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ON A CLASS OF MEANS OF SEVERAL VARIABLES

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(*communicated by G. Bennett*)

Abstract. The aim of this paper is to solve the comparison and equality problems of L–conjugate means of $n \geqslant 2$ variables defined by

$$L_{\varphi}^{*}(x_1, x_2, \ldots, x_n) := \varphi^{-1}\left(\frac{\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n) - \varphi(L(x_1, x_2, \ldots, x_n))}{n-1}\right),$$

where $L : I^n \to I$ is a symmetric mean on the open real interval I and $\varphi : I \to \mathbb{R}$ is continuous strictly monotonic function. The homogeneous L–conjugate means are also described. In the last section, the arithmetic mean is characterized as being the only mean that is conjugate arithmetic and quasiarithmetic.

1. Introduction

Let $I \subset \mathbb{R}$ be an open interval and let $n \geqslant 2$ be a given natural number. A function $M : I^n \to I$ is called a *mean of* n *variables* on I if it possesses the following properties

(i) If $x_1, x_2, \ldots, x_n \in I$ and $x_k \neq x_l$ for some $k, l \in \{1, 2, \ldots, n\}$ then

$$\min\{x_i \mid i = 1, 2, \ldots, n\} < M(x_1, x_2, \ldots, x_n) < \max\{x_i \mid i = 1, 2, \ldots, n\};$$

(ii) $M(x_1, x_2, \ldots, x_n)$ is symmetric for all variables $x_1, x_2, \ldots, x_n \in I$;

(iii) M is continuous on I^n.

Let $CM(I)$ denote the set of all *continuous* and *strictly monotonic* real functions defined on the interval I.

DEFINITION 1. *Let* $L : I^n \to I$ *be a fixed mean on* I. *A mean* $M : I^n \to I$ *is called an* L–conjugate mean of n variables *on* I *if there exists* $\varphi \in CM(I)$ *for which*

$$M(x_1, x_2, \ldots, x_n) = \varphi^{-1}\left(\frac{\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n) - \varphi(L(x_1, x_2, \ldots, x_n))}{n-1}\right) =: L_{\varphi}^{*}(x_1, x_2, \ldots, x_n) \tag{1.1}$$

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for all $x_1, x_2, \dots, x_n \in I$. *Then the function* φ *is called the* generating function *of the* L*–conjugate mean* L_φ^* *of* n *variables.*

It can be easily seen that for any $\varphi \in CM(I)$ $M = L_\varphi^* : I^n \to I$ is a mean of n variables on I, that is, the properties *(i)*, *(ii)*, and *(iii)* are fulfilled. For $n = 2$ the definition of L–conjugate mean with two variables was introduced by Daróczy [3].

In this paper we examine the *problem of comparison* of two L–conjugate means of n variables and generalize the result obtained in [3]. Using these results, the homogeneous L–conjugate means are also determined. In the last section we determine those means that are quasiarithmetic and also conjugate arithmetic.

2. Comparison

Let $L : I^n \to I$ be a fixed mean of n variables on I. The problem of *comparison* for L–conjugate means of n variables is the following: What conditions are necessary and sufficient for a pair of functions $\varphi, \psi \in CM(I)$ in order that

$$L_\varphi^*(x_1, x_2, \dots, x_n) \leqslant L_\psi^*(x_1, x_2, \dots, x_n) \tag{2.1}$$

be satisfied for all $x_1, x_2, \dots, x_n \in I$?

This question is answered by our main result contained in the following

THEOREM 1. *Let* $\varphi, \psi \in CM(I)$. *Then the inequality (2.1) holds for all* $x_1, x_2, \dots,$ $x_n \in I$ *if and only if* $\varepsilon_\psi \psi \circ \varphi^{-1}$ *is convex, where* $\varepsilon_\psi = 1$ *if* ψ *is increasing and* $\varepsilon_\psi = -1$ *if* ψ *is decreasing.*

Proof. We prove the theorem if ψ is increasing. The proof in the other case is similar. So let $\psi \in CM(I)$ be *increasing*. Then (2.1) implies

$$\psi \circ \varphi^{-1} \left(\frac{\varphi(x_1) + \cdots + \varphi(x_n) - \varphi(L(x_1, \dots, x_n))}{n-1} \right) \leqslant \frac{\psi(x_1) + \cdots + \psi(x_n) - \varphi(L(x_1, \dots, x_n))}{n-1}$$

for all $x_1, \dots, x_n \in I$. From this, with the notations $\varphi(x_i) =: u_i$ ($u_i \in \varphi(I) = J$ if $i = 1, 2, \dots, n$) and $f := \psi \circ \varphi^{-1}$ ($f \in CM(J)$), we have

$$(n-1)f \left(\frac{u_1 + u_2 + \cdots + u_n - M(u_1, u_2, \dots, u_n))}{n-1} \right) + f(M(u_1, u_2, \dots, u_n)) \leqslant f(u_1) + f(u_2) + \cdots + f(u_n), \tag{2.2}$$

where

$$M(u_1, u_2, \dots, u_n) := \varphi(L(\varphi^{-1}(u_1), \varphi^{-1}(u_2), \dots, \varphi^{-1}(u_n)) \tag{2.3}$$

is a mean of n variables on J. The inequality (2.2) is fulfilled for all $u_1, u_2, \dots, u_n \in J$. Define $N : J^2 \to J$ by

$$N(u, v) = M(\underbrace{u, \dots u}_{n-1}, v) \qquad (u, v \in J). \tag{2.4}$$

Setting $u_1 = \cdots = u_n = u,\ v = u_n$ in (2.2) we get

$$(n-1)f\left(\frac{(n-1)u+v-N(u,v)}{n-1}\right)+f(N(u,v)) \leqslant (n-1)f(u)+f(v) \qquad (u,v\in J). \quad (2.5)$$

We need the following lemma.

LEMMA 1. *Let $M : J^n \to J$ $(n \geqslant 2$ fix$)$ be a mean of n variables on J and let $N : J^2 \to J$ be defined by (2.4). Then the sequence of functions defined by the iteration*

$$N_1(u,v) := N(u,v)$$
$$N_{k+1}(u,v) := N\left(\frac{(n-1)u+v-N_k(u,v)}{n-1}, N_k(u,v)\right) \quad (k \geqslant 1)$$

is convergent and

$$\lim_{k\to\infty} N_k(u,v) = \frac{(n-1)u+v}{n}. \tag{2.6}$$

Proof. It is easy to see that if $u < v$ $(u,v \in J)$ then $u < N(u,v) < v$ and from this we have

$$u < \frac{(n-1)u+v-N(u,v)}{n-1} < v.$$

This means that the sequence $N_k(u,v)$ $(k \in \mathbb{N})$ is well-defined. If $u = v$ then the assertion clearly holds, since $N(u,u) = u$ $(u \in J)$. Let $u < v$ $(u,v \in J)$ be fixed. It can be easily seen that for the closed intervals

$$I_k := [\alpha_k(u,v), \omega_k(u,v)]$$

with the notations

$$\alpha_k(u,v) := \min\left\{\frac{(n-1)u+v-N_k(u,v)}{n-1}, N_k(u,v)\right\},$$
$$\omega_k(u,v) := \max\left\{\frac{(n-1)u+v-N_k(u,v)}{n-1}, N_k(u,v)\right\}$$

we have $I_{k+1} \subset I_k$ $(k \in \mathbb{N})$ and hence the seuences $\alpha_k(u,v)$ and $\omega_k(u,v)$ strictly increase and decrease, respectively.

Moreover

$$\frac{(n-1)\frac{(n-1)u+v-N_k(u,v)}{n-1} + N_k(u,v)}{n} = \frac{(n-1)u+v}{n} \in I_k$$

for all $k \in \mathbb{N}$, i.e.,

$$\frac{(n-1)u+v}{n} \in \bigcap_{k=1}^{\infty} I_k. \tag{2.7}$$

Let

$$\sup_{k\in\mathbb{N}} \alpha_k(u,v) = \lim_{k\to\infty} \alpha_k(u,v) = \alpha(u,v)$$

and

$$\inf_{k\in\mathbb{N}} \omega_k(u,v) = \lim_{k\to\infty} \omega_k(u,v) = \omega(u,v)$$

then

$$\alpha_l(u,v) \leqslant \alpha(u,v) \leqslant \omega(u,v) \leqslant \omega_s(u,v) \tag{2.8}$$

for all $l,s\in\mathbb{N}$. We show that

$$\alpha(u,v) = \omega(u,v) = \frac{(n-1)u+v}{n}.$$

If there existed $u<v$ such that $\alpha(u,v)<\omega(u,v)$ then, by the property of means,

$$\alpha(u,v) < N(\alpha(u,v),\omega(u,v)) < \omega(u,v)$$

and

$$\alpha(u,v) < N(\omega(u,v),\alpha(u,v)) < \omega(u,v)$$

would hold. On the other hand, the continuity of N and the convergence $(\alpha_k(u,v),\omega_k(u,v)) \to (\alpha(u,v),\omega(u,v))$ $(k\to\infty)$ imply the existence of $n_0\in\mathbb{N}$ for which

$$\alpha(u,v) < N(\alpha_{n_0}(u,v),\omega_{n_0}(u,v)) < \omega(u,v)$$

and

$$\alpha(u,v) < N(\omega_{n_0}(u,v),\alpha_{n_0}(u,v)) < \omega(u,v),$$

that is,

$$N_{n_0+1}(u,v) \in]\alpha(u,v),\omega(u,v)[.$$

Now $N_{n_0+1}(u,v)$ equals either $\alpha_{n_0+1}(u,v)$ or $\omega_{n_0}(u,v)$, which contradicts (2.8). Thus $\alpha(u,v)=\omega(u,v)$ is the only number that belongs to $\bigcap_{k=1}^{\infty} I_k$, that is, by (2.7),

$$\alpha(u,v) = \omega(u,v) = \frac{(n-1)u+v}{n}.$$

From this fact we get the assertion of Lemma 1. □

Now we continue the proof of Theorem 1.

Substituting v by $N_k(u,v)$ and u by $\frac{(n-1)u+v-N_k(u,v)}{n-1}$ in inequality (2.5), we get

$$\begin{aligned}(n-1)f\left(\frac{(n-1)u+v-N_{k+1}(u,v)}{n-1}\right)&+f\left(N_{k+1}(u,v)\right)\\ &\leqslant (n-1)f\left(\frac{(n-1)u+v-N_k(u,v)}{n-1}\right)+f\left(N_k(u,v)\right) \qquad (u,v\in J)\end{aligned}$$

for all $k\geqslant 1$. Hence, applying (2.5) and the above inequality repeatedly, we obtain

$$(n-1)f\left(\frac{(n-1)u+v-N_k(u,v)}{n-1}\right)+f\left(N_k(u,v)\right) \leqslant (n-1)f(u)+f(v) \tag{2.9}$$

for all $k \in \mathbb{N}$ and $u, v \in J$. Using the assertion of Lemma 1, by taking the limit $k \to \infty$ in (2.9) we deduce that

$$(n-1)f\left(\frac{(n-1)u+v-\frac{(n-1)u+v}{n}}{n-1}\right)+f\left(\frac{(n-1)u+v}{n}\right) \leqslant (n-1)f(u)+f(v),$$

that is,

$$f\left(\frac{(n-1)u+v}{n}\right) \leqslant \frac{(n-1)f(u)+f(v)}{n}$$

for all $u, v \in J$. With $\alpha := \frac{n-1}{n} \in]0,1[$ we have that $f(\alpha u + (1-\alpha)v) \leqslant \alpha f(u) + (1-\alpha)f(v)$ for all $u, v \in J$, that is f is α-convex. Using a result of Daróczy-Páles [5], we get that f is Jensen-convex on J. By the continuity of f, this yields that f is convex on J. This proves the necessity of the condition.

Now suppose that $\psi \in CM(I)$ is increasing and $f := \psi \circ \varphi^{-1}$ is convex on the interval $\varphi(I) = J$. Let $x_1, x_2, \dots, x_n \in I$ be arbitrary and define $u_i := \varphi(x_i)$ $(i = 1, 2, \dots, n)$. Then we obtain

$$M(u_1, u_2, \dots, u_n) = \varphi(L(\varphi^{-1}(u_1), \dots, \varphi^{-1}(u_n)) = \sum_{i=1}^{n} \lambda_i u_i$$

for some $\lambda_1, \dots, \lambda_n \geqslant 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. Thus, by the convexity of f,

$$f\left(\sum_{i=1}^{n} \lambda_i u_i\right) \leqslant \sum_{i=1}^{n} \lambda_i f(u_i)$$

and

$$f\left(\frac{\sum_{i=1}^{n}(1-\lambda_i)u_i}{n-1}\right) \leqslant \sum_{i=1}^{n} \frac{1-\lambda_i}{n-1} f(u_i).$$

From the above inequalities we obtain

$$\begin{aligned}
(n-1)f&\left(\frac{u_1+u_2+\cdots+u_n-M(u_1,u_2,\dots,u_n)}{n-1}\right)+f(M(u_1,u_2,\dots,u_n)) \\
&= (n-1)f\left(\frac{\sum_{i=1}^{n}(1-\lambda_i)u_i}{n-1}\right)+f\left(\sum_{i=1}^{n}\lambda_i u_i\right) \\
&\leqslant (n-1)\sum_{i=1}^{n}\frac{1-\lambda_i}{n-1}f(u_i)+\sum_{i=1}^{n}\lambda_i f(u_i) \leqslant \sum_{i=1}^{n} f(u_i),
\end{aligned}$$

from which (with the substitution $u_i = \varphi(x_i)$) the inequality (2.1) follows. □

REMARK. *Observe that the necessary and sufficient condition obtained in Theorem 1 is also necessary and sufficient for the comparison of the quasi–arithmetic means generated by φ and ψ resp.*

COROLLARY 1. *Let $\varphi, \psi \in CM(I)$. The equality*

$$L_{\varphi}^{*}(x_1, x_2, \ldots, x_n) = L_{\psi}^{*}(x_1, x_2, \ldots, x_n) \tag{2.10}$$

holds for all $x_1, x_2, \ldots, x_n \in I$ if and only if there exist real constants $\alpha \neq 0$ and β such that

$$\psi(x) = \alpha\varphi(x) + \beta \tag{2.11}$$

for all $x \in I$.

Proof. Due to (2.10), we have $L_{\varphi}^{*} \leqslant L_{\psi}^{*}$ and $L_{\varphi}^{*} \geqslant L_{\psi}^{*}$. Thus, by Theorem 1, both $\varepsilon_{\psi}\psi \circ \varphi^{-1} =: f$ is convex and concave in $\varphi(I) = J$, that is, for all values of $u, v \in J$ and $0 < \lambda < 1$

$$f(\lambda u + (1-\lambda)v) = \lambda f(u) + (1-\lambda)f(v).$$

This implies $f(u) = \alpha u + \beta \; u \in J$ for some constants $\alpha \neq 0$ and β. With the notation $u = \varphi(x)$ $(x \in I)$ we obtain (2.11). Conversely, if ψ is of the form (2.11) one can easily check equality (2.10). □

DEFINITION 2. *Let $\varphi, \psi \in CM(I)$. Then ψ and φ are called* equivalent *if there exist real numbers $\alpha \neq 0$ and β for which (2.11) holds for all $x \in I$. Notation: $\psi \sim \varphi$ or $\psi(x) \sim \varphi(x)$ $(x \in I)$.*

3. Homogenous means

The following definition is well-known.

DEFINITION. *If $\mathbb{R}_+$ denotes the set of positive real numbers and $M : \mathbb{R}_+^n \to \mathbb{R}_+$ is a mean of n variables on $\mathbb{R}_+$ then this mean is called* homogenous *if*

$$M(tx_1, tx_2, \ldots, tx_n) = tM(x_1, x_2, \ldots, x_n) \tag{3.1}$$

holds for all $x_1, x_2, \ldots, x_n, t \in \mathbb{R}_+$.

THEOREM 2. *Let $L : \mathbb{R}_+^n \to \mathbb{R}_+$ be a fixed homogenous mean of n variables on $\mathbb{R}_+$ and let $\varphi \in CM(\mathbb{R}_+)$. Then the L–conjugate mean $L_{\varphi}^{*} : \mathbb{R}_+^n \to \mathbb{R}_+$ of n variables on $\mathbb{R}_+$ is homogenous if and only if*

$$\varphi(x) \sim l_p(x) \qquad (x \in \mathbb{R}_+), \tag{3.2}$$

where

$$l_p(x) := \begin{cases} x^p & \text{if } p \neq 0 \\ \log x & \text{if } p = 0 \end{cases} \qquad (x \in \mathbb{R}_+). \tag{3.3}$$

Proof. Let $L : \mathbb{R}_+^n \to \mathbb{R}$ be a homogenous mean and $\varphi \in CM(\mathbb{R}_+)$ for which

$$L_{\varphi}^{*}(tx_1, tx_2, \ldots, tx_n) = tL_{\varphi}^{*}(x_1, x_2, \ldots, x_n) \tag{3.4}$$

holds for all $x_1, x_2, \ldots, x_n, t \in \mathbb{R}_+$.

For a fixed $t \in \mathbb{R}_+$, let

$$\psi_t(x) := \varphi(tx) \qquad (x \in \mathbb{R}_+). \tag{3.5}$$

Clearly, $\psi_t \in CM(\mathbb{R}_+)$ and, by (3.4),

$$L^*_{\psi_t}(x_1, x_2, \ldots, x_n) = L^*_{\varphi}(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n, t \in \mathbb{R}_+$. Thus, by Corollary 1, there exist real numbers $\alpha(t) \neq 0$ and $\beta(t)$ such that $\psi_t(x) = \alpha(t)\varphi(x) + \beta(t)$ for all $x \in \mathbb{R}_+$, which implies, by (3.5),

$$\varphi(tx) = \alpha(t)\varphi(x) + \beta(t) \tag{3.6}$$

for all elements $x \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ and $\alpha(t) \neq 0$. The functional equation (3.6) and its solutions are known (see [6],p.69; or [3]). □

THEOREM 3. *Let $L : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a fixed homogenous mean of n variables on $\mathbb{R}_+$. Then an L–conjugate mean $M : \mathbb{R}^n_+ \to \mathbb{R}_+$ of n variables on $\mathbb{R}_+$ is homogenous if and only if there exists $p \in \mathbb{R}$ such that*

$$M(x_1, x_2, \ldots, x_n) = L^*_p(x_1, x_2, \ldots, x_n),$$

where

$$L^*_p(x_1, x_2, \ldots, x_n) = \begin{cases} \left(\dfrac{x_1^p + x_2^p + \cdots + x_n^p - L^p(x_1, x_2, \ldots, x_n)}{n-1} \right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \left(\dfrac{x_1 x_2 \ldots x_n}{L(x_1, x_2, \ldots, x_n)} \right)^{\frac{1}{n-1}} & \text{if } p = 0 \end{cases} \tag{3.7}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}_+$. The one parameter family of means of n variables $L^*_p : \mathbb{R}^n_+ \to \mathbb{R}_+$ is increasing in p, that is, if $p \leqslant q$ then

$$L^*_p(x_1, x_2, \ldots, x_n) \leqslant L^*_q(x_1, x_2, \ldots, x_n). \tag{3.8}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}_+$.

Proof. By Theorem 2, (3.7) are the homogenous L–conjugate means of n variables. For the proof of the inequality (3.8), see [3], Theorem 5 (in the case $n = 2$). □

4. Conjugate arithmetic means which are quasi–arithmetic means

The best-known mean is the arithmetic mean $A : I^n \to I$ of n variables defined by

$$A(x_1, x_2, \ldots, x_n) := \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{4.1}$$

for all $x_1, x_2, \ldots, x_n \in I$. A mean $M : I^n \to I$ of n variables on I is called a *quasi–arithmetic mean of n variables on I* if there exists $\psi \in CM(I)$ such that

$$M(x_1, x_2, \ldots, x_n) = \psi^{-1}\left(\frac{\psi(x_1) + \psi(x_2) + \cdots + \psi(x_n)}{n}\right) = A_\psi(x_1, x_2, \ldots, x_n) \quad (4.2)$$

for all $x_1, x_2, \ldots, x_n \in I$ (see [1], [2], [6], [10], [11]). The following problem seems to be natural: For which $\varphi \in CM(I)$ will the A–conjugate mean (or conjugate arithmetic mean) of n variables $A^*_\varphi : I^n \to I$ be also a quasi–arithmetic mean of n variables on the interval I? This means that if φ is the required generating function, then there exists $\psi \in CM(I)$ such that

$$A^*_\varphi(x_1, x_2, \ldots, x_n) = A_\psi(x_1, x_2, \ldots, x_n) \quad (4.3)$$

holds for all $x_1, x_2, \ldots, x_n \in I$. In more details, for the unknown functions $\varphi, \psi \in CM(I)$, the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n) - \varphi(\frac{x_1+x_2+\cdots+x_n}{n})}{n-1}\right) = \psi^{-1}\left(\frac{\psi(x_1) + \psi(x_2) + \cdots + \psi(x_n)}{n}\right) \quad (4.4)$$

holds for all $x_1, x_2, \ldots, x_n \in I$, where $n \geqslant 2$ is a fixed natural number. For the case $n = 2$ see [3] and [4], therefore we suppose that $n \geqslant 3$.

LEMMA 2. *Let $\varphi, \psi \in CM(I)$ and $n \geqslant 2$. If the functional equation (4.4) holds and φ is continuously differentiable on I with $\varphi'(x) \neq 0$ for $x \in I$ then ψ is also continuously differentiable on I.*

Proof. For arbitrary $x, y \in I$ let $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = y$. Then

$$g_1(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + (n-1)\varphi(y) - \varphi(\frac{x+(n-1)y}{n})}{n-1}\right)$$

is continuously differentiable and

$$\frac{\partial g_1(x, y)}{\partial x} = \frac{\varphi'(x) - \varphi'(\frac{x+(n-1)y}{n})\frac{1}{n}}{\varphi'(g_1(x, y))}.$$

On the other hand, by (4.4),

$$\psi(x) = n\psi(g_1(x, y)) - (n-1)\psi(y) \quad (4.5)$$

for all $x, y \in I$. Let $x_0 \in I$ be fixed then

$$\psi(x_0) = n\psi(g_1(x_0, y)) - (n-1)\psi(y)$$

for all $y \in I$. Since ψ is monotonic, there exists $y_0 \in I$ such that ψ is differentiable at $g_1(x_0, y_0)$. Since

$$\psi(x) = n\psi(g_1(x, y_0)) - (n-1)\psi(y_0),$$

by the chain rule, the right hand side is differentiable at x_0, that is, ψ is differentiable at x_0. Thus, differentiating (4.5) with respect to x, we have

$$\psi'(x) = n\psi'(g_1(x,y))\frac{\varphi'(x) - \varphi'(\frac{x+(n-1)y}{n})\frac{1}{n}}{\varphi'(g_1(x,y))}$$

for all $x, y \in I$. Since $\frac{\partial g_1}{\partial x}(x,x) = 1 - \frac{1}{n} \neq 0$ and $\psi' : I \to \mathbb{R}$ is a *measurable* function, therefore the previous equation implies (cf. Járai [8] (Theorem 2) or [7] and [9]) that ψ' is continuous on I. $\square$

THEOREM 4. *Let $n \geqslant 3$ be fixed and $M : I^n \to I$ a conjugate arithmetic mean of n variables on I which has a continuously differentiable generating function. Then M is a quasi–arithmetic mean of n variables on I if and only if*

$$M(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all $x_1, x_2, \ldots, x_n \in I$.

Proof. Let $\varphi \in CM(I)$ be continuously differentiable on I for which $M = A^*_\varphi$ on I^n and let $N := \{x \mid x \in I, \varphi'(x) = 0\}$. Then N is closed and $I \cap (\mathbb{R} \setminus N)$ is open, nonvoid. Let $K :=]\alpha, \beta[\subset I$ be a maximal component of $I \cap (\mathbb{R} \setminus N)$. Then either $K = I$ or $K \neq I$ and in this case at least of one the endpoints of K (for example β) belongs to I. Clearly, $\varphi'(x) \neq 0$ if $x \in K$. Suppose that $A^*_\varphi : I^n \to I$ is a quasi–arithmetic mean of n variables on I. Then there exists $\psi \in CM(I)$ such that $A^*_\varphi = A_\psi$ on the set I. By the lemma, then ψ is continuously differentiable on K, thus the equation

$$\frac{1}{n}\sum_{i=1}^{n} \psi(x_i) = \psi(A^*_\varphi(x_1, x_2, \ldots, x_n))$$

can be differentiated with respect to x_k $(k \in \{1, 2, \ldots, n\})$, which implies

$$\frac{1}{n}\psi'(x_k) = \psi'(A^*_\varphi(x_1, x_2, \ldots, x_n))\frac{\varphi'(x_k) - \varphi'(\frac{x_1+x_2+\cdots+x_n}{n})\frac{1}{n}}{\varphi'(A^*_\varphi(x_1, x_2, \ldots, x_n))}$$

for all $x_1, x_2, \ldots, x_n \in K$. From this, by the symmetry, we have

$$\psi'(x_k)\left(\varphi'(x_l) - \varphi'\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\frac{1}{n}\right) = \psi'(x_l)\left(\varphi'(x_k) - \varphi'\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\frac{1}{n}\right)$$

for any x_k and x_l $(k \neq l)$. Now, for arbitrary $x, y, z \in K$, let $x_1 := x, x_2 := y$, and $x_3 = x_4 = \cdots = x_n = z$ (it is possible because $n \geqslant 3$). Then, in the case $k = 1, l = 2$, the previous equation implies,

$$\frac{1}{n}\varphi'\left(\frac{x + y + (n-2)z}{n}\right)(\psi'(y) - \psi'(x)) = \varphi'(x)\psi'(y) - \varphi'(y)\psi'(x) \tag{4.6}$$

for all $x,y,z \in K$. We show that φ' and ψ' are nonzero constant functions on K, which would imply, by the continuity of φ', $\varphi'(\beta) \neq 0$, therefore necessarily $K = I$.

Assume that (4.6) is valid. There are two possible cases: *(i)* ψ' is constant (clearly nonzero) on K; *(ii)* ψ' is not constant on K, that is, there exist $a < b$ $(a,b \in K)$ such that $\psi'(a) \neq \psi'(b)$.

In the case *(i)*, by (4.6), φ' is constant (nonzero) on K.

Now we show that the case *(ii)* is impossible. Since ψ' is continuous, in this case there exist numbers $a^*, b^* \in K$ $(a^* < b^*)$ for which $\psi'(a^*) \neq \psi'(b^*)$, furthermore $a^* - (b^* - a^*) \in K$ and $b^* + (b^* - a^*) \in K$ hold at the same time. Thus, by $n \geqslant 3$,

$$a^*, b^* \in \frac{a^* + b^* + (n-2)K}{n}$$

follows, that is, substituting $x = a^*$ and $y = b^*$ into (4.6), we have

$$\varphi'\left(\frac{a^* + b^* + (n-2)z}{n}\right) = n\frac{\varphi'(a^*)\psi'(b^*) - \varphi'(b^*)\psi'(a^*)}{\psi'(b^*) - \psi'(a^*)} =: c \neq 0$$

for all $z \in K$. Thus φ' is constant on $[a^*, b^*]$, which implies, by (4.6), that ψ' is a nonzero constant on $[a^*, b^*]$, which is a contradiction.

Therefore φ' and ψ' are nonzero constant functions on I, which yields that $\varphi(x) = \alpha x + \beta$ $(x \in I)$ for some $\alpha \neq 0$, $\beta \in \mathbb{R}$. Hence, $M = A^*_\varphi = A$ on the set I^n.
□

Notice that the functional equation (4.4) has further solutions in the case when $n = 2$ (see [3], [4]). The regularity assumptions in [3], [4] are similar to those of this paper. It is a natural problem to solve equation (4.4) without any further regularity condition for a fixed $n \geqslant 3$, however this problem is left open in this paper. In the next result, we weaken the regularity assumptions, but we assume (4.4) to be valid not only for fixed $n \in \mathbb{N}$.

THEOREM 5. *Let $\varphi \in CM(I)$. The A–conjugate mean of n variables $A^*_\varphi : I^n \to I$ equals the quasi–arithmetic mean of n variables $A^*_\psi : I^n \to I$ generated by $\psi \in CM(I)$ for all $n \in \mathbb{N}$ if and only if $\varphi \sim x$ $(x \in I)$, that is, $A^*_\varphi = A$ on the set I^n for all $n \in \mathbb{N}$.*

Proof. In this case for all n and $x_1, x_2, \dots, x_n \in I$ $(n \geqslant 2)$

$$\varphi^{-1}\left(\frac{\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n) - \varphi(\frac{x_1+x_2+\cdots+x_n}{n})}{n-1}\right) = \psi^{-1}\left(\frac{\psi(x_1) + \psi(x_2) + \cdots + \psi(x_n)}{n}\right). \quad (4.7)$$

Let $n = 2N$ $(N \in \mathbb{N})$ and $x_1 = x_2 = \cdots = x_N =: x, x_{N+1} = x_{N+2} = \cdots = x_{2N} =: y$. Then from (4.7)

$$\varphi^{-1}\left(\frac{N\varphi(x) + N\varphi(y) - \varphi(\frac{x+y}{2})}{2N-1}\right) = \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right)$$

follows for all $x, y \in I$ and $N \in \mathbb{N}$. This implies, taking the limit $N \to \infty$,

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) = \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)$$

for all $x, y \in I$. Thus

$$\psi(x) = \alpha\varphi(x) + \beta \qquad (\alpha \neq 0, x \in I).$$

Hence (4.7) can be rewritten as

$$\varphi^{-1}\left(\frac{\varphi(x_1)+\varphi(x_2)+\cdots+\varphi(x_n)-\varphi(\frac{x_1+x_2+\cdots+x_n}{n})}{n-1}\right) = \varphi^{-1}\left(\frac{\varphi(x_1)+\varphi(x_2)+\cdots+\varphi(x_n)}{n}\right).$$

Thus

$$\varphi\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) = \frac{\varphi(x_1)+\varphi(x_2)+\cdots+\varphi(x_n)}{n},$$

from which we obtain that $\varphi(x) = \gamma x + \delta$ $(\gamma \neq 0)$, that is, $\varphi(x) \sim x$ $(x \in I)$. Therefore $A_\varphi^* = A$, and in this case (4.7) holds. □

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