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A SHARP INEQUALITY AND THE INRADIUS CONJECTURE

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Abstract. We supply a proof of the following statement: The inradius of a triangle contained inside a (closed) unit square never exceeds $\frac{\sqrt{5}-1}{4}$.

1. Introduction

Suppose that $\mathcal{T}$ is a triangle contained inside a unit square $\mathcal{S}$, how large can the inradius of $\mathcal{T}$ be. That is, how large can the radius of the circle inscribed in $\mathcal{T}$ be.

If Δ is the area of $\mathcal{T}$, s is its semi-perimeter and ρ is its inradius then $\Delta = s\rho$, and the question is to find the maximum of the ratio $\frac{\Delta}{s}$ subject to the constraint that $\mathcal{T}$ be confined to the unit square.This question was posed by Funar [2][3] in 1984 when it was conjectured that

$$\rho \leqslant \frac{\sqrt{5}-1}{4}.$$

The purpose of the present note is to obtain the following sharp inequality.

PROPOSITION 1. *If* $0 \leqslant x \leqslant 1$ *and* $0 \leqslant y \leqslant 1$ *then*

$$(1+\sqrt{5})(1-xy) \leqslant \sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{(1-x)^2+(1-y)^2}. \tag{1}$$

As an application, we obtain a proof of the inradius inequality.

THEOREM 2. *Let* $\mathcal{T}$ *be any triangle contained in the unit square* $\mathcal{S}$ *and let* ρ *be the inradius of* $\mathcal{T}$. *Then*

$$\rho \leqslant \frac{\sqrt{5}-1}{4}. \tag{2}$$

Equality holds if $\mathcal{T}$ *is isosceles with side lengths* $1, \frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}$.

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2. The sharp inequality

Before proceeding with the proof of the main inequality, let us note some simple facts about the function F defined by

$$F(x,y)=\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2}-(1+\sqrt{5})(1-xy).$$

First of all we have the symmetry relation $F(x,y)=F(y,x)$. Also the convexity of $\varphi(x)=\sqrt{1+x^2}$ implies that $\varphi(x)+\varphi(1-x)\geqslant 2\varphi(\frac{1}{2})$ and hence that $F(x,0)\geqslant 0$. If we fix $y\in[0,1]$ and let $f(x)=F(x,y)$ be the resulting function defined for $0\leqslant x\leqslant 1$, then we calculate that

$$f'(x)=(1+\sqrt{5})y+\frac{x}{\sqrt{1+x^2}}-\frac{(1-x)}{\sqrt{(1-x)^2+(1-y)^2}}$$

and

$$f''(x)=\frac{1}{[\sqrt{1+x^2}]^3}+\frac{(1-y)^2}{[\sqrt{(1-x)^2+(1-y)^2}]^3}.$$

Thus f' is increasing and f is a convex function.We now go to the proof of Proposition 1.

Proof. Let ε be the positive number defined by $\varepsilon^2=1-\frac{6\sqrt{3}}{(1+\sqrt{5})^2}$ so that $\varepsilon<0.088$. For convenience put $c=1+\sqrt{5}$. We consider three cases.

Case 1. Suppose $1\geqslant y\geqslant\frac{1}{4}$. Set $g(y)=c^2y^2(y^2-2y+2)-1$. Then $g'(y)=2c^2y(2y^2-3y+2)>0$ so that g is increasing. As $g(\frac{1}{4})>0$, we have $g(y)>0$ for $y\geqslant\frac{1}{4}$. It follows that $cy-\frac{1}{\sqrt{y^2-2y+2}}>0$ and so $f'(0)>0$. Since f' is an increasing function we must have $f'(x)>0$ and so f is increasing on $[0,1]$. But $f(0)=F(0,y)=F(y,0)\geqslant 0$. Hence $F(x,y)=f(x)\geqslant 0$ on the set $\{(x,y):0\leqslant x\leqslant 1,\frac{1}{4}\leqslant y\leqslant 1\}$.

Case 2. Suppose $0\leqslant x\leqslant\frac{1}{4}$ and $0\leqslant y\leqslant\varepsilon$. From the expression for f' we have

$$f'(x)\leqslant(1+\sqrt{5})\varepsilon+\frac{x}{\sqrt{1+x^2}}-\frac{(1-x)}{\sqrt{x^2-2x+2}}.$$

In particular $f'(\frac{1}{4})\leqslant(1+\sqrt{5})\varepsilon+\frac{1}{\sqrt{17}}-\frac{3}{5}<0$. As f' is increasing, $f'(x)<0$ when $0\leqslant x\leqslant\frac{1}{4}$. Consequently we have that f is decreasing on $[0,\frac{1}{4}]$. But $f(\frac{1}{4})=F(\frac{1}{4},y)=F(y,\frac{1}{4})\geqslant 0$, by case 1. Hence $F(x,y)=f(x)\geqslant 0$ on the set $\{(x,y):0\leqslant x\leqslant\frac{1}{4}$ and $0\leqslant y\leqslant\varepsilon\}$ .If we recall the symmetry property of F we see that inequality (2) has been established at all points of the unit square outside the set $A=\{(x,y):\varepsilon\leqslant x\leqslant\frac{1}{4},\ \varepsilon\leqslant y\leqslant\frac{1}{4}\}$.

Case 3. Suppose now that $(x,y)\in A$. Then $xy\geqslant\varepsilon^2$. Let $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$ be the vertices of the unit square in the xy-plane. For $x,y\in[0,1]$, consider the points $M(1,y)$, $N(x,1)$ and let $\mathscr{T}$ be the triangle OMN. Then the side lengths of $\mathscr{T}$ are given by $a=|OM|=\sqrt{1+y^2}$, $b=|ON|=\sqrt{1+x^2}$ and $c=|MN|=\sqrt{(1-x)^2+(1-y)^2}$. If we put $P=2s=a+b+c$ for the perimeter of $\mathscr{T}$, then P equals the right-hand side in (2). The area Δ of $\mathscr{T}$ may be calculated by removing from the square the triangles OAM, MBN and OCN and one easily gets that $\Delta = \frac{1-xy}{2}$. If in Heron's formula, [1] $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, we apply the inequality between the arithmetic and geometric mean of three non-negative numbers we obtain

$$\Delta \leqslant \sqrt{s(\frac{s}{3})^3} = \frac{P^2}{12\sqrt{3}} \tag{3}$$

which is the isoperimetric inequality for triangles. Since $\Delta = \frac{1-xy}{2}$ we obtain

$$P \geqslant \sqrt{6\sqrt{3}}\sqrt{1-xy} = \frac{\sqrt{6\sqrt{3}}}{\sqrt{1-xy}}(1-xy) \geqslant (1+\sqrt{5})(1-xy)$$

because we are assuming that $\sqrt{1-xy} \leqslant \frac{\sqrt{6\sqrt{3}}}{(1+\sqrt{5})}$. This establishes inequality (2) when $xy \geqslant \varepsilon^2$ and completes the proof of Proposition 1. □

3. The inradius inequality

We now give the proof of Theorem 2.

Proof. We break down the first step of the proof into the following statements:

(a) *If a circle $\mathscr{C}$ of radius r is contained in a triangle $\mathscr{T}$ of inradius ρ then $r \leqslant \rho$ with equality if and only if $\mathscr{C}$ is the incircle of $\mathscr{T}$.*

For if $\mathscr{C}$ is contained in $\mathscr{T}$ and is not itself the incircle of $\mathscr{T}$, we may draw tangents to $\mathscr{C}$ parallel to those sides of $\mathscr{T}$ not tangent to $\mathscr{C}$.The constructed tangents produce a triangle $\mathscr{T}'$ similar to $\mathscr{T}$ and having incircle $\mathscr{C}$. Since the ratio of the inradii of $\mathscr{T}$ and $\mathscr{T}'$ equals the ratio of similitude our assertion follows.

(b) *If a triangle $\mathscr{T}_1$ of inradius r_1 is contained in another triangle $\mathscr{T}_2$ of inradius r_2 then $r_1 \leqslant r_2$.*

This is immediate from (a). For example, if $\mathscr{T}_1$ is contained in a unit square $\mathscr{S}$ and lies completely on one side of a diagonal of $\mathscr{S}$, then $r_1 \leqslant \frac{1}{2+\sqrt{2}} < \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{4}$.

(c) *If a triangle $\mathscr{T}_1$ is contained in a square $\mathscr{S}$ there is another triangle $\mathscr{T}_2$ containing $\mathscr{T}_1$ and having its vertices on the sides of $\mathscr{S}$.*

For if $\mathscr{T}_1 = ABC$ and one of its vertices, say C, is not on the perimeter of $\mathscr{S}$, extend AC to meet the perimeter of $\mathscr{S}$ in C'. If $\mathscr{T}' = ABC'$ does not have all its vertices on the perimeter of $\mathscr{S}$ repeat the construction for $\mathscr{T}'$ and so forth.Then $\mathscr{T}_1 \subseteq T' \subseteq T'' \subseteq T''' = \mathscr{T}_2$.

(d) *If a triangle $\mathscr{T}_1 = ABC$ has all its vertices on the perimeter of the square $\mathscr{S}$, there is a triangle with larger inradius having one vertex at a vertex of $\mathscr{S}$ and the two other vertices on the sides of $\mathscr{S}$ opposite that vertex.*

For if $\mathscr{S} = EFGH$, then the interior of one side of $\mathscr{S}$, say EF, contains no vertex of $\mathscr{T}_1$. Slide triangle ABC so that the vertex nearest to EF coincides with E or F.This gives a new triangle $\mathscr{T}'$ congruent to $\mathscr{T}_1$ and having one vertex at, say, E. Keeping the vertex E of $\mathscr{T}'$ fixed we extend the sides of $\mathscr{T}'$ issuing from E to meet the perimeter of the square.If the extended sides meet the (closed) edges HG and GF we have the required triangle. If the extended sides meet the same edge of $\mathscr{S}$, say GF, then the extended triangle is included in triangle EFG which, again, is of the required type: one vertex at F and the other vertices E on EH and G on GH.

It follows that of all triangles iscribed in a square, the one with largest inradius must have one of its vertices at a vertex of the square and, the two other vertices one on each of the (closed) sides of the square opposite that vertex.We may assume the square to be $OABC$ and the triangle to be OMN as described in Proposition 1 above. The problem then becomes that of finding the maximum value of the inradius ρ of OMN. But

$$\max \rho = \max \frac{2\Delta}{P} = \max_{0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1} \frac{1 - xy}{\sqrt{1+x^2} + \sqrt{1+y^2} + \sqrt{(1-x)^2 + (1-y)^2}}.$$

By Proposition 1 we get $\max \rho \leqslant \frac{1}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{4}$. Since equality holds when $x = \frac{1}{2}$, $y = 0$ it follows that $\max \rho = \frac{\sqrt{5}-1}{4}$. This completes the proof of Theorem 2. □

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