Mathematical
Inequalities
& Applications
Volume 4, Number 3 (2001), 357–368

ON SOME INEQUALITIES RELATED TO HARDY'S INTEGRAL INEQUALITY

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(*communicated by B. Mond*)

Abstract. Some Hardy–type inequalities related to those derived by Pachpatte and Love are given.

One of the many fundamental mathematical discoveries of G. H. Hardy is the following integral inequality [2, Theorem 330]:

If $p > 1$, $m \neq 1$, $f(t)$ *is non-negative measurable, and* F *is defined on* $(0, \infty)$ *by*

$$F(x) = \int_0^x f(t)dt \quad \text{for} \quad m > 1, \qquad F(x) = \int_x^\infty f(t)dt \quad \text{for} \quad m < 1,$$

then

$$\int_0^\infty x^{-m} F(x)^p dx < \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{-m+p} f(x)^p dx$$

unless $f(t) \equiv 0$. *The constant on the right is the best possible.*

A typical theorem in [1] is:

Let $m > 1$, $p \geqslant 1$, *and all hypotheses involving* k *hold for* $k = 1, 2, \ldots, n$. *Let* $r_k(x)$ *and* $w(x)$ *be positive and locally absolutely continuous in* $(0, \infty)$. *Let* $z(x)$ *be differentiable in* $(0, \infty)$ *with* $z'(x) > 0$ *and* $z(0+) > 0$. *Let*

$$I_k f(x) = \int_0^x \frac{r_k(t) z'(t)}{r_k(x) z(x)} f(t) dt$$

and

$$0 < \frac{1}{\alpha_k} \leqslant \underset{0<x<\infty}{ess\ inf} \left\{ 1 + \frac{1}{m-1} \frac{z(x)}{z'(x)} \left(p \frac{r_k'(x)}{r_k(x)} - \frac{w'(x)}{w(x)} \right) \right\},$$

If $f(x)$ *is non-negative measurable on* $(0, \infty)$, $F_0(x) = z(x) f(x)$,

$$F_k(x) = z(x) I_k I_{k-1} \ldots I_1 f(x)$$

Mathematics subject classification (2000): 26D15, 26D20.

Key words and phrases: Hardy-type integral inequalities, integrable functions, weight functions.

each exist for some positive x *and hence for all, and*

$$\frac{w(x)}{z(x)^{m-1}} F_k(x)^p \to 0 \quad as \quad x \to 0+,$$

then

$$\left(\int_0^\infty \frac{z'(x)}{z(x)^m} w(x) F_n(x)^p dx\right)^{\frac{1}{p}} \leqslant A \left(\int_0^\infty \frac{z'(x)}{z(x)^m} w(x) F_0(x)^p dx\right)^{\frac{1}{p}}$$

where

$$A = \left(\frac{p}{|m-1|}\right)^n \prod_{k=1}^n \alpha_k .$$

In this paper we wish to further develop theorems of this kind.

THEOREM 1. *Let* $m > 1$, $p \geqslant 1$, $q \geqslant 0$ *and* $X > 0$. *Let* $s(x)$, $w(x)$ *and* $z(x)$ *be absolutely continuous and positive on* $[0, X]$, *with* $z'(x)$ *essentially bounded and positive.*

If $f(x)$ *is non-negative and* L^p *on* $(0, X)$,

$$F(x) = \frac{1}{s(x)} \int_0^x \frac{s(t) z'(t)}{z(t)} f(t)\, dt, \quad for \ \ 0 \leqslant x \leqslant X,$$

and

$$0 < \frac{1}{\alpha} \leqslant \operatorname*{ess\,inf}_{0<x<X} \left\{ 1 + \frac{1}{m-1} \frac{z(x)}{z'(x)} \left((p+q) \frac{s'(x)}{s(x)} - \frac{w'(x)}{w(x)} \right) \right\},$$

then

$$\int_0^X \frac{z'(x)}{z(x)^m} w(x) F(x)^{p+q} dx \leqslant \left(\frac{(p+q)\alpha}{m-1}\right)^p \int_0^X \frac{z'(x)}{z(x)^m} w(x) F(x)^q f(x)^p dx. \tag{1}$$

This conclusion also holds with X *replaced by* ∞ *if the hypotheses on* s, w, z *and* z' *hold locally in* $[0, \infty)$; *but the integrals may not then be convergent.*

Proof. (i) Since s, w and z are continuous and positive on $[0, X]$, they have positive lower and upper bounds thereon; therefore $s(t)/z(t)$ is bounded. Since z' is bounded above, so also is $s(t)z'(t)/z(t)$. Further f is integrable, so the integral in $F(x)$ exists and is absolutely continuous in $[0, X]$.

Because $s(x)$ has positive lower bound, $1/s(x)$ is absolutely continuous on $[0, X]$, and consequently $F(x)$ is absolutely continuous. In particular $F(x)$ is bounded.

From the boundedness of $1/z(t)$ we have boundedness of $1/z(x)^m$. Also $z'(x)$ and $w(x)$ are bounded. So the whole integrand in the left side of (1) is bounded and the integral on the left of (1) is convergent.

(ii) Since $z(x)$ has positive lower bound and is absolutely continuous, $1/z(x)$ is absolutely continuous; consequently so also is $(1/z(x))^{m-1} = z(x)^{1-m}$.

Now $w(x)F(x)^{p+q}$ is also absolutely continuous on $[0, X]$; so the following integration by parts is valid:

$$\int_0^X w(x) F(x)^{p+q} (1-m) \frac{z'(x)}{z(x)^m} dx = \left[w(x) F(x)^{p+q} z(x)^{1-m} \right]_0^X$$

$$-\int_0^X z(x)^{1-m}\left\{w'(x)F(x)^{p+q}+w(x)(p+q)F(x)^{p+q-1}F'(x)\right\}dx$$
$$= w(X)F(X)^{p+q}z(X)^{1-m}$$

$$-\int_0^X z(x)^{1-m}\left\{w'(x)F(x)^{p+q}+w(x)(p+q)F(x)^{p+q-1}\left(\frac{z'(x)}{z(x)}f(x)-\frac{s'(x)}{s(x)}F(x)\right)\right\}dx.$$

It follows that

$$(m-1)\int_0^X \frac{z'(x)}{z(x)^m}w(x)F(x)^{p+q}dx+z(X)^{1-m}w(X)F(X)^{p+q}$$
$$=\int_0^X \frac{z'(x)}{z(x)^m}w(x)\left\{\frac{z(x)}{z'(x)}\left(\frac{w'(x)}{w(x)}-(p+q)\frac{s'(x)}{s(x)}\right)\right.$$
$$\left.\times F(x)^{p+q}+(p+q)F(x)^{p+q-1}f(x)\right\}dx \tag{2}$$
$$=\int_0^X \frac{z'(x)}{z(x)^m}w(x)\left[(m-1)-(m-1)\left\{1+\frac{1}{m-1}\frac{z(x)}{z'(x)}\left((p+q)\frac{s'(x)}{s(x)}-\frac{w'(x)}{w(x)}\right)\right\}\right.$$
$$\left.\times F(x)^{p+q}+(p+q)F(x)^{p+q-1}f(x)\right]dx. \tag{3}$$

Now (2) is finite, as shown in (i); so (3) is also finite. And

$$\frac{z'(x)}{z(x)^m}w(x)(p+q)F(x)^{p+q-1}f(x)$$

is integrable since $f(x)$ is integrable, $z'(x)$ is bounded, and the other factors in this expression are absolutely continuous, including $1/z(x)^m$. Writing

$$W(x)=\frac{z'(x)}{z(x)^m}w(x) \text{ and } S(x)=1+\frac{1}{m-1}\frac{z(x)}{z'(x)}\left((p+q)\frac{s'(x)}{s(x)}-\frac{w'(x)}{w(x)}\right),$$

(3) is equal to

$$\int_0^X W(x)\{(m-1)-(m-1)S(x)\}F(x)^{p+q}dx+(p+q)\int_0^X W(x)F(x)^{p+q-1}f(x)dx, \tag{4}$$

this separation into the sum of two integrals being correct because the integral (3) and the second integral in (4) exist as (finite) Lebesgue integrals, so that, by additivity, the first integral in (4) exists and (3) and (4) are equal.

(iii) Having thus proved that (2), (3) and (4) are all equal, we obtain, on dividing by the positive number $m-1$,

$$\int_0^X W(x)F(x)^{p+q}dx$$
$$\leqslant \int_0^X W(x)\{1-S(x)\}F(x)^{p+q}dx+\frac{p+q}{m-1}\int_0^X W(x)F(x)^{p+q-1}f(x)dx,$$

so that

$$\int_0^X W(x)S(x)F(x)^{p+q}dx \leqslant \frac{p+q}{m-1}\int_0^X W(x)F(x)^{p+q-1}f(x)dx.$$

Since $S(x) \geqslant 1/\alpha$ almost everywhere,

$$\frac{m-1}{(p+q)\alpha}\int_0^X W(x)F(x)^{p+q}dx \leqslant \int_0^X W(x)F(x)^{p+q-1-\frac{q}{p}}F(x)^{\frac{q}{p}}f(x)\,dx$$
$$\leqslant \left(\int_0^X W(x)F(x)^{p+q}dx\right)^{1-\frac{1}{p}}\left(\int_0^X W(x)F(x)^q f(x)^p dx\right)^{\frac{1}{p}} \qquad (5)$$

by Hölder's inequality.

(iv) Since $z'(x) > 0$ almost everywhere in $[0,X]$, and $w(x)/z(x)^m > 0$ everywhere, $W(x) > 0$ almost everywhere. So if the first factor on the right in (5) were to vanish, we should have $F(x) = 0$ almost everywhere, and therefore everywhere since F is continuous. This would make $f(t) = 0$ almost everywhere, so that both sides in (5) would vanish and (5) would hold trivially. We can thus exclude this situation, and suppose that the first factor on the right in (5) is positive (and it is finite as shown in (i)). Dividing through (5) by it,

$$\frac{m-1}{(p+q)\alpha}\left(\int_0^X W(x)F(x)^{p+q}dx\right)^{\frac{1}{p}} \leqslant \left(\int_0^X W(x)F(x)^q f(x)^p dx\right)^{\frac{1}{p}}.$$

Theorem 1 follows immediately from this.

THEOREM 2. *Let $m > 1$, $p+r \geqslant 1$, $p \geqslant 0$, $q \geqslant 0$ and $r \geqslant 0$. Let s, w, z, z', X and F be defined as in Theorem 1, $f(x)$ be L^{p+r} on $(0,X)$ and*

$$0 < \frac{1}{\alpha} \leqslant \operatorname*{ess\,inf}_{0<x<X}\left\{1 + \frac{1}{m-1}\frac{z(x)}{z'(x)}\left((p+q+r)\frac{s'(x)}{s(x)} - \frac{w'(x)}{w(x)}\right)\right\},$$

then

$$\int_0^X \frac{z'(x)}{z(x)^m}w(x)F(x)^{p+q}f(x)^r dx \leqslant \left(\frac{(p+q+r)\alpha}{m-1}\right)^p \int_0^X \frac{z'(x)}{z(x)^m}w(x)F(x)^q f(x)^{p+r}dx.$$

This conclusion also holds with X replaced by ∞ if the hypotheses on s, w, z and z' hold locally in $[0,\infty)$ and X is replaced by ∞ in the other hypotheses.

Proof. Write $W(x) = z'(x)w(x)/z(x)^m$. By Hölder's inequality, and Theorem 1 with p replaced by $p+r$,

$$\int_0^X W(x)F(x)^{p+q}f(x)^r dx = \int_0^X W(x)\left(F(x)^{\frac{qr}{p+r}}f(x)^r\right)\left(F(x)^{p+q-\frac{qr}{p+r}}\right)dx$$
$$\leqslant \left(\int_0^X W(x)F(x)^q f(x)^{p+r}dx\right)^{\frac{r}{p+r}}\left(\int_0^X W(x)F(x)^{p+q+r}dx\right)^{\frac{p}{p+r}}$$
$$\leqslant \left(\int_0^X W(x)F(x)^q f(x)^{p+r}dx\right)^{\frac{r}{p+r}}\left(\frac{(p+q+r)\alpha}{m-1}\right)^p\left(\int_0^X W(x)F(x)^q f(x)^{p+r}dx\right)^{\frac{p}{p+r}}.$$

This gives the stated inequality.

THEOREM 3. *Let* $m > 1$, $p \geqslant 1$, $X > 0$ *and all hypotheses involving* k *hold for* $k = 1, 2, \ldots, n$. *Let* $s_k(x)$, $w(x)$ *and* $z(x)$ *be absolutely continuous and positive on* $[0, X]$, *with* $z'(x)$ *essentially bounded and positive. For any* ϕ *non-negative and* L^p *on* $(0, X)$ *let*

$$\tilde{I}_k\phi(x) = \frac{1}{s_k(x)} \int_0^x \frac{s_k(t)z'(t)}{z(t)}\phi(t)dt, \quad \text{for } 0 \leqslant x \leqslant X.$$

If $f(x)$ *is non-negative and* L^p *on* $(0, X)$,

$$\tilde{F}_k(x) = \tilde{I}_k\tilde{I}_{k-1}\ldots\tilde{I}_1 f(x), \quad \tilde{F}_0(x) = f(x),$$

and

$$0 < \frac{1}{\alpha_k} \leqslant \underset{0<x<X}{ess\ inf}\left\{1 + \frac{1}{m-1}\frac{z(x)}{z'(x)}\left(p\frac{s_k'(x)}{s_k(x)} - \frac{w'(x)}{w(x)}\right)\right\},$$

then

$$\int_0^X \frac{z'(x)}{z(x)^m} w(x)\tilde{F}_n(x)^p dx \leqslant \left(\frac{p}{m-1}\right)^{np}\left(\prod_{i=1}^n \alpha_i\right)^p \int_0^X \frac{z'(x)}{z(x)^m} w(x) f(x)^p dx.$$

This conclusion also holds with X *replaced by* ∞ *if the hypotheses on* s_k, w, z *and* z' *hold locally in* $[0, \infty)$ *and* X *is replaced by* ∞ *in the other hypotheses; but then the integrals may not be convergent.*

Proof. Since $z(x)$ is continuous and positive on $[0, X]$ it has positive lower and upper bounds thereon; and the same holds for $s_k(x)$ and $w(x)$. Since $s_k(x)z'(x)$ is essentially bounded and $\phi(t)$ is in L^p on $[0, X]$, $\tilde{I}_k\phi(x)$ exists and is continuous on $[0, X]$. In particular $\tilde{I}_k\phi(x)$ is in L^p on $(0, X)$, and clearly non-negative. It follows that $\tilde{F}_k(x)$ exists and is in L^p on $(0, X)$ and non-negative.

In Theorem 1 replace q by 0, $s(x)$ by $s_k(x)$, and $f(x)$ by $\tilde{F}_{k-1}(x)$. Then α is replaced by α_k and $F(x)$ by $\tilde{I}_k\tilde{F}_{k-1}(x)$. The requirements of Theorem 1 being satisfied, that theorem gives

$$\int_0^X \frac{z'(x)}{z(x)^m} w(x)\tilde{F}_k(x)^p dx \leqslant \left(\frac{p\alpha_k}{m-1}\right)^p \int_0^X \frac{z'(x)}{z(x)^m} w(x)\tilde{F}_{k-1}(x)^p dx.$$

Iteration of this inequality gives the stated conclusion of Theorem 3.

THEOREM 4. *Let* p, k, n, $s_k(x)$, $w(x)$ *and* $z(x)$ *be as in Theorem 3. For any* $\phi(t)$ *non-negative and* L^p *on* $[0, X]$ *let*

$$\bar{I}_k\phi(x) = \frac{1}{s_k(x)} \int_0^x s_k(t)z'(t)\phi(t)dt, \quad \text{for } 0 \leqslant x \leqslant X.$$

If $f(x)$ *is non-negative and* L^p *on* $(0, X)$,

$$\bar{F}_k(x) = \bar{I}_k\bar{I}_{k-1}\ldots\bar{I}_1 f(x), \quad \bar{F}_0(x) = f(x), \quad m > (n-1)p + 1$$

and

$$0 < \frac{1}{\alpha_k} \leqslant \underset{0<x<X}{ess\ inf}\left\{1 + \frac{1}{m-(n-k)p-1}\frac{z(x)}{z'(x)}\left(p\frac{s_k'(x)}{s_k(x)} - \frac{w'(x)}{w(x)}\right)\right\},$$

then

$$\int_0^X \frac{z'(x)}{z(x)^m} w(x)\bar{F}_n(x)^p dx \leqslant \left(\prod_{i=1}^{n} \frac{p\alpha_i}{m-(n-i)p-1}\right)^p \int_0^X \frac{z'(x)}{z(x)^{m-np}} w(x) f(x)^p\, dx.$$

This conclusion also holds with X replaced by ∞ if the hypotheses on s_k, w, z and z' hold locally in $[0,\infty)$ and X is replaced by ∞ in the other hypotheses; but the integrals may not then be convergent.

Proof. Let $g(x) = z(x)^{-1}f(x)$, non-negative and L^p on $[0,X]$, so that

$$\bar{I}_k g(x) = \frac{1}{s_k(x)}\int_0^x \frac{s_k(t)z'(t)}{z(t)} z(t)g(t)dt = \tilde{I}_k(f(x)).$$

As in Theorem 3 this is non-negative and continuous, hence L^p on $[0,X]$. Similarly

$$\bar{F}_k(x) = \bar{I}_k\bar{F}_{k-1}(x) = \frac{1}{s_k(x)}\int_0^x \frac{s_k(t)z'(t)}{z(t)} z(t)\bar{F}_{k-1}(t)dt = \tilde{I}_k(z(x)\bar{F}_{k-1}(x)). \qquad (6)$$

Since $\bar{F}_0(x)$ is non-negative and L^1 and $z(x)$ is positive and continuous, (6) gives that $\bar{F}_1(x)$ is non-negative and continuous, and so in L^p on $[0,X]$. Hence inductively $\bar{F}_k(x)$ is non-negative and in L^p. The requirements of Theorem 1 are thus satisfied. Replacing m in that theorem by $m-(n-k)p \geqslant m-(n-1)p > 1$, the case $q = 0$ of that theorem gives, using (6),

$$\begin{aligned}
&\int_0^X \frac{z'(x)}{z(x)^{m-(n-k)p}} w(x)\bar{F}_k(x)^p\, dx \\
&\quad\leqslant \left(\frac{p\alpha_k}{m-(n-k)p-1}\right)^p \int_0^X \frac{z'(x)}{z(x)^{m-(n-k)p}} w(x)\,(z(x)\bar{F}_{k-1}(x)^p)\, dx \\
&\quad= \left(\frac{p\alpha_k}{m-(n-k)p-1}\right)^p \int_0^X \frac{z'(x)}{z(x)^{m-(n-k+1)p}} w(x)\bar{F}_{k-1}(x)^p dx.
\end{aligned}$$

Iteration of this inequality gives the stated conclusion of Theorem 4.

THEOREM 5. *Let $m < 1$, $p \geqslant 1$, $q \geqslant 0$ and $X > 0$. Let $s(x)$, $w(x)$ and $z(x)$ be locally absolutely continuous in $[X,\infty)$ and have positive upper and lower bounds thereon. Let $z'(x)$ be essentially bounded and essentially positive on $[X,\infty)$.*

If $f(x)$ is non-negative and in $L^1 \cap L^p$ on $[X,\infty)$,

$$F(x) = \frac{1}{s(x)}\int_x^\infty \frac{s(t)z'(t)}{z(t)} f(t)dt,$$

and

$$0 < \frac{1}{\alpha} \leqslant \underset{X<x<\infty}{ess\ inf}\left\{1 + \frac{1}{1-m}\frac{z(x)}{z'(x)}\left(\frac{w'(x)}{w(x)} - (p+q)\frac{s'(x)}{s(x)}\right)\right\}.$$

then

$$\int_X^\infty \frac{z'(x)}{z(x)^m} w(x)F(x)^{p+q}dx \leqslant \left(\frac{(p+q)\alpha}{1-m}\right)^p \int_X^\infty \frac{z'(x)}{z(x)^m} w(x)F(x)^q f(x)^p dx, \tag{7}$$

and these integrals are convergent.

Proof. (i) For each $x \geqslant X$, $F(x)$ exists, because $1/s(x)$ exists, $s(t)/z(t)$ is bounded, $z'(t)$ is essentially bounded and $f(t)$ is integrable on (X, ∞). $F(x)$ is also positive and continuous.

Suppose that there is $\xi > X$ such that $f(x) = 0$ almost everywhere in (ξ, ∞). Let

$$\xi_0 = \inf\{\xi > X : f(x) = 0 \text{ a.e. in } x > \xi\}. \tag{8}$$

(ii) Suppose that $\xi_0 > X$. Let $\sigma = \inf\{s(t)/z(t) : t > X\}$; then

$$s(X)F(X) \geqslant \int_X^\infty \sigma z'(t)f(t)dt = \sigma \int_X^\infty z'(t)f(t)dt.$$

If the last integral were zero, $z'(t)f(t) = 0$ a.e. in (X, ∞). Since $z'(t) > 0$ a.e., $f(t) = 0$ a.e. in (X, ∞), whence $f(t) = 0$ a.e. in (X, ξ_0), contradicting the definition of ξ_0. So the integral is positive, and $s(X)F(X) > 0$. It follows that $F(X) > 0$.

(iii) Since $s(x)F(x) \to 0$ as $x \to \infty$, and $s(x)$ has positive lower bound, $F(x) \to 0$ as $x \to \infty$. Since also $z(x)^{1-m}$ and $w(x)$ are bounded above, $z(x)^{1-m}w(x)F(x)^{p+q} \to 0$ as $x \to \infty$. But by (ii) $z(X)^{1-m}w(X)F(X)^{p+q} > 0$, and therefore there is $Y_0 > X$ such that

$$0 \leqslant z(Y)^{1-m}w(Y)F(Y)^{p+q} \leqslant z(X)^{1-m}w(X)F(X)^{p+q} \quad \text{for all } Y \geqslant Y_0. \tag{9}$$

(iv) Fix $Y \geqslant Y_0$. Since $1/s(x)$ and $s(x)F(x)$ are absolutely continuous on $[X, Y]$, so also is $F(x)$. Thus $w(x)F(x)^{p+q}$ and $z(x)^{1-m}$ are absolutely continuous on $[X, Y]$. This permits the following integration by parts:

$$\begin{aligned}(1-m)\int_X^Y \frac{z'(x)}{z(x)^m} w(x)F(x)^{p+q}dx &= \int_X^Y w(x)F(x)^{p+q}\frac{d}{dx}(z(x)^{1-m})dx \\ &= \left[w(x)F(x)^{p+q}z(x)^{1-m}\right]_X^Y - \int_X^Y z(x)^{1-m}\frac{d}{dx}\{w(x)F(x)^{p+q}\}dx.\end{aligned}$$

Using

$$F'(x) = -\frac{s'(x)}{s(x)}F(x) - \frac{z'(x)}{z(x)}f(x),$$

we obtain

$$(1-m)\int_X^Y \frac{z'(x)}{z(x)^m} w(x)F(x)^{p+q}\,dx - \left[z(x)^{1-m}w(x)F(x)^{p+q}\right]_X^Y \tag{10}$$

$$= -\int_X^Y \frac{w(x)}{z(x)^{m-1}} \left\{ \frac{w'(x)}{w(x)} F(x)^{p+q} - (p+q)F(x)^{p+q-1}\left(\frac{s'(x)}{s(x)}F(x) + \frac{z'(x)}{z(x)} f(x)\right)\right\} dx$$

$$= -\int_X^Y \frac{w(x)}{z(x)^{m-1}} \left(\frac{w'(x)}{w(x)} - (p+q)\frac{s'(x)}{s(x)}\right) F(x)^{p+q}\,dx$$

$$+\ (p+q)\int_X^Y \frac{z'(x)}{z(x)^m} w(x)F(x)^{p+q-1} f(x)\,dx. \tag{11}$$

The separation into the sum of two integrals in (11) is valid because (10) and the latter integral in (11) exist since F is continuous and $f \in L^1$; so the former integral in (11) exists by linearity.

Let $W(x) = z'(x)z(x)^{-m}w(x)$. The former term in (11) can be re-written

$$-\,(1-m)\int_X^Y W(x)\frac{z(x)}{z'(x)}\frac{1}{1-m}\left(\frac{w'(x)}{w(x)} - (p+q)\frac{s'(x)}{s(x)}\right) F(x)^{p+q}\,dx$$

$$= -\,(1-m)\int_X^Y W(x)\left\{1 + \frac{1}{1-m}\frac{z(x)}{z'(x)}\left(\frac{w'(x)}{w(x)} - (p+q)\frac{s'(x)}{s(x)}\right) - 1\right\} F(x)^{p+q} dx$$

$$\leqslant -\,(1-m)\int_X^Y W(x)\left(\frac{1}{\alpha} - 1\right) F(x)^{p+q}\,dx.$$

Now using (9), the equality of (10) and (11) gives

$$(1-m)\int_X^Y W(x)F(x)^{p+q}\,dx \leqslant (1-m)\left(1-\frac{1}{\alpha}\right)\int_X^Y W(x)F(x)^{p+q}\,dx$$
$$+\,(p+q)\int_X^Y W(x)F(x)^{p+q-1}f(x)dx;$$

so that

$$\frac{1-m}{(p+q)\alpha}\int_X^Y W(x)F(x)^{p+q}\,dx \leqslant \int_X^Y W(x)F(x)^{p+q-1}f(x)\,dx \tag{12}$$
$$= \int_X^Y W(x)F(x)^{p+q-\frac{p+q}{p}}F(x)^{\frac{q}{p}}f(x)\,dx$$
$$\leqslant \left(\int_X^Y W(x)F(x)^{p+q}dx\right)^{1-\frac{1}{p}} \left(\int_X^Y W(x)F(x)^q f(x)^p dx\right)^{\frac{1}{p}}$$

if $p > 1$, by Hölder's Inequality. If also the first factor on the right is not zero, division by it gives

$$\frac{1-m}{(p+q)\alpha}\left(\int_X^Y W(x)F(x)^{p+q}\,dx\right)^{\frac{1}{p}} \leqslant \left(\int_X^Y W(x)F(x)^q f(x)^p\,dx\right)^{\frac{1}{p}}. \tag{13}$$

If on the other hand that first factor is zero, (13) is trivial, its left side being zero. Thus (13) holds if $p > 1$. It also holds if $p = 1$, being then given by (12). We therefore obtain

$$\int_X^Y W(x)F(x)^{p+q}\,dx \leqslant \left(\frac{(p+q)\alpha}{1-m}\right)^p \int_X^Y W(x)F(x)^q f(x)^p\,dx \tag{14}$$

for all $p \geqslant 1$ and $Y \geqslant Y_0$. Making $Y \to \infty$, this gives (7) whenever $p \geqslant 1$ and $\xi_0 > X$ (sec (ii)). Since $f(x) = 0$ for almost all $x > \xi_0$, the right side of (14) attains its limit for some finite value of Y; thus the right side of (7) is convergent, and hence so is the left, by (14).

(v) If $\xi = X$, $f(x) = 0$ for almost all $x > X$. Consequently $F(x) = 0$ for all $x \geqslant X$, and both sides of (7) are trivially zero.

(vi) It remains to consider what happens when $\xi_0 = \infty$. For positive integers $n > X$ let $f_n(t) = f(t)$ for $t \leqslant n$, and $f_n(t) = 0$ for $t > n$. Also let

$$F_n(x) = \frac{1}{s(x)} \int_x^\infty \frac{s(t)z'(t)}{z(t)} f_n(t)dt \quad \text{for } x > X.$$

Then $f_n(t)$ fits the discussion in (i) to (iv), so that (14) gives

$$\int_X^\infty W(x)F_n(x)^{p+q}dx \leqslant \left(\frac{(p+q)\alpha}{1-m}\right)^p \int_X^\infty W(x)F_n(x)^q f_n(x)^p\,dx, \tag{15}$$

and these integrals are convergent. Now $f_n(x) \to f(x)$ as $n \to \infty$; it follows by monotonic convergence that $F_n(x) \to F(x)$. Hence $F_n(x)^{p+q} \to F(x)^{p+q}$ and $F_n(x)^q f_n(x)^p \to F(x)^q f(x)^p$. By monotonic convergence again, the two sides of (15) tend to the respective two sides of (7), which are convergent because $f \in L^p$ and the other functions involved are bounded. This establishes (7) and completes the proof of Theorem 5.

THEOREM 6. *Let* $m < 1$, $p + r \geqslant 1$, $q \geqslant 0$, $r > 0$ *and* $X > 0$. *Let* s, w, z, z' *and* F *be as in Theorem 5. If* f *is non-negative and in* $L^1 \cap L^p$ *on* (X, ∞), *and*

$$0 < \frac{1}{\alpha} \leqslant \operatorname*{ess\,inf}_{X<x<\infty} \left\{1 + \frac{1}{1-m}\frac{z(x)}{z'(x)}\left(\frac{w'(x)}{w(x)} - (p+q+r)\frac{s'(x)}{s(x)}\right)\right\},$$

then

$$\int_X^\infty \frac{z'(x)}{z(x)^m} w(x)F(x)^{p+q} f(x)^r dx \leqslant \left(\frac{(p+q+r)\alpha}{1-m}\right)^p \int_X^\infty \frac{z'(x)}{z(x)^m} w(x)F(x)^q f(x)^{p+r} dx,$$

and these integrals are convergent.

Proof. This is almost the same as the proof of Theorem 2; the differences are that $(0, X)$ is replaced by (X, ∞), $m - 1$ by $1 - m$, and Theorem 5 is used instead of Theorem 1.

THEOREM 7. *Let* $m < 1$, $p \geqslant 1$, $X > 0$, *and all hypotheses involving* k *hold for* $k = 1, 2, \ldots, n$. *Let* $s_k(x)$, $w(x)$ *and* $z(x)$ *be locally absolutely continuous in* $[X, \infty)$, *and have positive lower and upper bounds thereon. Let* $1/s_k(x)$ *be in* $L^1 \cap L^p$

on (X,∞). *Let* $z'(x)$ *be essentially bounded and essentially positive in* (X,∞). *For any* $\phi(x)$ *non-negative and in* $L^1 \cap L^p$ *on* (X,∞), *let*

$$\tilde{J}_k\phi(x) = \frac{1}{s_k(x)} \int_x^\infty \frac{s_k(t)z'(t)}{z(t)}\phi(t)dt, \quad \textit{for all } \ x \geqslant X.$$

If $f(x)$ *is non-negative and in* $L^1 \cap L^p$ *on* (X,∞),

$$\tilde{F}_k(x) = \tilde{J}_k\tilde{J}_{k-1}\dots\tilde{J}_1 f(x), \qquad \tilde{F}_0(x) = f(x),$$

and

$$0 < \frac{1}{\alpha_k} \leqslant \operatorname*{ess\,inf}_{X<x<\infty} \left\{1 + \frac{1}{1-m}\frac{z(x)}{z'(x)}\left(\frac{w'(x)}{w(x)} - p\frac{s'(x)}{s(x)}\right)\right\},$$

then

$$\int_X^\infty \frac{z'(x)}{z(x)^m} w(x)\tilde{F}_n(x)^p dx \leqslant \left(\frac{p}{1-m}\right)^{np} \prod_{i=1}^n \alpha_i^p \int_X^\infty \frac{z'(x)}{z(x)^m} w(x) f(x)^p dx.$$

Proof. Let K_k be an essential upper bound of $s_k(x)z'(x)/z(x)$ on (X,∞). Since $1/s_k(x)$ is bounded and ϕ is integrable, $\tilde{J}_k\phi(x)$ exists everywhere in (X,∞). And since $1/s_k(x)$ is locally absolutely continuous in $[X,\infty)$, so also is $\tilde{J}_k\phi(x)$. Further

$$\int_X^\infty \tilde{J}_k\phi(x)dx \leqslant \int_X^\infty \frac{dx}{s_k(x)} \int_x^\infty K_k\phi(t)dt \leqslant K_k \int_X^\infty \frac{dx}{s_k(x)} \int_X^\infty \phi(t)dt < \infty,$$

so that $\tilde{J}_k\phi(x)$ is integrable on (X,∞), and of course non-negative. Also

$$\left(\int_X^\infty \tilde{J}_k\phi(x)^p dx\right)^{\frac{1}{p}} \leqslant \left(\int_X^\infty \frac{dx}{s_k(x)^p}\left(\int_x^\infty K_k\phi(t)dt\right)^p\right)^{\frac{1}{p}}$$

$$\leqslant K_k \left(\int_X^\infty \frac{dx}{s_k(x)^p}\left(\int_X^\infty \phi(t)dt\right)^p\right)^{\frac{1}{p}} \leqslant K_k \int_X^\infty \phi(t)dt \left(\int_X^\infty \frac{dx}{s_k(x)^p}\right)^{\frac{1}{p}} < \infty$$

using Minkowski's integral inequality. Thus $\tilde{J}_k\phi(x)$ is in $L^1 \cap L^p$ on (X,∞). By successive application of these results, $\tilde{F}_k(x)$ is in $L^1 \cap L^p$ on (X,∞) and non-negative.

By Theorem 5 with $q=0$, since $\tilde{F}_k(x) = \tilde{J}_k\tilde{F}_{k-1}(x)$,

$$\int_X^\infty \frac{z'(x)}{z(x)^m} w(x)\tilde{F}_k(x)^p dx \leqslant \left(\frac{p\alpha_k}{1-m}\right)^p \int_X^\infty \frac{z'(x)}{z(x)^m} w(x)\tilde{F}_{k-1}(x)^p dx.$$

Iteration of this inequality with $k = 1,2,\dots,n$ gives the stated result, completing the proof of Theorem 7.

THEOREM 8. *Let* p, k, m, $s_k(x)$, $w(x)$, $z(x)$ *and* X *be as in Theorem 7. For any* ϕ *in* $L^1 \cap L^p$ *on* (X, ∞) *and non-negative, let*

$$\bar{J}_k\phi(x) = \frac{1}{s_k(x)} \int_x^\infty s_k(t)z'(t)\phi(t)dt \qquad \textit{for all } x \geqslant X.$$

If $f(x)$ *is in* $L^1 \cap L^p$ *on* (X, ∞) *and non-negative,*

$$\bar{F}_k(x) = \bar{J}_k\bar{J}_{k-1}\dots\bar{J}_1 f(x), \qquad \bar{F}_0(x) = f(x)$$

and

$$0 < \frac{1}{\alpha_k} \leqslant \underset{X<x<\infty}{\textit{ess inf}} \left\{1 + \frac{1}{1-m+(n-k)p}\frac{z(x)}{z'(x)}\left(\frac{w'(x)}{w(x)} - p\frac{s_k'(x)}{s_k(x)}\right)\right\},$$

then

$$\int_X^\infty \frac{z'(x)}{z(x)^m} w(x)\bar{F}_n(x)^p dx \leqslant \prod_{k=1}^n \left(\frac{p\alpha_k}{1-m+(n-k)p}\right)^p \int_X^\infty \frac{z'(x)}{z(x)^{m-np}} w(x)f(x)^p dx.$$

Proof. Let K_k be an essential upper bound of $s_k(t)z'(t)$ on (X, ∞). As in the proof of Theorem 7, but with the changed meaning of K_k, $\bar{J}_k\phi(x)$ is in $L^1 \cap L^p$ on (X, ∞), and non-negative. By successive applications of this, $\bar{F}_k(x)$ is in $L^1 \cap L^p$ on (X, ∞), and non-negative. Now

$$\bar{F}_k(x) = \bar{J}_k\bar{F}_{k-1}(x) = \frac{1}{s_k(x)} \int_x^\infty \frac{s_k(t)z'(t)}{z(t)} z(t)\bar{F}_{k-1}(t)dt = \tilde{J}_k(z(x)\bar{F}_{k-1}(x)),$$

and the requirements of Theorem 5, with $q = 0$ and m replaced by $m - (n-k)p$, $F(x)$ replaced by $\bar{F}_k(x)$ and $f(x)$ by $z(x)\bar{F}_{k-1}(x)$, are satisfied. Theorem 5 thus gives

$$\begin{aligned}
&\int_X^\infty \frac{z'(x)}{z(x)^{m-(n-k)p}} w(x)\bar{F}_k(x)^p\, dx \\
&\qquad \leqslant \left(\frac{p\alpha_k}{1-m+(n-k)p}\right)^p \int_X^\infty \frac{z'(x)}{z(x)^{m-(n-k)p}} w(x)\left(z(x)\bar{F}_{k-1}(x)\right)^p\, dx \\
&\qquad = \left(\frac{p\alpha_k}{1-m+(n-k)p}\right)^p \int_X^\infty \frac{z'(x)}{z(x)^{m-(n-k+1)p}} w(x)\bar{F}_{k-1}(x)^p\, dx.
\end{aligned}$$

Iteration of this inequality for $k = 1, 2, \dots, n$ gives the stated result, completing the proof of Theorem 8.

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(Received September 7, 2000)

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