Mathematical
Inequalities
& Applications
Volume 4, Number 3 (2001), 343–355

CARLEMAN–KNOPP TYPE INEQUALITIES VIA HARDY INEQUALITIES

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(*communicated by J. Pečarić*)

Abstract. Some new Carleman-Knopp type inequalities are proved as "end point" inequalities of modern forms of Hardy's inequalities. Both finite and infinite intervals are considered and both the cases $p \leqslant q$ and $q < p$ are investigated. The obtained results are compared with similar results in the literature and the sharpness of the constants is discussed for the power weight case. Moreover, some reversed Carleman-Knopp inequalities are derived and applied.

1. Introduction

It is well known that the classical Carleman-Knopp inequality[1] [1], [7] (see also [14])

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx \leqslant e\int_0^\infty f(x)dx \tag{1.1}$$

can be derived as a limiting case of the classical Hardy inequality. In the paper [5], the authors further investigated this idea by also proving inequalities (for $p \leqslant q$) of the type

$$\left(\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right]^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{1.2}$$

for general weights. The technique was to use operators of the form $\left(\frac{1}{x}\int_0^x f^\alpha(t)dt\right)^{\frac{1}{\alpha}}$ in a modern form of the well known Hardy inequality which then looks like

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f^\alpha(t)dt\right)^{\frac{q}{\alpha}} w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{1.3}$$

and taking $\alpha \to 0$ to obtain (1.2). Also, the more general inequality

$$\left(\int_0^\infty \left[\exp\left(\frac{1}{H(x)}\int_0^x h(t)\ln f(t)dt\right)\right]^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}, \tag{1.4}$$

Mathematics subject classification (2000): 26D15, 26D07.
Key words and phrases: Inequalities, Hardy's inequality, Carleman's inequality, Knopp's inequality.

[1]The discrete analogue of (1.1) was proved by Carleman and Knopp proved it in this continuous form.

where $0 \leqslant p, q < \infty$ and w, v are weight functions defined on $(0, \infty)$ and

$$H(x) := \int_0^x h(s)ds < \infty, \qquad x \in (0, \infty) \tag{1.5}$$

has been investigated by many authors for different choices of p, q and for different functions h e.g. one may refer to [2], [3], [4], [6], [8], [9], [11], [12] and [13] .

In this paper, we first prove that (1.4) is not, in principle, more general than (1.2). In fact, (1.4) is equivalent to (1.2) with some other weights, see (the reduction) Lemma 2.1. We also improve the results from [5] by proving inequalities of the type (1.3) where the intervals $(0, \infty)$ are replaced by the intervals $(0, b)$, $0 < b \leqslant \infty$ (see Theorems 3.2 and 3.3) i.e. inequalities of the form

$$\left(\int_0^b \left[\exp\left(\frac{1}{x} \int_0^x \ln f(t)dt \right) \right]^q w(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_0^b f^p(x)v(x)dx \right)^{\frac{1}{p}} \tag{1.6}$$

and the corresponding results for the case $0 < q < p < \infty$ $(p > 1)$ are also proved (see Theorems 4.2 and 4.3). An important observation is that for the power weight case when $q \to p$, we get the same upper bound as that obtained in [5] for the case $p = q$ which suggests that some kind of continuity is maintained for the sharp constant. Consequently, for the power weight case, the inequality (1.6) could be regarded as a version of Carleman-Knopp's inequality for all p and q (with the sharp constant maintained in all cases). Finally, some reversed Carleman-Knopp's type inequalities are proved and applied in Section 5. In particular, an equivalent norm in L_p- spaces is pointed out and the possibility to further generalize the results is briefly discussed.

2. The reduction lemma

LEMMA 2.1. *Let $0 < p, q < \infty$. Let w, v be weight functions defined on $(0, \infty)$ and f be a positive function defined on $(0, \infty)$. Moreover, let h be a strictly positive function on $(0, \infty)$, H be defined by (1.5) and $H(\infty) = \infty$. Then, the inequality*

$$\left(\int_0^\infty \left[\exp\left(\frac{1}{H(x)} \int_0^x h(t) \ln f(t)dt \right) \right]^q w(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty f^p(x)v(x)dx \right)^{\frac{1}{p}} \tag{2.1}$$

holds if and only if the inequality

$$\left(\int_0^\infty \left[\exp\left(\frac{1}{x} \int_0^x \ln f(t)dt \right) \right]^q w_h(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty f^p(x)v_h(x)dx \right)^{\frac{1}{p}} \tag{2.2}$$

holds with the same constant C and

$$w_h(x) = \frac{w(H^{-1}(x))}{H'(H^{-1}(x))}, \qquad v_h(x) = \frac{v(H^{-1}(x))}{H'(H^{-1}(x))}.$$

Proof. Consider the geometric mean operators

$$(G_h f)(x) := \exp\left(\frac{1}{H(x)}\int_0^x h(t)\ln f(t)dt\right)$$

and

$$(Gf)(x) := \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right).$$

We note that

$$(G_h f)(x) = (Gg)(H(x)),$$

where

$$g(y) = f(H^{-1}(y)).$$

Now, the inequality (2.1) reads

$$\left(\int_0^\infty [(Gg(H(x))]^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty [g(H(x))]^p v(x)dx\right)^{\frac{1}{p}}. \qquad (2.3)$$

By making the variable transformation $y = H(x)$, we see that (2.3) is equivalent to (2.2) and we are done.

REMARK. Lemma 2.1 means that for the case when h is continuous and strictly positive, inequalities of the type (2.1) can be obtained by only studying the basic inequality (2.2). In fact, the same is true also when h is not continuous (e.g. only measurable). In that case, we must just interpret the derivative and the inverse in some suitable generalized way (or just study (2.3) instead of (2.1)). Also, the case $h(x) \geqslant 0$ can be dealt with by making limiting approximations.

REMARK. Our proof above shows that a variant of Lemma 2.1 holds as well when the interval $(0,\infty)$ is replaced by a finite interval $(0,b)$, $b < \infty$.

EXAMPLE 2.2. (c.f. [4], [6]) Take $h(x) = x^k$ in Lemma 2.1 and put

$$(G_k f)(x) := \exp\left(\frac{k+1}{x^{k+1}}\int_0^x t^k \ln f(t)dt\right).$$

Then the inequality

$$\left(\int_0^\infty [(G_k f)(x)]^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}$$

is equivalent to

$$\left(\int_0^\infty [(Gg)(x)]^q w_k(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty g^p(x)v_k(x)dx\right)^{\frac{1}{p}}$$

where

$$w_k(x) = \frac{w\left([(k+1)x]^{\frac{1}{k+1}}\right)}{[(k+1)x]^{\frac{k}{k+1}}}, \qquad v_k(x) = \frac{v\left([(k+1)x]^{\frac{1}{k+1}}\right)}{[(k+1)x]^{\frac{k}{k+1}}}$$

and

$$g(x) = f\left([(k+1)x]^{\frac{1}{k+1}}\right).$$

3. The case $p \leqslant q$ revisited

In [5], the authors studied the inequalities (1.2) for $0 < \alpha < p \leqslant q < \infty$ and (1.1) (corresponding to $\alpha \to 0$) for $0 < p \leqslant q < \infty$ as a limiting case of Hardy inequality which holds for all positive functions defined on $(0,\infty)$. We give below modified versions of the same in which the inequalities hold for all positive functions defined on $(0,b)$, $0 < b \leqslant \infty$. The proofs of Theorems 3.1 and 3.2 go along the same lines as for the interval $(0,\infty)$ and so we omit them.

THEOREM 3.1. *Let $0 < \alpha < p \leqslant q < \infty$ and w, v be weights on $(0,b)$. Assume that for every $x > 0$*

$$\int_0^x v^{\frac{\alpha}{\alpha-p}}(t)dt < \infty.$$

Then, the inequality

$$\left(\int_0^b \left(\frac{1}{x}\int_0^x f^{\alpha}(t)dt\right)^{\frac{q}{\alpha}} w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{3.1}$$

holds for all positive functions defined on $(0,b)$ if and only if

$$D_{\alpha}(q,p) = \sup_{x\in(0,b)}\left(\int_x^b w(t)t^{\frac{-q}{\alpha}}dt\right)^{\frac{1}{q}}\left(\int_0^x v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{p-\alpha}{\alpha p}} < \infty. \tag{3.2}$$

Moreover, the best constant C in (3.1) satisfies

$$D_{\alpha}(q,p) \leqslant C \leqslant k_{\alpha}(q,p)D_{\alpha}(q,p),$$

where

$$k_{\alpha}(q,p) = \left(\frac{\alpha p + qp - q\alpha}{\alpha p}\right)^{\frac{1}{q}}\left(\frac{\alpha p + qp - q\alpha}{(p-\alpha)q}\right)^{\frac{p-\alpha}{\alpha p}}. \tag{3.3}$$

THEOREM 3.2. *Let $0 < p \leqslant q < \infty$ and w, v be weights on $(0,b)$. If*

$$\mathscr{B} = \lim_{\alpha\to 0}\sup_{x\in(0,b)}\left(\int_x^b w(t)\frac{t^{\frac{-q}{\alpha}}}{\alpha}dt\right)^{\frac{1}{q}}\left(\int_0^x v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{p-\alpha}{\alpha p}} < \infty, \tag{3.4}$$

then the inequality

$$\left(\int_0^b \left[\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right]^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{3.5}$$

holds for all positive functions f defined on the interval $(0,b)$. Moreover, if C is the best constant in (3.5), then

$$C \leqslant q^{\frac{1}{q}}e^{\frac{1}{q}}\mathscr{B}.$$

Note that the condition (3.1) is only sufficient for the inequality (3.2) to hold. However, for the power weight case, we have:

THEOREM 3.3. *Let* $0 < p \leqslant q < \infty$, $\xi,\eta \in \mathbb{R}$ *and* $0 < b \leqslant \infty$. *Then, the inequality*

$$\left(\int_0^b \left[\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right]^q x^{\xi}dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b f^p(x)x^{\eta}dx\right)^{\frac{1}{p}} \tag{3.6}$$

holds for some finite $C > 0$ *and all measurable functions* f *if and only if*

$$\begin{array}{ll} \xi \geqslant \frac{q}{p}(\eta+1)-1, & b<\infty, \\ \xi = \frac{q}{p}(\eta+1)-1, & b=\infty. \end{array} \tag{3.7}$$

Moreover, if C *is the best possible constant in (3.6), then*

$$C \leqslant \begin{cases} e^{\frac{\eta}{p}+\frac{1}{q}}b^{\frac{\xi+1}{q}-\frac{\eta+1}{p}} & ,b<\infty \\ e^{\frac{\eta}{p}+\frac{1}{q}} & ,b=\infty \end{cases}. \tag{3.8}$$

Proof. Let $b < \infty$. Assume that (3.7) holds and apply Theorem 3.2 with $w(x) = x^{\xi}$ and $v(x) = x^{\eta}$. A straightforward calculation shows that $\mathscr{B}$ defined by (3.4) is finite and can be estimated as follows

$$\mathscr{B} \leqslant e^{\frac{\eta}{p}} q^{-\frac{1}{q}} b^{\frac{\xi+1}{q}-\frac{\eta+1}{p}}.$$

We conclude that (3.6) holds and C can be estimated by (3.8).

Conversely, assume that (3.6) hold for some $C > 0$. and all measurable functions f. Moreover, assume that (3.7) holds for some $\xi < \frac{q}{p}(\eta+1)-1$. Now, we choose $f(x) = x^a,\ 0 \leqslant x \leqslant b,\ -\frac{(\eta+1)}{p} < a < -\frac{(\xi+1)}{q}$ and note that

$$\int_0^b f^p(x)x^{\eta}dx = \int_0^b x^{ap+\eta}dx < \infty$$

and

$$\begin{aligned} \int_0^b \left[\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right]^q x^{\xi}dx &= \int_0^b \exp\left(\frac{aq}{x}\int_0^x \ln t dt\right) x^{\xi}dx \\ &= e^{-aq}\int_0^b x^{aq+\xi}dx \\ &= \infty. \end{aligned}$$

This contradiction shows that our assumption is wrong so that, in fact, (3.7) holds.

Let $b = \infty$. The proof of the sufficient part only consists of some obvious modifications of the proof of the case $b < \infty$. Concerning the necessary part, we first consider the function

$$f(x) = \begin{cases} x^a & ,0 \leqslant x \leqslant 1 \\ 0 & ,x > 1, \end{cases}$$

with a choosen as above. Then, as before we find that $\xi \geqslant \frac{q}{p}(\eta+1)-1$ and it only remains to prove that

$$\xi \leqslant \frac{q}{p}(\eta+1)-1. \tag{3.9}$$

Assume on the contrary that (3.6) holds for some $C>0$ and all measurable positive functions and that

$$\xi > \frac{q}{p}(\eta+1)-1.$$

Choose

$$f(x)=\begin{cases} x^{a_1} & ,0\leqslant x\leqslant e \\ x^{a_2} & ,x>e, \end{cases}$$

where $a_1p+\eta>-1$ and $-\frac{(\xi+1)}{q}<a_2<-\frac{(\eta+1)}{p}$. Then

$$\int_0^\infty f^p(x)x^\eta dx=\int_0^e x^{a_1p+\eta}dx+\int_e^\infty x^{a_2p+\eta}dx<\infty$$

and

$$\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right]^q x^\xi dx \geqslant e^{-a_2q}\int_e^\infty x^{-a_2q+\xi}dx=\infty.$$

We conclude that (3.6) does not hold for any $C<\infty$ and this contradiction shows that (3.9) holds and the proof is complete.

4. The case $q<p$

First we state the following formal generalization of a modern form of Hardy's inequality:

THEOREM 4.1. *Let $0<\alpha<q<p<\infty$ and w,v be weight functions defined on $(0,b)$, $0<b\leqslant\infty$. Then the inequality*

$$\left(\int_0^b\left(\frac{1}{x}\int_0^x f^\alpha(t)dt\right)^{\frac{q}{\alpha}}w(x)dx\right)^{\frac{1}{q}}\leqslant C\left(\int_0^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.1}$$

holds for all positive functions f defined on $(0,b)$ if and only if

$$\mathscr{A}\equiv\left\{\int_0^b\left[\left(\int_x^b w(t)t^{-q/\alpha}dt\right)\left(\int_0^x v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{q-\alpha}{\alpha}}\right]^{\frac{p}{p-q}}v^{\frac{\alpha}{\alpha-p}}(x)dx\right\}^{\frac{p-q}{pq}}<\infty. \tag{4.2}$$

Moreover, the best constant C in (4.2) can be estimated by:

$$\left(\frac{q}{\alpha}\right)^{\frac{1}{q}}\left(\frac{p-q}{p-\alpha}\right)^{\frac{q-\alpha}{\alpha q}}\mathscr{A}\leqslant C\leqslant\left(\frac{q}{\alpha}\right)^{\frac{1}{q}}\left(\frac{p}{p-\alpha}\right)^{\frac{q-\alpha}{\alpha q}}\mathscr{A}.$$

For the case $\alpha = 1$, this is just the usual weighted Hardy's inequality (see [7, p.13]). Theorem 4.1 can be obtained from this special case by just replacing f with f^{α} and making some obvious substitutions.

REMARK. The inequality (4.1) is, in fact, a scale of inequalities for $\alpha \in (0,q)$. The case $\alpha \to 0$ needs attention. It can be observed that for this case $\int_x^b w(t)t^{-q/\alpha}dt \to 0$ and consequently the condition (4.2) is meaningless. On the other hand as $\alpha \to 0$, the inequality (4.1) becomes (1.5) and characterizations of such inequalities, for the case $b = \infty$, are available in the literature, see e.g. [4], [13]. However, the conditions there are fairly complicated.

For the limiting case $\alpha = 0$ we have the following result:

THEOREM 4.2. *Let* $0 < q < p < \infty$, $p > 1$ *and* w, v *be weight functions defined on* $(0,b)$, $0 < b \leqslant \infty$. *If* $\mathscr{A}^* < \infty$, *where*

$$\mathscr{A}^* \equiv \lim_{\alpha \to 0} \left\{ \int_0^b \left[\left(\int_x^b w(t) \frac{t^{-q/\alpha}}{\alpha} dt \right) \left(\int_0^x v^{\frac{\alpha}{\alpha - p}}(t)dt \right)^{\frac{q-\alpha}{\alpha}} \right]^{\frac{p}{p-q}} v^{\frac{\alpha}{\alpha-p}}(x)dx \right\}^{\frac{p-q}{pq}}, \tag{4.3}$$

then the inequality

$$\left(\int_0^b \exp\left(\frac{1}{x} \int_0^x \ln f(t)dt \right)^q w(x)dx \right)^{\frac{1}{q}} \leqslant C \left(\int_0^b f^p(x)v(x)dx \right)^{\frac{1}{p}} \tag{4.4}$$

holds for all positive functions f *defined on* $(0,b)$ *and the best constant* C *satisfies*

$$C \leqslant q^{\frac{1}{q}} e^{\frac{1}{p}} \mathscr{A}^*.$$

Proof. In view of the condition $\mathscr{A}^* < \infty$, we have in particular that

$$\int_0^t v^{\frac{\alpha}{\alpha-p}}(y)dy < \infty, \quad t \in (0,b).$$

Let $0 < \alpha < q$ and write $\tilde{w}(x) = x^{-q/\alpha}w(x)$. Then an application of Fubini's theorem gives

$$\begin{aligned} \int_0^b \left(\frac{1}{x} \int_0^x f^{\alpha}(t)dt \right)^{\frac{q}{\alpha}} w(x) &= \frac{q}{\alpha} \int_0^b \left[\int_0^x \left(\int_0^y f^{\alpha}(t)dt \right)^{\frac{q-\alpha}{\alpha}} f^{\alpha}(y)dy \right] \tilde{w}(x)dx \\ &= \frac{q}{\alpha} \int_0^b \left(\int_0^y f^{\alpha}(t)dt \right)^{\frac{q-\alpha}{\alpha}} f^{\alpha}(y) \left(\int_y^b \tilde{w}(x)dx \right) dy, \end{aligned}$$

which, by using Hölder's inequality for the product of three functions with exponents $(\frac{p}{p-q}, \frac{p}{\alpha}, \frac{p}{q-\alpha})$, gives

$$\int_0^b \left(\frac{1}{x}\int_0^x f^{\alpha}(t)dt\right)^{\frac{q}{\alpha}} w(x)dx$$

$$\leqslant q\left(\frac{p-\alpha}{\alpha}\right)^{\frac{\alpha-q}{p}} \left(\int_0^b \left(\int_y^b \frac{\tilde{w}(x)}{\alpha}dx\right)^{\frac{p}{p-q}} \left(\int_0^y v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{q-\alpha}{\alpha}\cdot\frac{p}{p-q}} v^{\frac{\alpha}{\alpha-p}}(y)dy\right)^{\frac{p-q}{p}}$$

$$\cdot\left(\int_0^b f^p(y)v(y)dy\right)^{\frac{\alpha}{p}} \left(\frac{p-\alpha}{\alpha}\right)^{\frac{q-\alpha}{p}}$$

$$\cdot\left(\int_0^b \left(\int_0^y f^{\alpha}(t)dt\right)^{\frac{p}{\alpha}} \left(\int_0^y v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{-p}{\alpha}} v^{\frac{\alpha}{\alpha-p}}(y)dy\right)^{\frac{q-\alpha}{p}}$$

$$= q\left(\frac{p-\alpha}{\alpha}\right)^{\frac{\alpha-q}{p}} (\mathscr{A}_{\alpha}^*)^q \left(\int_0^b f^p(y)v(y)dy\right)^{\frac{\alpha}{p}} \left(\int_0^b \left(\int_0^y f^{\alpha}(t)dt\right)^{\frac{p}{\alpha}} \tilde{w}^*(y)dy\right)^{\frac{q-\alpha}{p}}, \tag{4.5}$$

where

$$\tilde{w}^*(y) = \left(\frac{p-\alpha}{\alpha}\right)\left(\int_0^y v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{-p}{\alpha}} v^{\frac{\alpha}{\alpha-p}}(y)$$

and

$$\mathscr{A}_{\alpha}^* = \left\{\int_0^b \left[\left(\int_x^b w(t)\frac{t^{-q/\alpha}}{\alpha}dt\right)\left(\int_0^x v^{\frac{\alpha}{\alpha-p}}(t)dt\right)^{\frac{q-\alpha}{\alpha}}\right]^{\frac{p}{p-q}} v^{\frac{\alpha}{\alpha-p}}(x)dx\right\}^{\frac{p-q}{pq}}.$$

Now, we use Theorem 3.1 with $p=q$ (and weights $\tilde{w}^*$ and v) to estimate the last term in (4.5). Some straightforward calculations show that $D_{\alpha}(p,p)$, defined by (3.2) can be estimated by

$$D_{\alpha}(p,p) \leqslant 1$$

and also that

$$k_{\alpha}(p,p) = \left(\frac{p}{\alpha}\right)^{\frac{1}{p}}\left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}}$$

(c.f. (3.3)) and, consequently, the inequality

$$\left(\int_0^b \left(\int_0^y f^{\alpha}(t)dt\right)^{\frac{p}{\alpha}} \tilde{w}^*(y)dy\right)^{\frac{1}{p}} \leqslant \left(\frac{p}{\alpha}\right)^{\frac{1}{p}}\left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}}\left(\int_0^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.6}$$

holds.

By using (4.6) in (4.5), we obtain that

$$\int_0^b \left(\frac{1}{x}\int_0^x f^{\alpha}(t)dt\right)^{\frac{q}{\alpha}} w(x)dx \leqslant q\left(\frac{p-\alpha}{\alpha}\right)^{\frac{\alpha-q}{p}} (\mathscr{A}_{\alpha}^{*})^{q}\left(\int_0^b f^{p}(y)v(y)dy\right)^{\frac{\alpha}{p}} \cdot \left(\frac{p}{\alpha}\right)^{\frac{q-\alpha}{p}}\left(\frac{p}{p-\alpha}\right)^{(\frac{p-\alpha}{\alpha p})(q-\alpha)}\left(\int_0^b f^{p}(x)v(x)dx\right)^{\frac{q-\alpha}{p}}$$

which gives

$$\left(\int_0^b \left(\frac{1}{x}\int_0^x f^{\alpha}(t)dt\right)^{\frac{q}{\alpha}} w(x)dx\right)^{\frac{1}{q}} \leqslant q^{\frac{1}{q}}\left(\frac{p}{p-\alpha}\right)^{\frac{q-\alpha}{\alpha q}} \mathscr{A}_{\alpha}^{*}\left(\int_0^b f^{p}(x)v(x)dx\right)^{\frac{1}{p}}.$$

Now, by taking the limit $\alpha \to 0$, the last expression yields that the inequality (4.4) holds with

$$C \leqslant q^{\frac{1}{q}} e^{\frac{1}{p}} \lim_{\alpha\to 0} \mathscr{A}_{\alpha}^{*} = q^{\frac{1}{q}} e^{\frac{1}{p}} \mathscr{A}^{*}.$$

REMARK. Analogously to the case $p \leqslant q$ (c.f. Theorem 3.2), the condition $\mathscr{A}^{*} < \infty$ is sufficient for the inequality (4.4) to hold. However, for the power weight case, the condition is, again, necessary as well which we state in Theorem 4.3. An important point to note is that for the power weight case, b can not be infinity in Theorem 4.3 for otherwise $\mathscr{A}^{*} = \infty$. Consequently, we consider power weights defined on $(0,b)$, $0 < b < \infty$.

THEOREM 4.3. *Let* $0 < q < p < \infty$, $p > 1$, $\xi, \eta \in \mathbb{R}$ *and* $0 < b < \infty$. *Then, the inequality*

$$\left(\int_0^b \left[\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right]^{q} x^{\xi}dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b f^{p}(x)x^{\eta}dx\right)^{\frac{1}{p}} \tag{4.7}$$

holds if and only if

$$\xi > \frac{q}{p}(\eta+1)-1.$$

Moreover, the best possible constant C in (4.8) satisfies

$$C \leqslant \left(\frac{p-q}{p(\xi+1)-q(\eta+1)}\right)^{\frac{p-q}{pq}} e^{\frac{\eta}{p}+\frac{1}{q}} b^{\frac{\xi+1}{q}-\frac{\eta+1}{p}}. \tag{4.8}$$

Proof. The sufficient part is straightforward if we put $w(x) = x^{\xi}$ and $v(x) = x^{\eta}$ in Theorem 4.2 while the necessary part can be proved analogously as in Theorem 3.3 and so we omit the details.

REMARK. If we let $q \to p$ in (4.9), then the upper bound for the best constant C in (4.8) gives the same value as taking $p = q$ in (3.8). This means that there is a continuity in the upper bounds for the inequalities (3.6) and (4.8).

5. Final remarks and examples

REMARK. (General mean inequalities) The results of this paper and the paper [5] suggest that it can be of interest and possible to study more general inequalities of the type

$$\left(\int_0^b A(f(x))^q w(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b f^p(x)v(x)dx\right)^{\frac{1}{p}}, \tag{5.1}$$

where A is a more general mean operator than those studied in these papers i.e. the usual Hardy operator, the power mean operator (see e.g. Theorems 3.1 and 4.1) or geometric mean operator (see e.g. Theorems 3.2 and 4.2). A good candidate can be to take A to be general Gini-mean operators, see e.g. [15] and the references given there.

REMARK. (Reversed Carleman-Knopp type inequalities) If f is non-negative and non-increasing, then obviously

$$\frac{1}{x}\int_0^x f(t)dt \geqslant f(x). \tag{5.2}$$

We obtain easily reversed Carleman type inequalities by using (5.2) and other well-known inequalities in the literature.

First we state the following reversed Carleman-Knopp inequality:

EXAMPLE 5.1. Let $p > 0$. The inequality

$$\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right]^p dx \geqslant \int_0^\infty f^p(x)dx \tag{5.3}$$

holds for all non-negative, non-increasing functions f defined on $(0,\infty)$ and the inequality is sharp .

Obviously, (5.2) implies (5.3). The sharpness follows by considering the functions f_ε defined so that the right hand side is finite, $f_\varepsilon(x) = 1$ for $0 \leqslant x \leqslant 1$, $f_\varepsilon(x) > 0$ and $f_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0^+$.

REMARK. (An equivalent norm in L_p and $L_{p,q}$ spaces) Let f be a measurable function on a $\sigma-$ finite measure space (Ω,μ) and let f^* be its non-increasing rearrangement defined on $(0,\infty)$. Then, by taking $\xi = \eta = 0$ and $b = \infty$ in Theorem 3.3, we see in particular that the inequality

$$\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \ln f^*(t)dt\right)\right]^p dx \leqslant e\int_\Omega |f|^p\, d\mu. \tag{5.4}$$

holds since $\left(\int_\Omega |f|^p\, d\mu\right)^{1/p} = \left(\int_0^\infty |f^*|^p\, dx\right)^{1/p}$.

Further, since (5.3) also holds for f^*, we get from (5.3) and (5.5) that $\|f\|_{L^p(\Omega)} \cong \|f\|^*_{L^p(\Omega)}$, or more exactly,

$$\|f\|_{L^p(\Omega)} \leqslant \|f\|^*_{L^p(\Omega)} \leqslant e^{1/p} \|f\|_{L^p(\Omega)},$$

where

$$\|f\|^*_{L^p(\Omega)} = \left(\int_0^\infty \left[\exp\left(\frac{1}{x}\int_0^x \ln f^*(t)dt\right)\right]^p\right)^{\frac{1}{p}}$$

and the embedding constants 1 and $e^{1/p}$ are sharp.

It is obvious from our discussion above that it is possible to give a similar equivalent discription of the norm in general Lorentz $L_{p,q}$ spaces and not only for L_p spaces as indicated above.

REMARK. A version of the inequality (5.3) with weights and with different exponents p,q can also be obtained by using well-known results in the literature.

For example, by using a general result of Stepanov [17, Proposition 1] and (5.2) we obtain

EXAMPLE 5.2. Let $w(x)$ and $v(x)$ be weight functions and let f be a non-negative and non-increasing function. Then the inequality

$$\left(\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \geqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{5.5}$$

holds in each of the following cases:
(i) $0 < p \leqslant q < \infty$, and

$$\left(\int_0^\infty \left(\int_0^x w(t)dt\right)^{-\frac{r}{q}} \left(\int_0^x v(t)dt\right)^{\frac{r}{q}} v(x)dx\right)^{\frac{1}{r}} < \infty,$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.
(ii) $0 < q \leqslant p < \infty$, and

$$\sup_{x>0}\left(\int_0^x w(t)dt\right)^{-\frac{1}{q}} \left(\int_0^x v(t)dt\right)^{\frac{1}{p}} < \infty.$$

REMARK. (Reversed general mean inequalities) In view of Examples 5.1 and 5.2, the techniques of this paper and the paper [5], it should be of interest to study inequalities of the type

$$\left(\int_0^b A(f(x))^q w(x)dx\right)^{\frac{1}{q}} \geqslant C\left(\int_0^b f^p(x)v(x)dx\right)^{\frac{1}{p}}$$

for all those operators A for which (5.1) could be studied.

REMARK. (Concerning the constant $\mathscr{A}^*$ in Theorem 4.2) For the case $b = \infty$, we can make variable transformation and write $\mathscr{A}^*$ as

$$\mathscr{A}^* = q^{-\frac{1}{q}} \lim_{\alpha \to 0} \left(\int_0^\infty \left(\frac{q}{\alpha} \int_1^\infty \frac{w(tx)}{t^{q/\alpha}} dt \right)^{\frac{r}{q}} \left(\frac{1}{x} \int_0^x \left(\frac{1}{v(t)} \right)^{\frac{\alpha}{p-\alpha}} dt \right)^{\frac{1}{\alpha} \frac{p(q-\alpha)}{(p-q)}} (v(t))^{\frac{\alpha}{\alpha - p}} dt \right)^{\frac{1}{r}}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Since

$$\chi_{[1,\infty)}(t) \left(\frac{q}{\alpha} - 1 \right) t^{-q/\alpha} \to \delta_1(x) \quad as \quad \alpha \to 0,$$

where $\delta_1(x)$ is the Dirac delta function at $x = 1$ (point unit mass at $x = 1$), we see that $\mathscr{A}^*$ can be written in the simpler form

$$\mathscr{A}^* = q^{-\frac{1}{q}} \left(\int_0^\infty (w(x))^{\frac{r}{q}} \left(\exp \frac{1}{x} \int_0^x \ln \frac{1}{v(t)} dt \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}}.$$

(This observation was pointed out to us by Professor Vladimir Stepanov, see also [16].)

Acknowledgment. We thank the careful referee and Professor Vladimir Stepanov for some suggestions and generous advices which have improved the final version of this paper. The research of the first author is partially supported by the Department of Science and Technology (Ministry of HRD), India. The author also acknowledges with thanks the facilities provided to him by the Department of Mathematics, Luleå University of Technology, Sweden.

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(Received February 1, 2000)

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