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A NOTE ON OSTROWSKI'S INEQUALITY

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(*communicated by J. Pečarić*)

Abstract. It is shown in this paper that Ostrowski's inequelity is valid in every inner product space, and that it is a statement about projections.

Suppose that a, b and x are real n-tuples such that $a \neq 0$ and

$$\sum_{i=1}^{n} a_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i x_i = 1 . \tag{1}$$

Then

$$\sum_{i=1}^{n} x_i^2 \geqslant \frac{\sum_{i=1}^{n} a_i^2}{(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) - (\sum_{i=1}^{n} a_i b_i)^2} \tag{2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^{n} a_i^2 - a_k \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) - (\sum_{i=1}^{n} a_i b_i)^2}, k = 1, ..., n . \tag{3}$$

This result is known as Ostrowski's inequality (see A.M. Ostrowski [2]). For detailed account of this and related inequalities we refer the reader to D.S. Mitrinović, J.E. Pečarić, and A.M. Fink [1] (section 3 in Chapter IV) and references therein.

Recently there have been some extensions of this inequality. In particular, we mention C.E.M. Pearce, J. Pečarić, and S. Varošanec [3], where Ostrowski's inequality is given for L^2-functions, including also a possibility of multiple linear constraints (instead of (1)). However, although the standard proof of Ostrowski's inequality for n-tuples translates almost verbatim to the case of L^2-functions, if appears that the more general nature of Ostrowski's inequality has been unnoticed so far. In this article we want to fill that gap and to show that Ostrowski's inequality is actually a statement about projections and is valid in every inner product space.

Suppose that $(X; \langle \cdot, \cdot \rangle)$ is an inner product space and $\| \ \|$ denotes the coresponding norm. If M is a complete subspace of X, then there exist the orthogonal complement $M^\perp$ of M and the orthogonal projection P on M. Hence, $X = M \oplus M^\perp$, $P^2 = P$, $P(X) = M$, $P(x) = x$ for $x \in M$, and $P(x) = 0$, for $x \in M^\perp$.

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REMARK. It is important to notice that we did not require for X to be a Hilbert space. It is enough for M to be a complete space in order to have the orthogonal projection on M (see, for example, the proof of Theorem 4.82-A on p.246 in A.E. Taylor [4]).

As it is well-known, for every $x \in X$, Px and $x - Px$ are orthogonal vectors and, consequently,

$$\|x\|^2 = \|Px\|^2 + \|x - Px\|^2 . \tag{4}$$

In particular, we have that for every $x \in X$

$$\|x\|^2 \geqslant \|Px\|^2 \tag{5}$$

and the equality is valid if and only if $x - Px = 0$ (i.e., $x \in M$). We claim that Ostrowski's inequality can be interpreted in this language.

Let $a, b \in X$ be two nonzero vectors which are not proportional. Let $M = M_{(a,b)}$ be a subspace of X which is a linear span of a and b. Clearly, since M is two-dimensional, M is a complete subspace of X. By $P = P_{(a,b)}$ we denote the orthogonal projection on M.

Consider the set $S(a, b)$ of all $x \in X$ such that

$$\langle a, x\rangle = 0 \quad \text{and} \quad \langle b, x\rangle = 1 . \tag{6}$$

Obviously, (1) is just the special case of (6). We can describe completely the set $S(a, b)$.

LEMMA. *a) There is exactly one solution of (6) that belongs to M. This solution is a vector $y = y_{(a,b)} \in M$ given by*

$$y = \frac{\|a\|^2 b - \overline{\langle a, b\rangle}\, a}{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2} . \tag{7}$$

b) The set $S = S(a, b)$ is equal to $y + M^\perp$, i.e., $x \in X$ satisfies (6) if and only if $P(x) = y$.

Proof. a) Notice that, since a and b are nonzero and not proportional, we have that $\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2 \neq 0$. It is straightforward to check that y satisfies (6) (recall that $\langle a, b\rangle \langle b, a\rangle = \langle a, b\rangle \overline{\langle a, b\rangle} = |\langle a, b\rangle|^2$). Suppose now that $x \in M$ and x satisfies (6). Since $x \in M$ there are scalars α and β such that $x = \alpha a + \beta b$. Using (6) we get $\overline{\alpha}\|a\|^2 + \overline{\beta}\langle a, b\rangle = 0$ and $\overline{\alpha}\langle b, a\rangle + \overline{\beta}\|b\|^2 = 1$. It follows that β is real and

$$\beta = \frac{\|a\|^2}{\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2} , \tag{8}$$

while

$$\overline{\alpha} = -\frac{\langle a, b\rangle}{\|a\|^2}\beta , \tag{9}$$

which obviously proves the statement a) of Lemma.

b) Clearly, if $x \in y + M^\perp$, then x satisfies (6). Suppose that $x \in X$ satisfies (6). Then $x = Px + (x - Px)$, and, since a, $b \in M$, we have that $\langle a, x - Px\rangle = \langle b, x - Px\rangle = 0$. Therefore, Px satisfies (6) and $Px \in M$. By a) we conclude $Px = y$. $\square$

It is now easy to see that Ostrowski's inequality is just the special case of (5) for vectors $x \in y + M^{\perp}$. More precisely, we have the following theorem.

THEOREM. *Suppose that* $a, b \in X$ *are such that* $a \neq 0$ *and that* $S(a,b)$ *is nonempty. Then, for every* $x \in S(a,b)$, *we have that*

$$\|x\|^2 \geqslant \frac{\|a\|^2}{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2},$$

with equality if and only if $x = y$ *(where* y *is given by (7)).*

Proof. Notice first that under our assumptions b can not be zero (since there is some $x \in X$ such $\langle b,x\rangle = 1$) and a, b are not proportional. Therefore, all the notation above makes sense and Lemma applies. Using Lemma and (4) (or (5)) we get that for every $x \in S(a,b)$

$$\|x\|^2 = \|Px\|^2 + \|x - Px\|^2 = \|y\|^2 + \|x-y\|^2 \geqslant \|y\|^2, \tag{10}$$

with equality if and only if $x = y$.

Recall that $y = \alpha a + \beta b$, so

$$\|y\|^2 = \langle y,y\rangle = \alpha\langle a,y\rangle + \beta\langle b,y\rangle = \alpha \cdot 0 + \beta \cdot 1 = \beta,$$

which completes the proof (see (8)). □

Let us also mention that it is clear now that in this approach one does not have to start with two vectors, but with some arbitrary finite number of vectors. We leave the details to an interested reader.

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