997 when $n \geqslant 36$. This establishes (1.7) for the interval $(0, \frac{\pi}{2n+1}]$. Inequality (1.7) is also true for $\frac{\pi}{2n+1} < \theta < \frac{3\pi}{2n+1}$ because the derivative $Q_n'(\theta)$ is negative for this range of θ.

For $\frac{3\pi}{2n+1} \leqslant \theta < \pi$ we use again the estimates of (3.17), (3.18) and (3.19) to find that

$$Q_n(\theta) < 0.457 \quad \text{for} \quad n \geqslant 3.$$

By checking directly the case $n = 2$ we complete the proof of Theorem 3.

5. Remarks

In addition to (1.1), Rogosinski and Szegő [8] showed that

$$\frac{1}{1+p}+\sum_{k=1}^{n}\frac{\cos k\theta}{k+p}>0, \tag{1}$$

for all n and $0<\theta<\pi$, when $p\leqslant A$, where A is a number not exceeding $2(1+\sqrt{2})=4.8284\ldots$ and assume negative values for all $p>A$. Gasper [4] has proved that $A=4.5678\ldots$ and A is a root of a polynomial of seventh degree. The critical value A is determined by the case $n=3$ and this result is best possible. Thus there is no analogue of (1.3) and (1.6) in this more general case. However, there is an analogue of (1.4) for the sums of (5.1) in the case where p is a positive integer. Following the method of the proof of Theorem 2 it can be shown that for all positive integers n and for $0<\theta<\pi$ the inequality

$$\sum_{k=1}^{n}\frac{\cos k\theta}{k+p}<\mathrm{Ci}\left(\frac{\pi}{2}\right)+\sum_{k=1}^{\infty}\frac{\cos k\theta}{k+p} \tag{2}$$

holds and the constant $\mathrm{Ci}(\frac{\pi}{2})$ is the best possible. One has the immediate feeling that the proof of (5.2) is more complicated and to present all the details in full, we would need a much longer paper. In this paper we used the proof of Theorem 2 to give the principal ideas of the method and the relevant techniques which we need for the proof of (5.2).

In the case where p is a positive integer it is well known and easy to see that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\cos k\theta}{k+p} &= -\cos p\theta\left[\log\left(2\sin\frac{\theta}{2}\right)+\sum_{k=1}^{p}\frac{\cos k\theta}{k}\right]\\ &\quad+\sin p\theta\left[\frac{\pi-\theta}{2}-\sum_{k=1}^{p}\frac{\sin k\theta}{k}\right]\end{aligned} \tag{3}$$

(cf. [7]).

To see that inequality (5.2) is sharp we let

$$R_n^p(\theta)=-\sum_{k=n+1}^{\infty}\frac{\cos k\theta}{k+p}$$

then using (5.3) we have

$$\begin{aligned}R_n^p(\theta) &= \cos p\theta\left[\log\left(2\sin\frac{\theta}{2}\right)+\sum_{k=1}^{n+p}\frac{\cos k\theta}{k}\right]\\ &\quad-\sin p\theta\left[\frac{\pi-\theta}{2}-\sum_{k=1}^{n+p}\frac{\sin k\theta}{k}\right].\end{aligned} \tag{4}$$

Choosing $\theta_n = \dfrac{\pi}{2n+1}$ it follows from (5.4) that

$$\lim_{n\to\infty} R_n^p(\theta_n) = \mathrm{Ci}\left(\frac{\pi}{2}\right). \tag{5}$$

This represents the Gibbs's phenomenon for the convergence of the remainders $R_n^p(\theta)$ of the Fourier series (5.3) in $(0, \pi)$.

Finally, it is interesting to observe that while (5.5) is valid for $p = 0$, inequality (5.2) fails to hold in this case and this is explained by our Theorem 3.

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