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LOWER AND UPPER SOLUTIONS FOR SINGULAR DERIVATIVE DEPENDENT DIRICHLET PROBLEM

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Abstract. In this work, we consider the Dirichlet problem

$$u'' + f(t,u,u') = 0,$$
$$u(a) = 0, \quad u(b) = 0,$$

with f singular at $t = a$, $t = b$ and for $u = 0$ and extend previous results concerning the case f independent of u'. To this aim we extend the lower and upper solution method in order to work with solutions in $W^{1,1}(a,b) \cap W^{2,1}_{loc}(a,b)$ as well as with lower and upper solutions having unbounded derivatives.

1. Introduction

There has been a large literature on singular boundary value problems of the type

$$\begin{aligned} u'' + f(t,u) = 0, \\ u(a) = 0, \; u(b) = 0, \end{aligned} \tag{1.1}$$

where f can be singular at $t = a$, $t = b$ and also for $u = 0$. This kind of singularities appears for example in Emden-Fowler equations

$$\begin{aligned} u'' + \tfrac{f(t)}{u^\sigma} = h(t), \\ u(a) = 0, \; u(b) = 0, \end{aligned} \tag{1.2}$$

where $\sigma > 0$. Such equations have been used in several problems of applied mathematics [3, 7, 15, 22].

As it was already noticed by A. Rosenblatt in 1933 [28], these problems can be studied for more singular nonlinearities than L^p-Carathéodory functions. In 1953, G. Prodi [27] used lower and upper solutions for such singular problems. For more recent results, we can quote [5, 11, 32, 35] where the argument relies on the fact that $f(t,u)$ being positive, the solutions are concave, [15] where the authors study (1.2) in case $f(t) > 0$ and $h(t)$ can change sign and [13, 14, 20] for the general case (1.1). In [14], the authors consider the case where the "slope $\frac{f(t,u)}{u}$ is larger than the first eigenvalue

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λ_1 when u goes to zero and smaller than λ_1 when u goes to infinity". The singularity in $t = a$ and $t = b$ is of the same type as already considered in [27, 28, 32] i.e:
for any compact $S \subset \mathbb{R}_0^+$, there exists $h_S \in \mathscr{A}$ such that, for a.e. $t \in [a,b]$ and all $u \in S$, f satisfies

$$|f(t,u)| \leqslant h_S(t),$$

where $\mathscr{A} := \{h \in L^1_{loc}(a,b) \mid \int_a^b (t-a)(b-t)h(t)\,dt < \infty\}$. In that case the solutions are in $W^{2,\mathscr{A}} := \{u \in \mathscr{C}([a,b]) \mid u'' \in \mathscr{A}\}$.

In this work, we generalize [14] to the derivative dependent problem

$$\begin{aligned} &u'' + f(t,u,u') = 0,\\ &u(a) = 0, \; u(b) = 0. \end{aligned} \tag{1.3}$$

Our results are based on the lower and upper solutions method.

Although some of the ideas can be traced back to E. Picard [26], the method of lower and upper solutions was grounded by G. Scorza Dragoni [30]. This paper considers upper and lower solutions which are $\mathscr{C}^2$ and, in 1938, the same author extended his method to the L^1-Carathéodory case [31]. Upper and lower solutions with corners were considered by M. Nagumo in 1954 [25] (we can find some trace of this idea already in [26]). Since then a multitude of variants were introduced. Concerning a priori bounds on the derivative of solutions, the first result goes back to S. Bernstein [4]. In 1937, M. Nagumo [23] generalized these ideas introducing the so-called Nagumo condition which is both simple and very general. Later, H. Epheser [9] and I.T. Kiguradze [16] extended the Nagumo condition so as to deal with $W^{2,1}$-solutions (see also R.D. Moyer [21]). The idea to replace the Nagumo condition by the existence of bounding function, i.e. curves along which the vector field points one way, is due to M. Nagumo [24] (see also [1, 2]), while the idea of diagonals goes back to F. Sadyrbaev [29] (see also [10] for generalization and combination of these two notions in the continuous case). For a first use of bounding functions in the Carathéodory case, we refer to [12].

Our results concerning (1.3) can be described in the following way. First we consider the lower and upper solution method in case f is L^p-Carathéodory. In that case, we find in the literature several proofs of the existence of a solution assuming the lower and upper solutions have bounded derivative and using quite ingenious but not direct proof (see for example [9, 17, 34]). Here we remove this boundedness condition and prove the result using a simple modified problem as it is common in the lower and upper solution method. This relies on ideas from [6] and [19]. In the third section, we give our main results on (1.3) with f singular in $t = a$ and $t = b$. The first problem we have to solve is that, a Nagumo condition, as used in Section 2 (condition (b) of Theorem 2.2), forces f to be L^p-Carathéodory and gives an a priori bound on $\|u'\|_\infty$ on the solutions of (1.3). This is not natural for singular problems. For example, the problem

$$\begin{aligned} &t(1-t)u'' = 1,\\ &u(0) = 0, \; u(1) = 0, \end{aligned}$$

has $u(t) = t\ln t + (1-t)\ln(1-t)$ as solution which is in $W^{2,\mathscr{A}}(0,1) \subset W^{1,1}(0,1) \cap W^{2,1}_{loc}(0,1)$ but not in $\mathscr{C}^1([0,1])$. Hence, a more natural idea is to look for an a priori bound on u' in $L^1(a,b)$. We achieve this aim using two different approach: a Nagumo type condition and the use of bounding functions. A third approach can be found in [19, Theorem 3.2_1]. In the last section, we give an application in the spirit of [14] where f is singular in $t = a$, $t = b$ and $u = 0$. This result has to be compared with [19, Theorem 4.1_1].

2. Regular problem

In this section, we consider the problem

$$\begin{aligned} u'' &= f(t,u,u'), \\ u(a) = A_0,&\ \ u(b) = B_0, \end{aligned} \tag{2.1}$$

where $f : D \subset [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a L^p*-Carathéodory function* i.e.

(a) for a.e. $t \in [a,b]$, the function $f(t,\cdot,\cdot)$ with domain $\{(u,v) \in \mathbb{R}^2 \mid (t,u,v) \in D\}$ is continuous;
(b) for all $(u,v) \in \mathbb{R}^2$, the function $f(\cdot,u,v)$ with domain $\{t \in [a,b] \mid (t,u,v) \in D\}$ is measurable;
(c) for all $r > 0$, there exists $h \in L^p(a,b)$ such that for all $(t,u,v) \in D$ with $|u| + |v| \leqslant r$, $|f(t,u,v)| \leqslant h(t)$.

A function $f : D \subset [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ which satisfies condition (a) and (b) is called a *Carathéodory function*.

In the future, we denote by D^+g, D^-g, D_+g and D_-g the four Dini derivatives of a given real function g: '$+$' or '$-$' means limit from the right or from the left, whereas the upper or lower position of the symbol means upper or lower limit.

DEFINITION 2.1. A function $\alpha \in \mathscr{C}([a,b])$ is a $W^{2,1}$*-lower solution* of (2.1) if

(a) for any $t_0 \in]a,b[$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$,
or there exists an open interval $I_0 \subset]a,b[$ such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and, for a.e. $t \in I_0$,
$$\alpha''(t) \geqslant f(t,\alpha(t),\alpha'(t));$$
(b) $\alpha(a) \leqslant A_0$, $\alpha(b) \leqslant B_0$.

A function $\beta \in \mathscr{C}([a,b])$ is a $W^{2,1}$*-upper solution* of (2.1) if

(a) for any $t_0 \in]a,b[$, either $D^-\beta(t_0) > D_+\beta(t_0)$,
or there exists an open interval $I_0 \subset]a,b[$ such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and, for a.e. $t \in I_0$,
$$\beta''(t) \leqslant f(t,\beta(t),\beta'(t));$$
(b) $\beta(a) \geqslant A_0$, $\beta(b) \geqslant B_0$.

If the nonlinearity depends on u', the existence of a well-ordered pair of lower and upper solutions alone does not guarantee the existence of a solution (see for example [25]). A Nagumo condition allows us to deduce an a priori bound on the derivative from an a priori bound on the function. This is the aim of the next proposition where we consider a "Carathéodory version" of the Nagumo condition.

PROPOSITION 2.1. *Let* α, $\beta \in \mathscr{C}([a,b])$ *be such that* $\alpha \leqslant \beta$, $p \in [1,\infty]$ *and define* $q \in [1,\infty]$ *from* $\frac{1}{q}+\frac{1}{p}=1$. *Assume there exist* $r \geqslant \max\{\beta(b)-\alpha(a), \beta(a)-\alpha(b)\}/(b-a)$, $\varphi \in \mathscr{C}(\mathbb{R}^+,\mathbb{R}_0^+)$, $\psi \in L^p(a,b)$ *and* $R > r$ *such that*

$$\int_r^R \frac{s^{1/q}}{\varphi(s)}\,ds > \|\psi\|_{L^p}(\max_t \beta(t) - \min_t \alpha(t))^{1/q}. \tag{2.2}$$

Define

$$E := \{(t,u,v) \in [a,b]\times\mathbb{R}^2 \mid \alpha(t) \leqslant u \leqslant \beta(t)\}. \tag{2.3}$$

Then, for every L^p*-Carathéodory function* $f : E \to \mathbb{R}$ *such that for a.e.* $t \in [a,b]$ *and all* $(u,v) \in \mathbb{R}^2$ *with* $(t,u,v) \in E$ *and* $r \leqslant |v| \leqslant R$

$$|f(t,u,v)| \leqslant \psi(t)\varphi(|v|),$$

and for every solution u *of*

$$u'' = f(t,u,u'), \tag{2.4}$$

which is such that $\alpha \leqslant u \leqslant \beta$, *we have*

$$\|u'\|_\infty < R.$$

Proof. Let u be a solution of (2.4) such that $\alpha \leqslant u \leqslant \beta$. Observe first that, by definition of r, there exists $\tau \in [a,b]$ with $|u'(\tau)| \leqslant r$.

Now consider an interval $I = [t_0,t_1]$ or $[t_1,t_0]$ such that $u'(t) \geqslant r$ on I, $u'(t_0) = r$, $u'(t_1) = R$. Then we have

$$\begin{aligned}\int_r^R \frac{r^{1/q}}{\varphi(r)}\,dr &\leqslant \int_{t_0}^{t_1} \frac{u'^{1/q}(s)u''(s)}{\varphi(u'(s))}\,ds = \int_{t_0}^{t_1} \frac{u'^{1/q}(s)f(s,u(s),u'(s))}{\varphi(u'(s))}\,ds\\ &\leqslant \left|\int_{t_0}^{t_1} \psi(s)u'^{1/q}(s)\,ds\right| \leqslant \|\psi\|_{L^p}\left|\int_{t_0}^{t_1} u'(s)\,ds\right|^{1/q}\\ &\leqslant \|\psi\|_{L^p}(\max_t \beta(t) - \min_t \alpha(t))^{1/q}.\end{aligned}$$

This contradicts (2.2) and we deduce that $u'(t) < R$. In the same way we prove that $u'(t) > -R$. □

After this preliminary result, we can give our main result concerning regular problems.

THEOREM 2.2. *Let* A_0, $B_0 \in \mathbb{R}$. *Assume* α *and* $\beta \in \mathscr{C}([a,b])$ *are* $W^{2,1}$*-lower and upper solutions of problem* (2.1) *such that* $\alpha \leqslant \beta$. *Define* $A \subset [a,b]$ *(resp.* $B \subset [a,b]$*) to be the set of points where* α *(resp.* β*) is derivable.*

Let E *be defined by* (2.3), $p \in [1,\infty]$ *and* $f : E \to \mathbb{R}$ *be a* L^p*-Carathéodory function. Suppose there exists* $N \in L^1(a,b)$, $N > 0$ *such that, for a.e.* $t \in A$ *(resp. for a.e.* $t \in B$*),*

$$f(t,\alpha(t),\alpha'(t)) \geqslant -N(t), \quad (\text{resp. } f(t,\beta(t),\beta'(t)) \leqslant N(t)). \tag{2.5}$$

Assume moreover there exist $r \geqslant \max\{\frac{\beta(a)-\alpha(b)}{b-a}, \frac{\beta(b)-\alpha(a)}{b-a}\}$, $\varphi \in \mathscr{C}(\mathbb{R}^+, \mathbb{R}_0^+)$ *and* $\psi \in L^p(a,b)$ *satisfying*

(a) $\displaystyle\int_r^\infty \frac{s^{1/q}}{\varphi(s)}\,ds > \|\psi\|_{L^p}(\max_t \beta(t) - \min_t \alpha(t))^{1/q}$, *where* $q = \frac{p}{p-1} \in [1,\infty]$*;*

(b) for a.e. $t \in [a,b]$ *and all* $(u,v) \in \mathbb{R}^2$ *such that* $(t,u,v) \in E$ *and* $|v| \geqslant r$,

$$|f(t,u,v)| \leqslant \psi(t)\varphi(|v|).$$

Then, the problem (2.1) *has at least one solution* $u \in W^{2,p}(a,b)$ *such that for all* $t \in [a,b]$

$$\alpha(t) \leqslant u(t) \leqslant \beta(t).$$

Proof. The proof proceeds in several steps.

Step 1 – The modified problem. Let R be large enough so that

$$\int_r^R \frac{s^{1/q}}{\varphi(s)}\,ds > \|\psi\|_{L^p}(\max_t \beta(t) - \min_t \alpha(t))^{1/q}.$$

Increasing N if necessary, we can assume $N(t) \geqslant \psi(t)\max_{[0,R]}\varphi(v)$ on $[a,b]$. Define then

$$\begin{aligned}\bar{f}(t,u,v) &= \max\{\min\{f(t,\gamma(t,u),v), N(t)\}, -N(t)\},\\ \omega_1(t,\delta) &= \chi_A(t)\max_{|v|\leqslant\delta}|\bar{f}(t,\alpha(t),\alpha'(t)+v) - \bar{f}(t,\alpha(t),\alpha'(t))|,\\ \omega_2(t,\delta) &= \chi_B(t)\max_{|v|\leqslant\delta}|\bar{f}(t,\beta(t),\beta'(t)+v) - \bar{f}(t,\beta(t),\beta'(t))|,\end{aligned}$$

where χ_A and χ_B are the characteristic functions of the sets A and B and

$$\gamma(t,u) = \max\{\min\{u,\beta(t)\},\alpha(t)\}. \tag{2.6}$$

It is clear that ω_i are L^1-Carathéodory functions, nondecreasing in δ, such that $\omega_i(t,0) = 0$ and $|\omega_i(t,\delta)| \leqslant 2N(t)$.

We consider now the modified problem

$$\begin{aligned}u'' &= \bar{f}(t,u,u') - \omega(t,u),\\ u(a) &= A_0,\ u(b) = B_0,\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}\omega(t,u) &= -\omega_2(t,u-\beta(t)), && \text{if } u > \beta(t),\\ &= 0, && \text{if } \alpha(t) \leqslant u \leqslant \beta(t),\\ &= \omega_1(t,\alpha(t)-u), && \text{if } u < \alpha(t).\end{aligned}$$

Step 2 – Existence of a solution u *of* (2.7). Let us write (2.7) as an integral equation

$$u(t) = A_0 + \frac{B_0 - A_0}{b-a}(t-a) + \int_a^b G(t,s)[\bar{f}(s,u(s),u'(s)) - \omega(s,u(s))]ds,$$

where $G(t,s)$ is the Green function corresponding to the problem

$$\begin{aligned}u'' &= f(t),\\ u(a) &= 0,\ u(b) = 0.\end{aligned}$$

The operator $T : \mathscr{C}^1([a,b]) \to \mathscr{C}^1([a,b])$ defined by

$$(Tu)(t) = A_0 + \frac{B_0 - A_0}{b-a}(t-a) + \int_a^b G(t,s)[\bar{f}(s,u(s),u'(s)) - \omega(s,u(s))]ds,$$

is completely continuous and bounded. By Schauder's Theorem, T has a fixed point which is a solution of (2.7).

Step 3 – The solution u of (2.7) *satisfies* $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ *on* $[a,b]$. Assume $u-\alpha$ has a negative minimum at some point $t^* \in]a,b[$. Let $t_0 = \sup\{t > t^* \mid u(t) - \alpha(t) = u(t^*) - \alpha(t^*)\}$. Then $u(t_0) - \alpha(t_0) = \min_t(u(t) - \alpha(t)) < 0$, $u'(t_0) - D_-\alpha(t_0) \leqslant u'(t_0) - D^+\alpha(t_0)$ and, by definition of a $W^{2,1}$-lower solution, there exist an open interval I_0 and $t_1 \in I_0$, $t_1 > t_0$ such that $\alpha \in W^{2,1}(I_0)$, $t_0 \in I_0$, $u'(t_1) - \alpha'(t_1) > 0$ and for a.e. $t \in I_0$

$$\alpha''(t) \geqslant f(t, \alpha(t), \alpha'(t)).$$

Further $u'(t_0) - \alpha'(t_0) = 0$ and for t near enough t_0

$$|u'(t) - \alpha'(t)| \leqslant \alpha(t) - u(t).$$

As ω_1 is nondecreasing and $\bar{f}(t,\alpha(t),\alpha'(t)) \leqslant f(t,\alpha(t),\alpha'(t))$, we come to the contradiction

$$\begin{aligned} 0 &< u'(t_1) - \alpha'(t_1) = \int_{t_0}^{t_1} (u''(s) - \alpha''(s))ds \\ &\leqslant \int_{t_0}^{t_1} [\bar{f}(s,\alpha(s),u'(s)) - \bar{f}(s,\alpha(s),\alpha'(s)) - \omega_1(s,\alpha(s) - u(s))]ds \leqslant 0. \end{aligned}$$

Step 4 – The solution u of (2.7) *is such that* $\|u'\|_\infty < R$. Observe that condition (b) is satisfied with $f(t,u,v)$ replaced by $\bar{f}(t,u,v)$. Hence, we conclude by Proposition 2.1.

Conclusion. It follows from Step 3 and 4 that the solution u of (2.7) obtained in Step 2 solves (2.1). □

REMARK 2.1. Condition (2.5) is satisfied if α, $\beta \in W^{1,\infty}(a,b)$ or if f does not depend on v.

3. Singular Problems

Consider now the homogeneous Dirichlet problem

$$\begin{gathered} u'' = f(t,u,u'), \\ u(a) = 0, \; u(b) = 0. \end{gathered} \tag{3.1}$$

We first extend Proposition 2.1 in such a way to obtain a L^1-a priori bound on u' instead of a L^∞-bound.

To state such a result we shall need the following concept. A function $f : D \subset [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is said to be a L^1_{loc}*-Carathéodory function* if it is a Carathéodory function and for all $r > 0$, there exists $h \in L^1_{loc}(a,b)$ such that for a.e. $t \in [a,b]$ and all $(u,v) \in \mathbb{R}^2$, with $|u| + |v| \leqslant r$ and $(t,u,v) \in D$, we have

$$|f(t,u,v)| \leqslant h(t).$$

PROPOSITION 3.1. *Let α and $\beta \in \mathscr{C}([a,b])$ be such that $\alpha \leqslant \beta$ and define E from (2.3). Assume there exist $r > 0$, $\psi \in L^1_{loc}(a,b)$ and a nondecreasing function $\varphi \in \mathscr{C}(\mathbb{R}^+, \mathbb{R}^+_0)$ such that*

(a) $\displaystyle\int_r^{\infty} \frac{ds}{\varphi(s)} = \infty$;

(b) $\displaystyle\Phi^{-1}(2|\int_{\frac{a+b}{2}}^t \psi(s)\,ds|) \in L^1(a,b)$, *where* $\displaystyle\Phi(u) = \int_r^u \frac{ds}{\varphi(s)}$.

Then there exists $h \in L^1(a,b) \cap \mathscr{C}(]a,b[)$ such that, for every $a \leqslant a_1 \leqslant \frac{2a+b}{3} < \frac{a+2b}{3} \leqslant b_1 \leqslant b$, every L^1_{loc}-Carathéodory function $f : E \to \mathbb{R}$ which satisfies, for a.e. $t \in [a_1,b_1]$ and all $(u,v) \in \mathbb{R}^2$ with $(t,u,v) \in E$, $|v| \geqslant r$,

$$|f(t,u,v)| \leqslant \psi(t)\varphi(|v|),$$

and every solution u of (2.4) on $[a_1,b_1]$ such that $\alpha \leqslant u \leqslant \beta$, we have

$$|u'(t)| \leqslant h(t).$$

REMARK 3.1. Let us comment assumption (b):

(i) If $\psi \in L^1(a,b)$, condition (b) is implied by condition (a);

(ii) In case $\varphi(v) \equiv 1$, condition (b) becomes $\psi \in \mathscr{A}$;

(iii) Condition (b) is implied by $\Phi^{-1}(2(b-a)\psi(t)) \in \mathscr{A}$. This can be seen using Jensen inequality (see for example [8, Theorem II-2.2]). For $t \geqslant \frac{a+b}{2}$, we obtain

$$\begin{aligned}\Phi^{-1}(2\int_{\frac{a+b}{2}}^t \psi(s)\,ds) &= \Phi^{-1}(\frac{1}{b-a}\int_a^b 2(b-a)\psi(s)\chi_{[\frac{a+b}{2},t]}(s)\,ds)\\ &\leqslant \frac{1}{b-a}\int_{\frac{a+b}{2}}^t \Phi^{-1}(2(b-a)\psi(s))\,ds.\end{aligned}$$

If $\Phi^{-1}(2(b-a)\psi(t)) \in \mathscr{A}$, we have $\int_{\frac{a+b}{2}}^t \Phi^{-1}(2(b-a)\psi(s))\,ds \in L^1(\frac{a+b}{2},b)$ and it follows that $\Phi^{-1}(2\int_{\frac{a+b}{2}}^t \psi(s)\,ds) \in L^1(\frac{a+b}{2},b)$.

In the same way, we can write $\Phi^{-1}(2\int_t^{\frac{a+b}{2}} \psi(s)\,ds) \in L^1(a,\frac{a+b}{2})$.

Proof. Step 1 – Existence of a function $h \in \mathscr{C}(]a,b[)$ that satisfies the assertions of the Proposition. Let c, d be such that $\frac{2a+b}{3} \leqslant c < \frac{a+b}{2} < d \leqslant \frac{a+2b}{3}$ and $M > \max\{r, \frac{\max_t \beta - \min_t \alpha}{d-c}\}$. Define h_1 to be the solution of

$$h_1' = \psi(t)\varphi(h_1), \quad t \in [c,b[, \qquad h_1(c) = M,$$

h_2 the solution of

$$h_2' = -\psi(t)\varphi(h_2), \quad t \in]a,d], \qquad h_2(d) = M,$$

and

$$\begin{aligned} h(t) &= h_2(t), && \text{on }]a,c], \\ &= \max\{h_1(t), h_2(t)\}, && \text{on }]c,d[, \\ &= h_1(t), && \text{on } [d,b[. \end{aligned}$$

Let $a \leqslant a_1 \leqslant \frac{2a+b}{3} < \frac{a+2b}{3} \leqslant b_1 \leqslant b$, $f : E \to \mathbb{R}$ such that, for a.e. $t \in [a_1, b_1]$ and all $(u,v) \in \mathbb{R}^2$ with $(t,u,v) \in E$, $|v| \geqslant r$, $|f(t,u,v)| \leqslant \psi(t)\varphi(|v|)$, and u be a solution of (2.4) on $[a_1,b_1]$ such that $\alpha \leqslant u \leqslant \beta$. Observe that there exists $\tau \in [c,d]$ with $|u'(\tau)| \leqslant M$. Consider an interval $I = [t_0, t_1]$ such that $u'(t) \geqslant r$ on I, $u'(t_0) = M$ and $t_0 \geqslant c$. We have, for every $t \in [t_0, t_1]$,

$$\begin{aligned} \int_M^{u'(t)} \frac{d\sigma}{\varphi(\sigma)} &= \int_{t_0}^t \frac{u''(s)}{\varphi(u'(s))} ds \leqslant \int_{t_0}^t \psi(s)\, ds \\ &\leqslant \int_c^t \psi(s)\, ds = \int_c^t \frac{h_1'(s)}{\varphi(h_1(s))} ds = \int_M^{h_1(t)} \frac{d\sigma}{\varphi(\sigma)} \end{aligned}$$

and $u'(t) \leqslant h_1(t)$ on I and hence on $[\tau, b_1]$.

In the same way, we prove that $u' \leqslant h_2$ on $[a_1, \tau]$ and hence, $u'(t) \leqslant h(t)$ on $[a_1, b_1]$. The proof that, for any $t \in [a_1, b_1]$, $u'(t) \geqslant -h(t)$ is similar.

Step 2 – $h \in L^1(a,b)$. We compute

$$\Phi(h_1(t)) - \Phi(M) = \int_M^{h_1(t)} \frac{dr}{\varphi(r)} = \int_c^t \psi(s)\, ds.$$

Hence we write

$$\begin{aligned} h_1(t) &= \Phi^{-1}\left(\Phi(M) + \int_c^t \psi(s)\, ds\right) \\ &\leqslant \Phi^{-1}\left(\Phi(M) + \int_c^d \psi(s)\, ds + \int_{\frac{a+b}{2}}^t \psi(s)\, ds\right) \end{aligned}$$

and as Φ^{-1} is convex

$$h_1(t) \leqslant \frac{1}{2}\left\{\Phi^{-1}\left(2[\Phi(M) + \int_c^d \psi(s)\, ds]\right) + \Phi^{-1}\left(2\int_{\frac{a+b}{2}}^t \psi(s)\, ds\right)\right\},$$

from which we deduce $h_1 \in L^1(c,b)$. In the same way, we have

$$h_2(t) = \Phi^{-1}(\Phi(M) + \int_t^d \psi(s)\, ds)$$

and deduce $h_2 \in L^1(a,d)$. $\square$

REMARK 3.2. The condition φ nondecreasing is not essential. If it is not satisfied, we have to replace condition (b) by

$$\Phi^{-1}(\Phi(M) + \int_c^t \psi(s)\, ds) \in L^1(c,b), \quad \Phi^{-1}(\Phi(M) + \int_t^d \psi(s)\, ds) \in L^1(a,d)$$

where M, c and d are defined in the proof of Proposition 3.1.

THEOREM 3.2. *Let* α, $\beta \in \mathscr{C}([a,b])$ *be* $W^{2,1}$ *-lower and upper solutions of the problem* (3.1) *such that* $\alpha \leqslant \beta$. *Define* $A \subset [a,b]$ *(resp.* $B \subset [a,b]$ *) to be the set of points where* α *(resp.* β *) is derivable.*

Let E *be defined by* (2.3) *and* $f : E \to \mathbb{R}$ *be a* L^1_{loc} *-Carathéodory function. Suppose there exists* $N \in L^1_{loc}(a,b)$, $N > 0$ *such that, for a.e.* $t \in A$ *(resp. for a.e.* $t \in B$ *),*

$$f(t,\alpha(t),\alpha'(t)) \geqslant -N(t), \qquad (resp.\ f(t,\beta(t),\beta'(t)) \leqslant N(t)).$$

Assume moreover there exist $r > 0$, $\psi \in L^1_{loc}(a,b)$ *and a nondecreasing function* $\varphi \in \mathscr{C}(\mathbb{R}^+, \mathbb{R}^+_0)$ *such that*

(a) $\displaystyle\int_r^\infty \frac{ds}{\varphi(s)} = \infty$;

(b) $\displaystyle\Phi^{-1}\Big(2\Big|\int_{\frac{a+b}{2}}^t \psi(s)\,ds\Big|\Big) \in L^1(a,b)$, *where* $\displaystyle\Phi(u) = \int_r^u \frac{ds}{\varphi(s)}$;

(c) for a.e. $t \in [a,b]$ *and all* (u,v) *such that* $(t,u,v) \in E$ *and* $|v| \geqslant r$,

$$|f(t,u,v)| \leqslant \psi(t)\varphi(|v|).$$

Then the problem (3.1) *has at least one solution* u *such that for all* $t \in [a,b]$

$$\alpha(t) \leqslant u(t) \leqslant \beta(t).$$

Proof. Step 1 – The modified problem. Let $(a_n)_n$, $(b_n)_n \subset]a,b[$, $(A_n)_n$, $(B_n)_n \subset \mathbb{R}$ be such that

$$\lim_{n\to\infty} a_n = a,\ \lim_{n\to\infty} b_n = b,\ \lim_{n\to\infty} A_n = 0,\ \lim_{n\to\infty} B_n = 0,$$
$$\alpha(a_n) \leqslant A_n \leqslant \beta(a_n),\ \ \alpha(b_n) \leqslant B_n \leqslant \beta(b_n).$$

Consider the modified problem

$$\begin{gathered} u'' = f(t,u,u'),\\ u(a_n) = A_n,\ u(b_n) = B_n. \end{gathered} \tag{3.2}$$

We can assume that, for any n, $a_n \leqslant \frac{2a+b}{3} < \frac{a+2b}{3} \leqslant b_n$. Hence, by Theorem 2.2 and Proposition 3.1, for every n, problem (3.2) has a solution u_n satisfying on $[a_n,b_n]$

$$\alpha(t) \leqslant u_n(t) \leqslant \beta(t), \qquad |u_n'(t)| \leqslant h(t),$$

with h given by Proposition 3.1.

Step 2 – Existence of a solution u *of* (3.1). Using Arzela-Ascoli Theorem, we can find $(u_n^1)_n$, a subsequence of $(u_n)_n$, that converges in $\mathscr{C}^1([a_1,b_1])$. Proceeding by induction, for any $k \in \mathbb{N}$, we build $(u_n^k)_n$, a subsequence of $(u_n^{k-1})_n$, that converges in $\mathscr{C}^1([a_k,b_k])$. It follows that the diagonal sequence $(u_n^n)_n$ converges pointwise to some function u and that, for any compact $K \subset]a,b[$, the convergence takes place in $\mathscr{C}^1(K)$. Hence, u satisfies on $]a,b[$

$$u'' = f(t,u,u')$$

and

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \qquad |u'(t)| \leqslant h(t).$$

Let us prove that $\lim_{t\to a} u(t) = 0$. Fix $\epsilon > 0$ and choose $\delta > 0$ so that $\int_a^{a+\delta} h(s)\,ds \leqslant \epsilon/3$. Let us fix $t \in [a, a+\delta]$ and pick n large enough so that $t \in [a_n, b_n]$, $|u_n(t) - u(t)| \leqslant \epsilon/3$ and $|A_n| \leqslant \epsilon/3$. We compute then

$$\begin{aligned} |u(t)| &\leqslant |u(t) - u_n(t)| + |u_n(t) - u_n(a_n)| + |A_n| \\ &\leqslant |u(t) - u_n(t)| + \int_{a_n}^{t} h(s)\,ds + |A_n| \leqslant \epsilon. \end{aligned}$$

In the same way, we prove $\lim_{t\to b} u(t) = 0$. □

REMARK 3.3. By Remark 3.1, this result generalizes Theorem 2.2 for the homogeneous case as well as [14, Theorem 1].

As a first illustration consider the following example where Theorem 2.2 does not apply.

EXAMPLE 3.1. Consider the boundary value problem

$$\begin{gathered} u'' = \tfrac{1}{t^n}|u'|^a + u + t, \\ u(0) = 0, \; u(1) = 0, \end{gathered}$$

where $0 \leqslant a < 1$ and $0 < n < 2 - a$. Existence of a solution follows from Theorem 3.2 with $\alpha(t) = -1$, $\beta(t) = 0$, $\psi(t) = \frac{1}{t^n} + 1$, $\varphi(y) = \max\{1, y^a\}$.

Observe that we do not use the full power of condition (c) in Theorem 3.2 so that we can generalize it in the following way with the same proof.

THEOREM 3.3. *Under the assumptions of Theorem 3.2 with (c) replaced by*

(c') there exist $a \leqslant c < \frac{a+b}{2} < d \leqslant b$ *such that, for a.e.* $t \in [a,b]$ *and all* (u,v) *with* $(t,u,v) \in E$ *and* $|v| \geqslant r$*,*

$$\begin{aligned} f(t,u,v)sgn(v) &\geqslant -\psi(t)\varphi(|v|), \quad \text{if } t \in]a,d[, \\ f(t,u,v)sgn(v) &\leqslant \psi(t)\varphi(|v|), \quad \text{if } t \in]c,b[, \end{aligned}$$

the problem (3.1) *has at least one solution* u *such that, for all* $t \in [a,b]$*,*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t).$$

EXAMPLE 3.2. Consider the following example

$$\begin{gathered} u'' + |u|^{1/2} - \tfrac{1}{t^n}u'^{1/3} - \tfrac{1}{t} = 0, \\ u(0) = 0, \; u(\pi) = 0, \end{gathered}$$

where $n > 0$ is not upper bounded. We can apply Theorem 3.3 to prove the existence of a solution choosing $\alpha(t) = t \ln\frac{t}{\pi} - t$, $\beta(t) = 0$, $c = \pi/3$, $d = 2\pi/3$, $\varphi(v) = \max\{1, v^{1/3}\}$ and

$$\begin{aligned} \psi(t) &= \tfrac{1}{t} + \pi^{1/2}, &&\text{on } [0, \pi/3], \\ &= \tfrac{1}{t} + \pi^{1/2} + \tfrac{1}{t^n}, &&\text{on }]\pi/3, \pi]. \end{aligned}$$

Hence, there is a solution u such that for all $t \in [0,\pi]$, $\quad t \ln\frac{t}{\pi} - t \leqslant u(t) \leqslant 0$.

We can generalize the results of this section using the idea of bounding functions and diagonals. First we consider the analogue of Proposition 3.1.

PROPOSITION 3.4. *Let A_0, $B_0 \in \mathbb{R}$, α and $\beta \in \mathscr{C}([a,b])$ be such that $\alpha \leqslant \beta$ and define E from* (2.3). *Assume there exist μ_1, $\mu_2 \in \mathscr{C}([a,b])$, g_1, g_2, h_1, $h_2 \in W^{1,1}_{loc}(a,b)$ such that*

(a) $\mu_2(a) \leqslant A_0 \leqslant \mu_1(a)$, $\mu_1(b) \leqslant B_0 \leqslant \mu_2(b)$,
$D^+\mu_1 \geqslant g_2$, $D^-\mu_1 \geqslant g_1$, $D_-\mu_2 \leqslant h_1$, $D_+\mu_2 \leqslant h_2$ *on* $[a,b]$*;*
Then for any L^1_{loc}-Carathéodory function $f : E \to \mathbb{R}$ which satisfies

(b) for all $(t,u,v) \in E$

$f(t,u,v) \leqslant g_1'(t)$, *if* $u < \mu_1(t)$ *and* $g_1(t) - 1 < v < g_1(t)$,
$f(t,u,v) \geqslant g_2'(t)$, *if* $u > \mu_1(t)$ *and* $g_2(t) - 1 < v < g_2(t)$,
$f(t,u,v) \leqslant h_1'(t)$, *if* $u < \mu_2(t)$ *and* $h_1(t) + 1 > v > h_1(t)$,
$f(t,u,v) \geqslant h_2'(t)$, *if* $u > \mu_2(t)$ *and* $h_2(t) + 1 > v > h_2(t)$.

and every solution u of (2.1) *such that $\alpha \leqslant u \leqslant \beta$, we have*

$$\min\{g_1(t), g_2(t)\} \leqslant u'(t) \leqslant \max\{h_1(t), h_2(t)\}.$$

Proof. Let u be a solution of (2.1) such that $\alpha \leqslant u \leqslant \beta$.

Claim 1 – The function u is such that

$$\forall t \in [a,b], \text{ either } u(t) \leqslant \mu_2(t) \text{ or } u'(t) \leqslant h_2(t),$$

Assume on the contrary that for some $t_1 \in [a,b[$, $u(t_1) > \mu_2(t_1)$ and $u'(t_1) > h_2(t_1)$. Observe that since $u(b) = B_0$, we can find $t_2 \in]t_1,b]$ such that $\forall t \in [t_1,t_2[$, $u(t) > \mu_2(t)$, $u'(t) > h_2(t)$ and either $u(t_2) = \mu_2(t_2)$ or $u'(t_2) = h_2(t_2)$. In the first case, we obtain a contradiction with the fact that $u - \mu_2$ is nondecreasing as, for every $t \in]t_1,t_2[$,

$$D^+(u-\mu_2)(t) = u'(t) - D_+\mu_2(t) > h_2(t) - D_+\mu_2(t) \geqslant 0.$$

In the second case, changing t_1 if necessary, we can assume that $\forall t \in [t_1,t_2[$, $u'(t) \leqslant h_2(t) + 1$ and we obtain again a contradiction with the fact that $u' - h_2$ is nondecreasing as, for every $t \in]t_1,t_2[$,

$$u''(t) - h_2'(t) = f(t,u(t),u'(t)) - h_2'(t) \geqslant 0.$$

As a conclusion, for every $t \in [a,b]$, either $u(t) \leqslant \mu_2(t)$ or $u'(t) \leqslant h_2(t)$.

Claim 2 – The function u is such that

$$\forall t \in [a,b[, \text{ either } u(t) < \mu_2(t) \text{ or } u'(t) \leqslant h_2(t).$$

Let us suppose that for some $t_0 \in [a,b[$, $u(t_0) \geqslant \mu_2(t_0)$ and $u'(t_0) > h_2(t_0)$. Hence, for $t_1 \in]t_0,b]$ near enough t_0, $u(t_1) > \mu_2(t_1)$ and $u'(t_1) > h_2(t_1)$. A contradiction follows from Claim 1.

Claim 3 – The function u is such that

$$\forall t \in [a,b], \text{ either } u(t) \geqslant \mu_2(t) \text{ or } u'(t) \leqslant h_1(t),$$

and

$$\forall t \in]a,b], \text{ either } u(t) > \mu_2(t) \text{ or } u'(t) \leqslant h_1(t).$$

This claim is proved reversing the time and using the argument of Claim 1 and 2.

Conclusion – For $t \in [a,b]$, we deduce from the above claims that $u'(t) \leqslant \max\{h_1(t), h_2(t)\}$. We prove in a similar way that $u'(t) \geqslant \min\{g_1(t), g_2(t)\}$. □

THEOREM 3.5. *Let* α, $\beta \in \mathscr{C}([a,b])$ *be* $W^{2,1}$*-lower and upper solutions of the problem* (3.1) *such that* $\alpha \leqslant \beta$. *Define* $A \subset [a,b]$ *(resp.* $B \subset [a,b]$*) to be the set of points where* α *(resp.* β*) is derivable.*

Let E *be defined by* (2.3) *and* $f : E \to \mathbb{R}$ *be a* L^1_{loc}*-Carathéodory function. Suppose there exists* $N \in L^1_{loc}(a,b)$, $N > 0$ *such that, for a.e.* $t \in A$ *(resp. for a.e.* $t \in B$*),*

$$f(t,\alpha(t),\alpha'(t)) \geqslant -N(t), \qquad (\text{resp. } f(t,\beta(t),\beta'(t)) \leqslant N(t)).$$

Assume moreover there exist μ_1, $\mu_2 \in \mathscr{C}([a,b])$, g_1, g_2, h_1, $h_2 \in W^{1,1}_{loc}(a,b) \cap L^1(a,b)$ *such that*

(a) $\mu_2(a) \leqslant 0 \leqslant \mu_1(a)$, $\mu_1(b) \leqslant 0 \leqslant \mu_2(b)$,
$\max(\alpha,\mu_2) \leqslant \min(\beta,\mu_1)$ *on a neighbourhood of* a,
$\max(\alpha,\mu_1) \leqslant \min(\beta,\mu_2)$ *on a neighbourhood of* b,
$D^+\mu_1 \geqslant g_2$, $D^-\mu_1 \geqslant g_1$, $D_-\mu_2 \leqslant h_1$, $D_+\mu_2 \leqslant h_2$ *on* $[a,b]$*;*

(b) for all $(t,u,v) \in E$
$f(t,u,v) \leqslant g_1'(t)$, *if* $u < \mu_1(t)$ *and* $g_1(t) - 1 < v < g_1(t)$,
$f(t,u,v) \geqslant g_2'(t)$, *if* $u > \mu_1(t)$ *and* $g_2(t) - 1 < v < g_2(t)$,
$f(t,u,v) \leqslant h_1'(t)$, *if* $u < \mu_2(t)$ *and* $h_1(t) < v < h_1(t) + 1$,
$f(t,u,v) \geqslant h_2'(t)$, *if* $u > \mu_2(t)$ *and* $h_2(t) < v < h_2(t) + 1$.

Then the problem (3.1) *has at least one solution* u *such that, for all* $t \in [a,b]$,

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad \min\{g_1(t), g_2(t)\} \leqslant u'(t) \leqslant \max\{h_1(t), h_2(t)\}. \tag{3.3}$$

Proof. Step 1 – The modified problem. Let $(a_n)_n$, $(b_n)_n \subset]a,b[$, $(A_n)_n$, $(B_n)_n \subset \mathbb{R}$ be such that

$$\begin{gathered}\lim_{n\to\infty} a_n = a, \ \lim_{n\to\infty} b_n = b, \ \lim_{n\to\infty} A_n = 0, \ \lim_{n\to\infty} B_n = 0,\\ \max(\alpha,\mu_2)(a_n) \leqslant A_n \leqslant \min(\beta,\mu_1)(a_n),\\ \max(\alpha,\mu_1)(b_n) \leqslant B_n \leqslant \min(\beta,\mu_2)(b_n).\end{gathered}$$

Consider the modified problem

$$\begin{gathered}u'' = f(t,u,u'),\\ u(a_n) = A_n, \ u(b_n) = B_n.\end{gathered} \tag{3.4}$$

Adapting the arguments of Theorem 2.2 and using Proposition 3.4, we prove that (3.4) has a solution u_n satisfying for all $t \in [a_n, b_n]$

$$\alpha(t) \leqslant u_n(t) \leqslant \beta(t), \quad \min\{g_1(t), g_2(t)\} \leqslant u_n'(t) \leqslant \max\{h_1(t), h_2(t)\}.$$

Step 2 – Existence of a solution u *of* (3.1). We argue as in Step 2 of the proof of Theorem 3.2. □

REMARK 3.5. We can generalize the conditions of Theorem 3.5 in the spirit of [10] or [12] and improve those results but we choose to concentrate in this paper on the new ideas and not to give the maximal generality. Of course, we can state also the equivalent of Theorem 2.2 using bounding functions and diagonals as in Theorem 3.5.

Using Theorem 3.5, we can allow a stronger dependence in u' than previously as in the following example.

EXAMPLE 3.3. Consider the following modification of Example 3.2

$$\begin{gathered} u'' + |u|^{1/2} + \frac{\pi^{n-1}}{t^n} u'^4 - \frac{1}{t} = 0, \\ u(0) = 0, \ u(\pi) = 0, \end{gathered}$$

where $n > 0$. It is easy to see that $\alpha(t) = t \ln \frac{t}{\pi} - t$ and $\beta(t) = 0$ are still lower and upper solutions. Further, we can apply Theorem 3.5 with $\mu_1 = \mu_2 = 0$, $h_1 = h_2 = 1$, $g_1 = g_2 = -1$. Notice that as $\beta(t) = 0$, the application of the theorem to this example imposes the vector field points downward along the segments $u' = k$ and $u' = -k$ when $k \geqslant 1$. Hence, there is a solution u such that for all $t \in [0, \pi]$, $t \ln \frac{t}{\pi} - t \leqslant u(t) \leqslant 0$.

We can replace the condition (b) of Theorem 3.5 by asking a control on f only for $v = g_i(t)$ (resp. $v = h_i(t)$) if we control α' and β' by g_i and h_i as in the next result.

COROLLARY 3.6. *Under the assumptions of Theorem 3.5 with (b) replaced by*

(b') for a.e. $t \in [a, b]$ *and all* $u \in \mathbb{R}$ *with* $\alpha(t) \leqslant u \leqslant \beta(t)$

$$\begin{array}{ll} f(t,u,g_1(t)) \leqslant g_1'(t), & \text{if } u < \mu_1(t), \\ f(t,u,g_2(t)) \geqslant g_2'(t), & \text{if } u > \mu_1(t), \\ f(t,u,h_1(t)) \leqslant h_1'(t), & \text{if } u < \mu_2(t), \\ f(t,u,h_2(t)) \geqslant h_2'(t), & \text{if } u > \mu_2(t); \end{array}$$

(c) for a.e. $t \in A$, *(resp. for a.e.* $t \in B$*)*

$$\begin{array}{ll} \alpha'(t) > g_1(t) \text{ if } \alpha(t) < \mu_1(t), & (\text{resp. } \beta'(t) > g_1(t) \text{ if } \beta(t) < \mu_1(t)), \\ \alpha'(t) > g_2(t) \text{ if } \alpha(t) \geqslant \mu_1(t), & (\text{resp. } \beta'(t) > g_2(t) \text{ if } \beta(t) \geqslant \mu_1(t)), \\ \alpha'(t) < h_1(t) \text{ if } \alpha(t) < \mu_2(t), & (\text{resp. } \beta'(t) < h_1(t) \text{ if } \beta(t) < \mu_2(t)), \\ \alpha'(t) < h_2(t) \text{ if } \alpha(t) \geqslant \mu_2(t), & (\text{resp. } \beta'(t) < h_2(t) \text{ if } \beta(t) \geqslant \mu_2(t)), \end{array}$$

the problem (3.1) *has at least one solution* u *such that, for all* $t \in [a, b]$,

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \quad \min\{g_1(t), g_2(t)\} \leqslant u'(t) \leqslant \max\{h_1(t), h_2(t)\}.$$

Proof. We define

$$\begin{array}{rcll} \varphi(t,u) & = & g_1(t), & \text{if } u < \mu_1(t), \\ & = & g_2(t), & \text{if } u \geqslant \mu_1(t), \\ \psi(t,u) & = & h_1(t), & \text{if } u < \mu_2(t), \\ & = & h_2(t), & \text{if } u \geqslant \mu_2(t), \end{array}$$

and consider the modified problem

$$\begin{gathered} u'' = \tilde{f}(t,u,u'), \\ u(a) = 0, \ u(b) = 0, \end{gathered}$$

where

$$\begin{array}{rcll} \tilde{f}(t,u,v) & = & f(t,u,\varphi(t,u)), & \text{if } v < \varphi(t,u), \\ & = & f(t,u,v), & \text{if } \varphi(t,u) \leqslant v \leqslant \psi(t,u), \\ & = & f(t,u,\psi(t,u)), & \text{if } v > \psi(t,u). \end{array}$$

The result follows by application of Theorem 3.5. □

We can obtain Theorem 3.3 as a Corollary of Theorem 3.5. For example, we define μ_2 and h_1 in the following way. We set

$$\begin{aligned} \mu_2(t) &= \min_t \alpha(t), && \text{if } t \in [a,c[, \\ &= \tfrac{\max_t \beta - \min_t \alpha}{d-c}(t-c) + \min_t \alpha(t), && \text{if } t \in [c,d], \\ &= \max_t \beta(t), && \text{if } t \in]d,b], \end{aligned}$$

and we define h_1 on $[c,b[$ as the solution of

$$\begin{gathered} h_1'(t) = \psi(t)\varphi(h_1(t)+1), \\ h_1(c) = M \end{gathered}$$

and $h_1(t) = M$ on $[a,c[$ where $M \geqslant \max\{r, \frac{\max_t \beta - \min_t \alpha}{d-c}\}$. It is easy to verify that these functions verify the required assumptions and to define similarly the other functions.

4. Application

In this section we extend [14, Theorem 2] to the boundary value problem

$$\begin{gathered} u'' + f(t,u,u') = 0, \\ u(0) = 0, \;\; u(\pi) = 0. \end{gathered} \tag{4.1}$$

To extend the assumption "the slope $\frac{f(t,u)}{u}$ is larger than the first eigenvalue λ_1 when u goes to zero and smaller than λ_1 when u goes to infinity" used in [14], we consider the piecewise linear problem

$$\begin{gathered} u'' + B|u'| + Cu = 0, \\ u(0) = 1, \;\; u'(0) = 0, \end{gathered}$$

where B, $C > 0$. The solution of this problem is positive on $]-\frac{\Gamma(B,C)}{2}, \frac{\Gamma(B,C)}{2}[$ where

$$\begin{aligned} \Gamma(B,C) &= \frac{4}{\sqrt{B^2-4C}} \tanh^{-1}\left(\frac{\sqrt{B^2-4C}}{B}\right), && \text{if } B^2-4C > 0, \\ &= \frac{4}{\sqrt{4C-B^2}} \tan^{-1}\left(\frac{\sqrt{4C-B^2}}{B}\right), && \text{if } B^2-4C < 0, \\ &= \frac{4}{B}, && \text{if } B^2-4C = 0. \end{aligned}$$

Hence, if we use Theorem 3.2, [14, Theorem 2] extends in the following way with a quite similar proof. Other results are obtained using Theorems 3.3 or 3.5.

THEOREM 4.1. *Assume*

(i) *the function* $f :]0,\pi[\times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ *satisfies a Carathéodory condition and for each compact set* $L \subset]0,\infty[\times \mathbb{R}$ *there exists* $k_L \in \mathscr{A}$ *such that, for a.e.* $t \in]0,\pi[$ *and all* $(u,v) \in L$,

$$|f(t,u,v)| \leqslant k_L(t);$$

(ii) there exists B_1, $C_1 \geqslant 0$ *with* $\pi > \Gamma(B_1, C_1)$ *and, for any compact set* $K \subset]0, \pi[$, *there is* $\epsilon > 0$ *such that, for a.e.* $t \in K$, *all* $u \in]0, \epsilon]$ *and* $v \in \mathbb{R}$,

$$f(t, u, v) \geqslant B_1|v| + C_1 u;$$

(iii) there exists B_2, $C_2 \geqslant 0$ *with* $\pi < \Gamma(B_2, C_2)$, $M > 0$ *and* $k \in \mathscr{A}$ *such that, for a.e.* $t \in]0, \pi[$ *and all* $(u, v) \in [M, \infty[\times\mathbb{R}$,

$$f(t, u, v) \leqslant B_2|v| + C_2 u + k(t);$$

(iv) for any compact set $L \subset]0, \infty[$, *there exist* $\psi \in L^1_{loc}(0, \pi)$, $\varphi \in \mathscr{C}(\mathbb{R}^+, \mathbb{R}^+_0)$ *nondecreasing and* $r > 0$ *such that, for a.e.* $t \in]0, \pi[$, *all* $u \in L$ *and* $|v| \geqslant r$ *we have*

$$|f(t, u, v)| \leqslant \psi(t)\varphi(|v|)$$

and, for some $C > 2$,

$$\int_r^\infty \frac{ds}{\varphi(s)} = \infty,$$

$$\Phi^{-1}(C|\int_{\frac{\pi}{2}}^t \psi(s)\, ds|) \in L^1(0, \pi),$$

where $\Phi(u) = \int_r^u \frac{ds}{\varphi(s)}$.

Then the problem (4.1) *has at least one solution.*

REMARK 4.1. Assumption (ii) is equivalent to assume there exist $B_1, C_1 \geqslant 0$ and a function $a_1 \in \mathscr{C}^2_0([0, \pi], \mathbb{R}^+)$ such that:

(a) $t \in]0, \pi[$ implies $a_1(t) > 0$;

(b) $f(t, u, v) \geqslant B_1|v| + C_1 u$, for all $t \in]0, \pi[$, $0 < u \leqslant a_1(t)$, $v \in \mathbb{R}$;

(c) $a_1''(t) > 0$, for all $t \in [0, \pi/3] \cup [2\pi/3, \pi]$.

Proof of Theorem 4.1. Step 1 – Construction of lower solutions. Decreasing B_1 and C_1 if necessary, we can assume that $\Gamma(B_1, C_1) > \pi/3$. Let a_2 be the solution of

$$u'' + B_1|u'| + C_1 u = 0,$$
$$u(\pi/2) = 1, \; u'(\pi/2) = 0.$$

Observe that $a_2(\frac{\pi}{2} - \frac{\Gamma(B_1,C_1)}{2}) = a_2(\frac{\pi}{2} + \frac{\Gamma(B_1,C_1)}{2}) = 0$ and $a_2(t) > 0$ for all $t \in]\frac{\pi}{2} - \frac{\Gamma(B_1,C_1)}{2}, \frac{\pi}{2} + \frac{\Gamma(B_1,C_1)}{2}[$. Consider the function $\alpha_2(t) = A_2 a_2(t)$, where A_2 is chosen small enough so that, for a.e. $t \in]\frac{\pi}{2} - \frac{\Gamma(B_1,C_1)}{2}, \frac{\pi}{2} + \frac{\Gamma(B_1,C_1)}{2}[$, all $0 < u \leqslant \alpha_2(t)$ and $v \in \mathbb{R}$, we have

$$f(t, u, v) \geqslant B_1|v| + C_1 u.$$

Next, we choose a_1 from Remark 4.1 and let $\alpha_1(t) = A_1 a_1(t)$, where $A_1 \in]0, 1]$ is small enough so that for some points $t_1 \in]0, \frac{\pi}{3}[$, $t_2 \in]\frac{2\pi}{3}, \pi[$, one has :

$$\alpha_1(t) \geqslant \alpha_2(t), \quad \text{for all } t \in [0, t_1] \cup [t_2, \pi];$$
$$\alpha_2(t) \geqslant \alpha_1(t), \quad \text{for all } t \in [t_1, t_2].$$

Step 2 – Approximation problems. We define for each $n \in \mathbb{N}, n \geqslant 1$,

$$\eta_n(t) = \max\{\frac{\pi}{2^{n+1}}, \min(t, \pi - \frac{\pi}{2^{n+1}})\}, \ t \in]0,\pi[$$

and set

$$\tilde{f_n}(t,u,v) = \max\{f(\eta_n(t),u,v), f(t,u,v)\}.$$

We have that, for each index $n, \tilde{f_n} :]0,\pi[\times\mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is measurable in the first variable, continuous in the two other ones and

$$\tilde{f_n}(t,u,v) = f(t,u,v), \text{ for all } (t,u,v) \in K_n \times \mathbb{R}_0^+ \times \mathbb{R},$$

where

$$K_n = [\frac{\pi}{2^{n+1}}, \pi - \frac{\pi}{2^{n+1}}].$$

Hence the sequence of functions $(\tilde{f_n})_n$ converges to f uniformly on any set $K \times \mathbb{R}_0^+ \times \mathbb{R}$, where K is an arbitrary compact subset of $]0,\pi[$.

Next we define

$$f_n(t,u,v) = \min\{\tilde{f_1}(t,u,v), \cdots, \tilde{f_n}(t,u,v)\}.$$

Each of the functions f_i are defined on $]0,\pi[\times\mathbb{R}_0^+ \times \mathbb{R}$ and moreover

$$f_1(t,u,v) \geqslant f_2(t,u,v) \geqslant \cdots \geqslant f_n(t,u,v) \geqslant f_{n+1}(t,u,v) \geqslant \cdots \geqslant f(t,u,v).$$

The sequence $(f_n)_n$ converges to f uniformly on compact subsets of $]0,\pi[\times \mathbb{R}_0^+ \times \mathbb{R}$ since

$$f_n(t,u,v) = f(t,u,v), \text{ for all } t \in K_n, \ u \in \mathbb{R}_0^+, \ v \in \mathbb{R}.$$

Define now a decreasing sequence $(\epsilon_n)_n \subset \mathbb{R}_0^+$ such that

$$\lim_{n\to\infty} \epsilon_n = 0,$$

$$f(t,u,v) \geqslant B_1|v| + C_1 u, \text{ for all } t \in K_n, \ u \in]0,\epsilon_n], \ v \in \mathbb{R},$$

and consider the sequence of approximation problems

$$\begin{aligned} &u'' + f_n(t,u,u') = 0, \\ &u(0) = \epsilon_n, \ u(\pi) = \epsilon_n. \end{aligned} \tag{P_n}$$

Step 3 – A lower solution of (P_n). It is clear that for any $c \in]0,\epsilon_n]$

$$\tilde{f_n}(t,c,0) \geqslant f(\eta_n(t),c,0) \geqslant 0.$$

As the sequence $(\epsilon_n)_n$ is decreasing, we also have

$$f_n(t,\epsilon_n,0) = \min_{1\leqslant k\leqslant n} \tilde{f_k}(t,\epsilon_n,0) \geqslant 0.$$

It follows that $\alpha_3(t) := \epsilon_n$ is such that

$$\alpha_3''(t) + f_n(t,\alpha_3(t),\alpha_3'(t)) = f_n(t,\epsilon_n,0) \geqslant 0.$$

Step 4 – Existence of a solution u_1 *of* (P_1) *such that* $\max(\alpha_1(t),\alpha_2(t),\epsilon_1) \leqslant u_1(t)$. From assumption (iii), we can find $M \geqslant \max(\alpha_1(t),\alpha_2(t),\epsilon_1)$ and $k \in \mathscr{A}$ such that, for all $t \in]0,\pi[$, $u \in [M,\infty[$ and $v \in \mathbb{R}$,

$$f(t,u,v) \leqslant B_2|v| + C_2 u + k(t).$$

Also, one has

$$f(\eta_1(t),u,v) \leqslant B_2|v| + C_2 u + k(\eta_1(t)).$$

Hence, we can write

$$f_1(t,u,v) = \max\{f(\eta_1(t),u,v), f(t,u,v)\} \leqslant B_2|v| + C_2 u + \tilde{k}(t),$$

where $\tilde{k}(t) = \max\{k(t), k(\max\{\frac{\pi}{4}, \min(t,\frac{3\pi}{4})\}\}$. Choose β such that

$$\begin{gathered}\beta'' + B_2|\beta'| + C_2\beta + \tilde{k}(t) = 0,\\ \beta(0) = M,\ \beta(\pi) = M,\end{gathered}$$

and observe that β is well defined and bounded since $k \in \mathscr{A}$ (see for example [19]). It is easy to see now that

$$\beta'' + f_1(t,\beta,\beta') \leqslant \beta'' + B_2|\beta'| + C_2\beta + \tilde{k}(t) = 0.$$

By Theorem 3.2, we know that there is a solution u_1 of (P_1) such that

$$\max(\alpha_1(t),\alpha_2(t),\epsilon_1) \leqslant u_1(t) \leqslant \beta(t).$$

In fact, $\max(\alpha_1(t),\alpha_2(t),\epsilon_1)$ is a $W^{2,1}$-lower solution of (P_1),

$$|f_1(t,u,v)| \leqslant \varphi(|v|)(\psi(t) + \psi(\eta_1(t)))$$

and for ϵ such that $\frac{2}{1-\epsilon} \leqslant C$,

$$\begin{aligned}\Phi^{-1}(2|\int_{\frac{\pi}{2}}^{t}(\psi(s) \ &+\ \psi(\eta_1(s)))\,ds|) \leqslant \Phi^{-1}(2|\int_{\frac{\pi}{2}}^{t}\psi(s)\,ds| + 2|\int_{\frac{\pi}{2}}^{t}\psi(\eta_1(s))\,ds|)\\ &\leqslant\ (1-\epsilon)\Phi^{-1}(\frac{2}{1-\epsilon}|\int_{\frac{\pi}{2}}^{t}\psi(s)\,ds|) + \epsilon\Phi^{-1}(\frac{2}{\epsilon}|\int_{\frac{\pi}{2}}^{t}\psi(\eta_1(s))\,ds|)\\ &\leqslant\ (1-\epsilon)\Phi^{-1}(C|\int_{\frac{\pi}{2}}^{t}\psi(s)\,ds|) + \epsilon\Phi^{-1}(\frac{2}{\epsilon}|\int_{\frac{\pi}{2}}^{t}\psi(\eta_1(s))\,ds|).\end{aligned}$$

Hence, $\Phi^{-1}(2|\int_{\frac{\pi}{2}}^{t}(\psi(s) + \psi(\eta_1(s)))\,ds|) \in L^1(a,b)$.

Step 5 – The problem (P_n) *has at least one solution* u_n *such that*

$$\max(\alpha_1(t),\alpha_2(t),\epsilon_n) \leqslant u_n(t) \leqslant u_{n-1}(t).$$

Let us notice that u_{n-1} is an upper solution of (P_n). The claim follows by Theorem 3.2.

Step 6 – Existence of a solution of (4.1). Consider now the pointwise limit

$$\tilde{u}(t) = \lim_{n\to\infty} u_n(t).$$

It is clear that, for any $n \geqslant 1$ and all $t \in]0,\pi[$,

$$\max(\alpha_1(t), \alpha_2(t)) \leqslant \tilde{u}(t) \leqslant u_n(t).$$

Let now $K \subset]0,\pi[$ be a compact interval. There is an index $n^* = n^*(K)$ such that $K \subset K_n$ for all $n \geqslant n^*$ and therefore for these $n \geqslant n^*$ and $t \in K$,

$$0 = u_n''(t) + f_n(t, u_n(t), u_n'(t)) = u_n''(t) + f(t, u_n(t), u_n'(t)).$$

Hence the function u_n is a solution of the equation in (4.1) for all $t \in K$ and $n \geqslant n^*$. Define $\alpha(t) = \max_t\{\alpha_1(t), \alpha_2(t)\}$. Let φ and ψ be given by assumption (iv) corresponding to $L = [\min_{t\in K} \alpha(t), \max_{t\in K} \beta(t)]$. By Proposition 3.1, there exists $h \in L^1(0,\pi) \cap \mathscr{C}(]0,\pi[)$ such that, for all $t \in K$, and $n \geqslant n^*$, $|u_n'(t)| \leqslant h(t)$.

Observe now that, for some $k \in L^1(K)$, for a.e. $t \in K$, all $u \in [\alpha(t), u_{n^*}(t)]$ and $|v| \leqslant h(t)$, we have

$$|f(t,u,v)| \leqslant k(t).$$

Then by Arzelá-Ascoli theorem it is standard to conclude that $\tilde{u}$ is a solution of (4.1) on the interval K. Since K was arbitrary, we find that $\tilde{u} \in W^{2,1}_{loc}(]0,\pi[, \mathbb{R}_0^+)$ and, for all $t \in]0,\pi[$,

$$\tilde{u}''(t) + f(t, \tilde{u}(t), \tilde{u}'(t)) = 0.$$

Since

$$\tilde{u}(0) = \tilde{u}(\pi) = \lim_{n\to+\infty} \epsilon_n = 0,$$

it remains only to check the continuity of $\tilde{u}$ at $t = 0$ and $t = \pi$.

Let $\epsilon > 0$ be give. Take n_ϵ such that $u_{n_\epsilon}(0) < \epsilon$. By the continuity of $u_{n_\epsilon}(t)$ in $t = 0$, we can find a constant $\delta = \delta_\epsilon > 0$ such that

$$0 < u_{n_\epsilon}(t) < \epsilon, \text{ for any } 0 < t < \delta.$$

Hence, we obtain

$$0 \leqslant \tilde{u}(t) \leqslant u_{n_\epsilon}(t) < \epsilon, \text{ for any } 0 < t < \delta.$$

The same argument works in proving the continuity of $\tilde{u}(t)$ at $t = \pi$. □

REMARK 4.2. Using the results of [18] or [19], we can generalize Theorem 4.1, assuming B_i, C_i are functions respectively in $L^1(0,\pi)$ and in $\mathscr{A}$ but the condition $\Gamma(B_1, C_1) < \pi < \Gamma(B_2, C_2)$ becomes less transparent. In that way we extend [33].

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