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TOEPLITZ OPERATORS AND WEIGHTED NORM INEQUALITIES ON THE BIDISC

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To the memory of Professor K. Seddighi

(*communicated by L. Pick*)

Abstract. Let H^p be the Hardy space on the bidisc and $1 < p < \infty$. For a function ϕ in L^∞, we study the norm of the Hankel operator H_ϕ on H^p and the invertibility of the Toeplitz operator T_ϕ on H^p. The latter is strongly related to weighted norm inequalities on the bidisc.

1. Introduction

Let m be the normalized Lebesgue measure on the torus T^2. For $1 \leqslant p \leqslant \infty$, $L^p = L^p(T^2, m)$ denotes the Lebesgue space and $H^p = H^p(T^2, m) = \{f \in L^p\ ;\ \hat{f}(\ell, n) = 0 \text{ if } \ell < 0 \text{ or } n < 0\}$, that is, H^p denotes the usual Hardy space on T^2. Let $K^p = \{f \in L^p\ ;\ \hat{f}(\ell, n) = 0 \text{ if } \ell \leqslant 0 \text{ and } n \leqslant 0\}$. Then $K^p = \{f \in L^q\ ;\ \int f g dm = 0 \text{ if } g \in H^q\}$ where $1/p + 1/q = 1$. Put $H = H^p \cap L$ and $K = K^p \cap L$ where L denotes the set of all trigonometric polynomials on T^2. Suppose m_z and m_w denote the normalized Lebesgue measures on the torus $T = T_z$ and $T = T_w$. Then $T^2 = T_z \times T_w$ and $m = m_z \times m_w$.

Let P be a projection from L onto H with $P = 0$ on K. Then P can be extended boundedly to L^p for $1 < p < \infty$ and P is an orthogonal projection when $p = 2$. For a function ϕ in L^∞, the Hankel operator determined by ϕ is

$$H_\phi f = (I - P)(\phi f) \quad (f \in H^p)$$

and the Toeplitz operator determined by ϕ is

$$T_\phi f = P(\phi f) \quad (f \in H^p).$$

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For the bounded linear operator A on H^p or L^p, $\|A\|_p$ denotes the norm of A. When $p=2$, and H^p is the Hardy space on the disc, Z.Nehari [10] proved that $\|H_\phi\|_2 = \|\phi + H^\infty\|$. It is not difficult to generalize this for $p \neq 2$, that is,

$$\|\phi + H^\infty\| \leqslant \|H_\phi\|_p \leqslant \|1-P\|_q \|\phi + H^\infty\|.$$

This is different from a formula in [1, Theorem 2.11] because the Hankel operator is different from ours. In Section 2, we generalize the formula above for $\|H_\phi\|_p$ to the Hardy space on the bidisc. When $p=2$, this is known (see [2],[4],[8] and [9]). When H^p is the Hardy space on the disc, R. Rochberg [13] showed that T_ϕ is invertible on H^p if and only if $\phi = k\phi_0$ and $\phi_0 = \bar{h}_0/h_0$ where k is an invertible function in H^∞ and h_0 is an outer function in H^p with $|h_0|^p$ satisfying the (A_p)-condition (see [10, Theorem 1]). If $p=2$, this reduces to a theorem of H. Widom and A. Devinatz. In Section 4, for some special symbols ϕ we generalize the above theorem of R.Rochberg to the Hardy space on the bidisc. In Section 3, we define (A_p)-condition on T^2 and give a theorem of Hunt, Muckenhoupt and Wheeden on T^2. This is used in Section 4.

An order relation can be introduced in Z^2. Let L_r be a line with rational slope r in the plane. S_r denotes all lattice points on one side of L_r, together with those on the right side ray of L_r from the origin. When L is a real axis, that is, $L = L_0$, then $S_0 = \{(m,0)\ ;\ m>0\} \cup \{(m,n)\ ;\ n>0\}$. When L is an imaginary axis, that is, $L = L_{-\infty}$, then $S_{-\infty} = \{(0,n)\ ;\ n>0\} \cup \{(m,n)\ ;\ m>0\}$. This order is non-archimedean, and Z^2 has the smallest positive element (m_0,n_0) in S_r. We assume that S_r contains Z_+^2, that is, $-\infty \leqslant r \leqslant 0$. When $-\infty < r < 0$, $|m_0|$ and $|n_0|$ have no common factor except 1 and $r = n_0/m_0$, and let $(m_1,n_1) = (0,1)$. When $r=0$, $(m_0,n_0) = (1,0)$ and let $(m_1,n_1) = (0,1)$. When $r = -\infty$, $(m_0,n_0) = (0,1)$ and let $(m_1,n_1) = (1,0)$. For each half plane S_r, put

$$Z = Z_r = z^{m_0} w^{n_0}$$

and

$$W = W_r = z^{m_1} w^{n_1}.$$

Hence $Z_0 = W_{-\infty} = z$ and $W_0 = Z_{-\infty} = w$, and if $-\infty < r < 0$ then $W_r = w$.

For each r with $-\infty \leqslant r \leqslant 0$, put H_r^p = the norm closed linear span of $\cup_{j=-\infty}^{\infty} Z_r^j H^p$ in L^p if $1 \leqslant p < \infty$ and $\mathbf{H}_r^\infty$ = the weak* closed linear span of $\cup_{j=-\infty}^{\infty} Z_r^j H^\infty$ in L^∞. $\mathscr{L}_r^p$ and $\mathscr{H}_r^p$ denote the norm closure of the set of trigonometric polynomials and analytic polynomials, respectively, of Z_r in L^p if $1 \leqslant p < \infty$. $\mathscr{L}_r^\infty$ and $\mathscr{H}_r^\infty$ denote the weak* closure. Then

$$\mathbf{H}_r^p = \mathscr{L}_r^p + \mathscr{L}_r^p W + \cdots + \mathscr{L}_r^p W^{n-1} + W^n \mathbf{H}_r^p.$$

Let $\mathscr{E}$ be a conditional expectation from $\mathbf{H}_r^\infty$ onto $\mathscr{L}_r^\infty$. Then $\mathscr{E}$ is multiplicative on $\mathbf{H}_r^\infty$ and $\mathbf{H}_r^\infty + \overline{W\mathbf{H}_r^\infty}$ is weak* dense in L^∞. Hence $\mathbf{H}_r^\infty$ is an extended weak* Dirichlet algebra with respect to $\mathscr{E}$ (see [7]). For $r=0$ and $r=-\infty$, we will write that $\mathbf{H}_0^p = \mathbf{H}_w^p$, $\mathbf{H}_{-\infty}^p = \mathbf{H}_z^p$, $W\mathbf{H}_0^p = w\mathbf{H}_w^p$, $W\mathbf{H}_{-\infty}^p = z\mathbf{H}_z^p$, $\mathscr{L}_0^\infty = \mathscr{L}_z^\infty$ and $\mathscr{L}_{-\infty}^\infty = \mathscr{L}_w^\infty$ where $\mathscr{L}_z^\infty = L^\infty(T_z, dm_z)$ and $\mathscr{L}_w^\infty = L^\infty(T_w, dm_w)$. $H^p(T_z, dm_z)$ and $H^p(T_w, dm_w)$ denote one variable Hardy spaces. Let P^w be a projection from L

onto $(\mathbf{H}_w^\infty) \cap L$ with $P^w = 0$ on $\overline{w\mathbf{H}_w^\infty} \cap L$, and put $P_0^w f = wP^w(\bar{w}f)$. P^z and P_0^z can be defined similarly.

All results in this paper can be generalized easily to the Hardy space on the polydisc.

2. Hankel operator

Theorem 1 for $p = 2$ is known in [2],[3],[8] and [9]. Proposition 2 for $p = 2$ is written in [8]. However its proof had some gap except $r = 0$ and $r = -\infty$ (see[9]).

LEMMA 1. *Suppose* $1/p + 1/q = 1$. *If* h *is a function in* $\mathbf{H}_r^1$, *then there exists an* f *in* $\mathbf{H}_r^p$ *and a* g *in* $\mathbf{H}_r^q$ *such that* $h = fg$, $\|f\|_p \leqslant \|h\|_1$ *and* $\|g\|_q \leqslant \|h\|_1$.

Proof. In Section 1, we noted that $\mathbf{H}_r^\infty$ is an extended weak* Dirichlet algebra. If h is in $\mathbf{H}_r^1$ then $\chi_{E(h)}\mathscr{E}(\log|h|) > -\infty$ *a.e.* where $E(h)$ is the support set of h. Hence Theorem 4′ in [7] implies the lemma for $p = 2$. For $p \neq 2$, we can prove it similarly because Theorem 4′ can be shown for arbitrary p (see[7, Section 5]).

LEMMA 2. *Suppose* $1/p + 1/q = 1$. *If* ϕ *is a function in* L^∞, *then*

$$\begin{aligned}\|\phi + \mathbf{H}_r^\infty\| &= \sup\{|\int \phi h dm| \; ; \; h \in W\mathbf{H}_r^1 \text{ and } \|h\|_1 \leqslant 1\} \\ &= \sup\{|\int \phi f g dm| \; ; f \in \mathbf{H}_r^p, \; g \in W\mathbf{H}_r^q, \; \|f\|_p \leqslant 1 \text{ and } \|g\|_q \leqslant 1\}\end{aligned}$$

Proof. This is a result of Lemma 1 and Hahn-Banach theorem because the annihilator of $W\mathbf{H}_r^1$ in L^∞ is $\mathbf{H}_r^\infty$.

THEOREM 1. *Suppose* $1 < p < \infty$, $1/p + 1/q = 1$ *and* ϕ *is function in* L^∞. *Then*

$$\begin{aligned}\max(\|\phi + \mathbf{H}_z^\infty\|, \; \|\phi + \mathbf{H}_w^\infty\|) &\leqslant \|H_\phi\| \\ &\leqslant \|1 - P\|_q\{\|P_0^w\|_q\|\phi + \mathbf{H}_w^\infty\| + \|P_0^z\|_q\|\phi + \mathbf{H}_z^\infty\|\}.\end{aligned}$$

Proof. We will give the lower estimate of $\|H_\phi\|$. Since $\bar{z}^n\mathbf{H}_w^q = \mathbf{H}_w^q$ for any positive integer n, $H^p \times w\mathbf{H}_w^q = \bar{z}^n H^p \times w\mathbf{H}_w^q$. Hence

$$H^p \times K^q \supset \{\bigcup_{n=0}^{\infty} \bar{z}^n H^p\} \times w\mathbf{H}_w^q.$$

Using Lemma 2 in the second equality,

$$\begin{aligned}\|H_\phi\| &= \sup\{|\int (1-P)\phi f g dm| \; ; f \in H^p, \; g \in L^q, \; \|f\|_p \leqslant 1 \text{ and } \|g\|_q \leqslant 1\} \\ &\geqslant \sup\{|\int \phi f g dm| \; ; f \in H^p, \; g \in K^q, \; \|f\|_p \leqslant 1 \text{ and } \|g\|_q \leqslant 1\} \\ &\geqslant \sup\{|\int \phi f g dm| \; ; f \in \mathbf{H}_w^p, \; g \in w\mathbf{H}_w^q, \; \|f\|_p \leqslant 1 \text{ and } \|g\|_q \leqslant 1\} \\ &= \|\phi + \mathbf{H}_w^\infty\|.\end{aligned}$$

Similarly we can show that $\|H_\phi\| \geqslant \|\phi + \mathbf{H}_z^\infty\|$.

Now we will give the upper estimate of $\|H_\phi\|$. Here for $F \in L^p$ and $G \in L^q$, put $\langle F, G\rangle = \int F\bar{G}dm$.

$$\begin{aligned} \|H_\phi\| &= \sup\{|\langle H_\phi f, g\rangle| \,;\, f \in H^p,\ g \in L^q,\ \|f\|_p \leqslant 1 \text{ and } \|g\|_q \leqslant 1\} \\ &= \sup\{|\langle \phi f, (I-P)g\rangle| \,;\, f \in H^p,\ g \in L^q,\ \|f\|_p \leqslant 1 \text{ and } \|g\|_q \leqslant 1\} \\ &\leqslant \|I-P\|_q \sup\{|\langle \phi f, \bar{h}\rangle| \,;\, f \in H^p,\ h \in K^q,\ \|f\|_p \leqslant 1 \text{ and } \|h\|_q \leqslant 1\}. \end{aligned}$$

Since $K^q = w\mathbf{H}_w^q + z\mathbf{H}_z^q$, by Lemma 2

$$\begin{aligned} &\sup\{|\langle \phi f, \bar{h}\rangle| \,;\, f \in H^p,\ h \in K^q,\ \|f\|_p \leqslant 1 \text{ and } \|h\|_q \leqslant 1\} \\ &\quad\leqslant \sup\{|\langle \phi f, \overline{P_0^w h}\rangle| \,;\, f \in H^p,\ h \in K^q,\ \|f\|_p \leqslant 1 \text{ and } \|h\|_q \leqslant 1\} \\ &\qquad + \sup\{|\langle \phi f, \overline{P_0^z h}\rangle| \,;\, f \in H^p,\ h \in K^q,\ \|f\|_p \leqslant 1 \text{ and } \|h\|_q \leqslant 1\} \\ &\quad\leqslant \|P_0^w\|_q \sup\{|\int \phi f k dm| \,;\, f \in H^p,\ k \in w\mathbf{H}_w^q,\ \|f\|_p \leqslant 1 \text{ and } \|k\|_q \leqslant 1\} \\ &\qquad + \|P_0^z\|_q \sup\{|\int \phi f k dm| \,;\, f \in H^p,\ k \in \mathbf{H}_z^q,\ \|f\|_p \leqslant 1 \text{ and } \|k\|_g \leqslant 1\} \\ &\quad\leqslant \|P_0^w\|_q \sup\{|\int \phi f k dm| \,;\, f \in \mathbf{H}_w^p,\ k \in w\mathbf{H}_w^q,\ \|f\|_p \leqslant 1 \text{ and } \|k\|_q \leqslant 1\} \\ &\qquad + \|P_0^z\|_q \sup\{|\int \phi f k dm| \,;\, f \in \mathbf{H}_z^p,\ k \in z\mathbf{H}_z^q,\ \|f\|_p \leqslant 1 \text{ and } \|k\|_q \leqslant 1\} \\ &\quad\leqslant \|P_0^w\|_q\|\phi + \mathbf{H}_w^\infty\| + \|P_0^z\|_q\|\phi + \mathbf{H}_z^\infty\|. \end{aligned}$$

This completes the proof.

COROLLARY 1. *Suppose $p = 2$ and ϕ is a function in $\mathbf{H}_w^\infty$. Then*

$$\|H_\phi\| = \|\phi + \mathbf{H}_z^\infty\|.$$

Proof. Apply Theorem 1 as $p = 2$.

COROLLARY 2. *Suppose $p = 2$ and ϕ is a continuous function on T^2. Then $\lim_{n\to\infty}\|H_{\phi z^n}\| = \|\phi + \mathbf{H}_w^\infty\|$ and $\lim_{n\to\infty}\|H_{\phi w^n}\| = \|\phi + \mathbf{H}_z^\infty\|$.*

Proof. By Theorem 1 for $p = 2$.

$$\begin{aligned} \|\phi + \mathbf{H}_w^\infty\| &= \|\phi z^n + \mathbf{H}_w^\infty\| \\ &\leqslant \|H_{\phi z^n}\| \leqslant \|\phi z^n + \mathbf{H}_w^\infty\| + \|\phi z^n + \mathbf{H}_z^\infty\| \\ &= \|\phi + \mathbf{H}_w^\infty\| + \|\phi + \bar{z}^n\mathbf{H}_z^\infty\|. \end{aligned}$$

Since ϕ is continuous, $\lim_{n\to\infty}\|\phi + \bar{z}^n\mathbf{H}_z^\infty\| = 0$.

PROPOSITION 2. *Suppose* $1 < p < \infty$ *and* ϕ *is a function in* L^∞. *Then*

$$\sup_{-\infty \leqslant r \leqslant 0} \|\phi + \mathbf{H}_r^\infty\| \leqslant \|H_\phi\| \leqslant \|P\|_p \|\phi + H^\infty\|.$$

Proof. For the lower estimate of $\|H_\phi\|$, the proof is almost parallel to the case of $r = 0$ and $-\infty$ which were proved in Theorem 1. When $p = 2$, the proposition was proved in [8, Theorem 1] with a gap (see [9]). The point is to prove that the linear span of $\bigcup_{n=-\infty}^{\infty} Z^n H^p$ is dense in $\mathbf{H}_r^p$ when $r \neq 0$ and $r \neq -\infty$. For $-\infty < r < 0$,

$$ZH^p \supset \{z^s w^t \ ; \ (s,t) \in \mathbf{Z}^2, \ s \geqslant m_0, \ t \geqslant n_0\}$$

and

$$Z^{-1}H^p \supset \{z^s w^t \ ; \ (s,t) \in \mathbf{Z}^2, \ s \geqslant -m_0, \ t \geqslant -n_0\}.$$

where $Z = Z_r = z^{m_0} w^{n_0}$, $r = n_0/m_0$ and $n_0 < 0$, $m_0 > 0$. Hence $\bigcup_{n=-\infty}^{\infty} Z^n H^p$ contains $\{z^m w^n \ ; \ (m,n) \in S_r \cup$ (the left side ray of L_r from the origin)$\}$ and the linear span of $\{z^m w^n \ ; \ (m,n) \in S_r \cup$ (the left side ray of L_r from the origin)$\}$ is dense in $\mathbf{H}_r^p$. This implies that the linear span of $\bigcup_{n=-\infty}^{\infty} Z^n H^p$ is dense in $\mathbf{H}_r^p$.

COROLLARY 3. *Suppose* $p = 2$ *and* $\bar{\phi}$ *is a function in* $H_0^\infty = \{f \in H^\infty \, ; \int f \, dm = 0\}$. *Then*

$$\left(\int |\phi|^2 dm \right)^{1/2} \leqslant \|H_\phi\| \leqslant \|\phi\|_\infty.$$

Proof. If $\bar{\phi}$ is in H_0^∞, then $\bar{\phi}$ is orthogonal to $\mathbf{H}_r^2$ and hence $\|\phi + \mathbf{H}_r^\infty\| \geqslant \left(\int |\phi|^2 dm \right)^{1/2}$.

COROLLARY 4. *Suppose* $1 < p < \infty$ *and* $\phi = \phi_w \phi_z$ *is a function in* L^∞ *where* ϕ_w *is unimodular in* $L^\infty(T_w, m_w)$ *and* ϕ_z *is unimodular in* $L^\infty(T_z, m_z)$. *Then*

$$\max(\|\phi_z + H^\infty\|, \ \|\phi_w + H^\infty\|) \leqslant \|H_\phi\| \leqslant \|P\|_p \|\phi + H^\infty\|.$$

Proof. This is a corollary of Proposition 2 by the following equality:

$$\|\phi_z \phi_w + \mathbf{H}_z^\infty\| = \|\phi_z + \mathbf{H}_z^\infty\| = \|\phi_z + H^\infty(T_z, m_z)\| = \|\phi_z + H^\infty\|.$$

3. Weighted norm inequality

If W is a nonnegative function in L^1, then

$$\left(\frac{1}{m(E\times F)}\int_{E\times F} W^{-\frac{1}{p-1}}dm\right)^{1-p}$$
$$\leqslant \frac{1}{m_z(E)}\int_E dm_z\left(\frac{1}{m_w(F)}\int_F W^{-\frac{1}{p-1}}dm_w\right)^{1-p} \leqslant \frac{1}{m(E\times F)}\int_{E\times F} Wdm.$$

where $1<p<\infty$ and $E\times F$ is a Borel set on T^2. If $W(z,w)=W_1(z)W_2(w)$, and W_1 and W_2 are nonnegative functions in $L^1(T_z)=L^1(T,m_z)$ and $L^1(T_w)=L^1(T,m_w)$, respectively, then

$$\left(\frac{1}{m(E\times F)}\int_{E\times F} W^{-\frac{1}{p-1}}dm\right)^{1-p}=\left(\frac{1}{m_z(E)}\int_E W_1^{\frac{1}{p-1}}dm_z\right)^{1-p}\left(\frac{1}{m_w(F)}\int_F W_2^{-\frac{1}{p-1}}dm_w\right)^{1-p},$$
$$\frac{1}{m_z(E)}\int_E dm_z\left(\frac{1}{m_w(F)}\int_F W^{-\frac{1}{p-1}}dm_w\right)^{1-p}=\frac{1}{m_z(E)}\int_E W_1dm_z\left(\frac{1}{m_w(F)}\int_F W_2^{-\frac{1}{p-1}}dm_w\right)^{1-p}$$

and

$$\frac{1}{m(E\times F)}\int_{E\times F} Wdm=\frac{1}{m_z(E)}\int_E W_1dm_z\frac{1}{m_w(F)}\int_F W_2dm_w.$$

Suppose $1<p<\infty$ and W is a nonnegative function in L^1. We say that W satisfies (A_p)-condition for w if there exists a positive finite constant γ such that

$$\frac{1}{m(E\times I)}\int_{E\times I} Wdm \leqslant \gamma\frac{1}{m_z(E)}\int_E dm_z\left(\frac{1}{m_w(I)}\int_I W^{-\frac{1}{p-1}}dm_w\right)^{1-p}$$

where E is a Borel set in T_z and I is an interval in T_w. Similarly (A_p)-condition for z can be defined. If W satisfies (A_p)-condition for w and z, then we say that W satisfies (A_p)-condition.

Using a theorem of Hunt, Muckenhoupt and Wheeden [6] on T, we give the generalization to T^2. This is known essentially in [3]. This will be used in Section 4.

LEMMA 3. *Suppose $1<p<\infty$ and W is a nonnegative function in L^1.*

$$\int |f|^pWdm \leqslant \gamma_p\int |f+\bar{g}|^pWdm \quad (f\in \mathbf{H}_w^\infty,\ g\in w\mathbf{H}_w^\infty)$$

with γ_p independent of f and g if and only if W satisfies (A_p)-condition for w.

Proof. It is easy to see that W satisfies (A_p)-condition for w if and only if for $a.e.m_z$ W satisfies (A_p)-condition of one variable. In a theorem of Hunt, Muckenhoupt and Wheeden (cf. [5, Theorem 6.1 in Chapter VI]), the constant Γ_p of (A_p)-condition and the constant γ_p of a weighted norm inequality are equivalent, that is, $0<\varepsilon\leqslant \Gamma_p/\gamma_p\leqslant 1/\varepsilon$. This implies the lemma with Fubini's theorem.

THEOREM 3. *Suppose* $1 < p < \infty$ *and* W *is a nonnegative function in* L^1.

$$\int |f|^p W dm \leqslant \gamma_p \int |f + \bar{g}|^p W dm \quad (f \in H,\ g \in K)$$

with γ_p *independent of* f *and* g *if and only if* W *satisfies* (A_p)*-condition.*

Proof. Suppose that

$$\int |f|^p W dm \leqslant \gamma_p \int |f + \bar{g}|^p W dm \quad (f \in H,\ g \in K).$$

Then for any nonnegative integer n,

$$\int |\bar{z}^n f|^p W dm \leqslant \gamma_p \int |\bar{z}^n f + \bar{z}^n \bar{g}|^p W dm.$$

Since $K^\infty \supset w\mathbf{H}_w^\infty$, if $g \in w\mathbf{H}_w^\infty$ then $z^n g \in w\mathbf{H}_w^\infty$ and $\bar{z}^n f \in \mathbf{H}_w^\infty$ for any $n \geqslant 0$. Since $\bigcup_{n=1}^{\infty} \bar{z}^n H$ is dense in $\mathbf{H}_w^\infty$, for $F \in \mathbf{H}_w^\infty$ and $G \in w\mathbf{H}_w^\infty$

$$\int |F|^p W dm \leqslant \gamma_p \int |F + \bar{G}|^p W dm.$$

Now Lemma 3 implies that W satisfies (A_p)-condition for w. The same argument implies that W satisfies (A_p)-condition for z. Conversely, suppose that W satisfies (A_p)-condition. By Lemma 3, for the weight W we have weighted norm inequalities for $\mathbf{H}_w^\infty + \bar{w}\bar{\mathbf{H}}_w^\infty \to \mathbf{H}_w^\infty$ and $\mathbf{H}_z^\infty + \bar{z}\bar{\mathbf{H}}_z^\infty \to \mathbf{H}_z^\infty$. This implies the weighted norm inequality for $H + \bar{K} \to H$. In fact, this is a simple result of the following decomposition. If $f \in H$ and $g \in K$, then $\bar{g} = g_1 + g_2$ where $g_1 \in \mathbf{H}_w^\infty \cap \bar{K}$ and $g_2 \in \bar{w}\bar{\mathbf{H}}_w^\infty \cap \bar{K}$, and so

$$f + \bar{g} = (f + g_1) + g_2$$

where $f + g_1 \in \mathbf{H}_w^\infty \cap L$. Then $f \in H^\infty$ and $g_1 \in \bar{z}\bar{\mathbf{H}}_z^\infty$.

Our (A_p)-condition on T^2 seems to be strange if we compare with that on T. A natural (A_p)-condition on T^2 may be the following : There exists a positive constant γ such that

$$\frac{1}{m(I \times I)} \int_{I \times I} W dm \leqslant \gamma \left(\frac{1}{m(I \times I)} \int_{I \times I} W^{\frac{1}{p-1}} dm \right)^{1-p}.$$

However this is too weak for one weighted norm inequality.

COROLLARY 5. *Suppose* $1 < p < \infty$ *and* $W = W_w W_z$ *where* W_w *is a nonnegative function in* $L^1(T_w)$ *and* W_z *is a nonnegative function in* $L^1(T_w)$.

$$\int |f|^p W dm \leqslant \gamma_p \int |f + \bar{g}|^p w dm \quad (f \in H,\ g \in K)$$

if and only if W_w *and* W_z *satisfy one variable* (A_p)*-condition.*

4. Toeplitz operator

For ϕ in L^∞, $\mathbf{T}^w_\phi$ and $\mathbf{T}^z_\phi$ are Toeplitz operators on $\mathbf{H}^p_w$ and $\mathbf{H}^p_z$, respectively. That is , $\mathbf{T}^w_\phi f = \mathbf{P}^w(\phi f)$ $(f \in \mathbf{H}^p_w)$ and $\mathbf{T}^z_\phi f = \mathbf{P}^z(\phi f)(f \in \mathbf{H}^p_z)$. $\sigma(T_\phi)$, $\sigma(\mathbf{T}^w_\phi)$ and $\sigma(\mathbf{T}^z_\phi)$ denote the spectrums, respectively. For a nonzero function h in H^p, we call it an outer function if

$$\int \log|h|dm = \log\left|\int hdm\right|$$

(cf. [14, p73]). For a nonzero function h in $\mathbf{H}^p_w$, we call it a w-outer function if

$$\int_{T^2} \log|h|dm = \int_T \log\left|\int_T hdm_w\right|dm_z > -\infty.$$

Similary we can define an z-outer function. When a function in H^p is outer for w and z, it is called weakly outer (see [11]). If h is an outer function in H^p, then h is weakly outer.

In order to prove Lemma 4, we use a general theory of an extended weak $*$ Dirichlet algebra [7]. However we can also prove this using a general theory of a weak $*$ Dirichlet algebra. Since

$$\int_T \log\left|\int_T hdm_w\right|dm_z = \int_{T^2} \log|\mathscr{E}(h)|dm$$

where $\mathscr{E}$ is a conditional expectation from $\mathbf{H}^\infty_w$ onto $\mathscr{L}^\infty_z = \mathbf{H}^\infty_w \cap \bar{\mathbf{H}}^\infty_w$, if h is a w-outer function, then $h\mathbf{H}^\infty_w$ is dense in $\mathbf{H}^p_w$ [7].

LEMMA 4. *Suppose* $1 < p < \infty$ *and* ϕ *is a function in* L^∞. *If* $\mathbf{T}^w_\phi$ *is left invertible on* $\mathbf{H}^p_w$, $\phi = k\phi_0$ *where* k *is invertible in* $\mathbf{H}^\infty_w$, ϕ_0 *is a unimodular function, and* $\mathbf{T}^w_{\phi_0}$ *is left invertible on* $\mathbf{H}^p_w$.

Proof. If $\mathbf{T}^w_\phi$ is left invertible on $\mathbf{H}^p_w$, then there exist a positive constant ε such that

$$\int |\phi f + \bar{g}|^p dm \geqslant \varepsilon \int |f|^p dm$$

for $f \in \mathbf{H}^p_w$ and $g \in w\mathbf{H}^p_w$. As $g = 0$, $\int |\phi|^p|f|^p dm \geqslant \varepsilon \int |f|^p dm$ for $f \in \mathbf{H}^p_w$. If v is a nonnegative function in L^1 with $\log v \in L^1$, then by Theorem 4$'$ in [7] $v^{1/p} = |f|$ for some function f in $\mathbf{H}^1_w \cap L^p$. By Theorem 5 in [7], $\mathbf{H}^1_w \cap L^p = \mathbf{H}^p_w$ and so $f \in \mathbf{H}^p_w$. Hence

$$\int |\phi|^p v dm \geqslant \varepsilon \int v dm$$

for all $v \in L^1$ with $v \geqslant 0$. This implies that ϕ is invertible in L^∞. Again by Theorem 4$'$ $\phi = k\phi_0$ for some $k \in \mathbf{H}^\infty_w$ and for some unimodular function ϕ_0. It is easy to see that k is invertible in $\mathbf{H}^\infty_w$. Since $\mathbf{T}^w_\phi = \mathbf{T}^w_{\phi_0}\mathbf{T}^w_k$ and $\mathbf{T}^w_k$ is invertible, $\mathbf{T}^w_{\phi_0}$ is left invertible.

LEMMA 5. *Suppose* $1 < p < \infty$ *and* ϕ *is a function in* L^∞. $\mathbf{T}^w_\phi$ *is invertible on* $\mathbf{H}^p_w$ *if and only if there exist an invertible function* k *in* $\mathbf{H}^\infty_w$ *and a* w*-outer function* h *in* $\mathbf{H}^p_w$ *such that* $\phi = k\bar{h}/h$ *and* $|h|^p$ *satisfies* (A_p)*-condition for* w.

Proof. By Lemma 4, we may assume that ϕ is a unimodular function. If $\mathbf{T}^w_\phi$ is invertible on $\mathbf{H}^p_w$, then there exist $f \in \mathbf{H}^p_w$ and $g \in w\mathbf{H}^p_w$ such that $\phi f = 1 + \bar{g}$. Since $(\mathbf{T}^w_\phi)^*$ is invertible on $\mathbf{H}^q_w$ where $1/p + 1/q = 1$, there exist $f' \in \mathbf{H}^q_w$ and $g' \in w\mathbf{H}^q_w$ such that $\bar{\phi} f' = 1 + \bar{g}'$. Therefore $ff' = \overline{(1+g)}\ \overline{(1+g')}$ belongs to $\mathbf{H}^{p/2}_w \cap \bar{\mathbf{H}}^{q/2}_w$. When $p \geqslant 2$, $\mathbf{H}^{p/2}_w \cap \bar{\mathbf{H}}^{q/2}_w = \mathbf{H}^{p/2}_w \cap \bar{\mathbf{H}}^{p/2}_w = \mathscr{L}^{p/2}_z$. Hence ff' belongs to $\mathscr{L}^{p/2}_z$ and so $ff' = 1$ *a.e.* on T^2 because $g, g' \in w\mathbf{H}^q_w$. When $1 < p < 2$, $\mathbf{H}^{p/2}_w \cap \bar{\mathbf{H}}^{q/2}_w = \mathscr{L}^{q/2}_z$. This also implies $ff' = 1$ *a.e.* on T^2. Hence $f \in \mathbf{H}^p_w$ and $f^{-1} \in \mathbf{H}^q_w$, and $1+g \in \mathbf{H}^p_w$ and $(1+g)^{-1} \in \mathbf{H}^q_w$. Since $\phi = (1+\bar{g})/f$ and $|\phi| = 1$ *a.e.* on T^2, $|f| = |1+g|$ *a.e.* on T^2 and so $1 + g = \alpha f$ for some unimodular $\alpha \in \mathscr{L}^\infty_z$. Therefore $\phi = \bar{h}/h$ where $h = \beta f$ for some unimodular $\beta \in \mathscr{L}^\infty_z$. Then h is a w-outer function in $\mathbf{H}^p_w$.

Since $\mathbf{T}^w_\phi$ is invertible on $\mathbf{H}^p_w$, there exist positive constants γ and ε such that

$$\gamma \|\phi f + \bar{g}\|_p \geqslant \|\mathbf{T}^w_\phi f\|_p \geqslant \varepsilon \|f\|_p$$

where $f \in \mathbf{H}^\infty_w$ and $g \in w\mathbf{H}^\infty_w$. Then

$$\gamma^p \int |h^{-1} f + \bar{h}^{-1}\bar{g}|^p |h|^p dm \geqslant \varepsilon^p \int |h^{-1} f|^p |h|^p dm$$

and hence we can show that

$$\gamma^p \int |F + \bar{G}|^p |h|^p dm \geqslant \varepsilon^p \int |F|^p |h|^p dm$$

where $F \in \mathbf{H}^\infty_w$ and $G \in w\mathbf{H}^\infty_w$, because h is w-outer. By Lemma 3, $|h|^p$ satisfies (A_p)-condition for w.

Conversely if $\phi = \bar{h}/h$ and $|h|^p$ satisfies (A_p)-condition for w, then by Lemma 3

$$\gamma_p \int |f + \bar{g}|^p |h|^p dm \geqslant \int |f|^p |h|^p dm \quad (f \in \mathbf{H}^\infty_w,\ g \in w\mathbf{H}^\infty_w)$$

and so

$$\gamma_p \int |\phi h f + \bar{h}\bar{g}|^p dm \geqslant \int |hf|^p dm \quad (f \in \mathbf{H}^\infty_w,\ g \in w\mathbf{H}^\infty_w).$$

Since h is a w-outer function in $\mathbf{H}^p_w$, $h\mathbf{H}^\infty_w$ is dense in $\mathbf{H}^p_w$ by [7] and so

$$\gamma_p \int |\phi F + \bar{G}|^p dm \geqslant \int |F|^p dm \quad (F \in \mathbf{H}^\infty_w,\ G \in w\mathbf{H}^\infty_w).$$

This implies that $\mathbf{T}^w_\phi$ is left invertible because $L^p/\bar{w}\bar{\mathbf{H}}^p_w \cong \mathbf{H}^p_w$. Since $[\mathbf{T}^w_\phi \mathbf{H}^p_w]_p \supseteq [\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p$, we will prove that $[\mathbf{P}^w(\bar{h}\mathbf{H}^\infty_w)]_p = \mathbf{H}^p_w$. Then the invertibility of $\mathbf{T}^w_\phi$ follows. For any n, we can write $h = \sum_{j=0}^{n} h_j w^j + w^{n+1} k_{n+1}$ where $h_j \in \mathscr{L}^p_z (0 \leqslant j \leqslant n)$ and $k_{n+1} \in \mathbf{H}^p_w$. Since h is a w-outer function, $|h_0| > 0$ *a.e.* on T^2 and

$h_0 \in [\mathbf{P}^w(\bar{h}\mathbf{H}_w^\infty)]_p$. $[\mathbf{P}^w(\bar{h}\mathbf{H}_w^\infty)]_p$ is invariant under multiplication by $u \in \mathscr{L}_z^\infty$ and so $\mathscr{L}_z^p \subset [\mathbf{P}^w(\bar{h}\mathbf{H}_w^\infty)]_p$. Since $\mathbf{P}^w(w\bar{h}) = w\bar{h}_0 + \bar{h}_1$, $w\bar{h}_0 \in [\mathbf{P}^w(\bar{h}\mathbf{H}_w^\infty)]_p$ and so $w\mathscr{L}_z^p \subset [\mathbf{P}^w(\bar{h}\mathbf{H}_w^\infty)]_p$. Since $\mathbf{P}^w(w^2\bar{h}) = w^2\bar{h}_0 + w\bar{h}_1 + \bar{h}_2$, similarly we can show that $w^2\mathscr{L}_z^p \subset [\mathbf{P}^w(\bar{h}\mathbf{H}_w^\infty)]_p$. By repeating this method, we can prove that $\mathbf{H}_w^p \subseteq [\mathbf{P}^w(\bar{h}\mathbf{H}_w^\infty)]_p$.

THEOREM 4. *Suppose $1 < p < \infty$ and ϕ is a function in L^∞. If T_ϕ is invertible on H^p, then*

$$\phi = k_w \frac{\bar{h}_w}{h_w} = k_z \frac{\bar{h}_z}{h_z}$$

where k_t is invertible in $\mathbf{H}_t^\infty$ for $t = w, z$ and h_t is a t-outer function in $\mathbf{H}_t^p$ for $t = w, z$ such that $|h_t|^p$ satisfies (A_p)-condition for $t = w, z$.

Proof. If T_ϕ is invertible on H^p, then there exist positive constants γ and ε such that

$$\gamma \|\phi f + \bar{g}\|_p \geqslant \|T_\phi f\|_p \geqslant \varepsilon \|f\|_p$$

where $f \in H$ and $g \in K$. As in the proof of Theorem 3, for any nonegative integer n

$$\gamma^p \int |\phi \bar{z}^n f + \bar{z}^n g|^p dm \geqslant \varepsilon^p \int |\bar{z}^n f|^p dm$$

where $f \in H$ and $g \in (w\mathbf{H}_w^\infty) \cap L$, and so we can show that

$$\gamma \|\phi f + \bar{g}\|_p \geqslant \varepsilon \|f\|_p$$

where $f \in \mathbf{H}_w^\infty$ and $g \in w\mathbf{H}_w^\infty$. This implies that $\mathbf{T}_\phi^w$ is left invertible on $\mathbf{H}_w^p$. Since $\mathbf{P}^w(\phi\mathbf{H}_w^\infty) \supset P(\phi\mathbf{H}_w^\infty) \supset P(\phi H)$ and $T_\phi H^p = H^p$, $\mathbf{T}_\phi^w \mathbf{H}_w^p$ is dense in $\mathbf{H}_w^p$ and so $\mathbf{T}_\phi^w$ is invertible on $\mathbf{H}_w^p$. Now Lemma 5 implies that $\phi = k_w\bar{h}_w/h_w$ where k_w is invertible in $\mathbf{H}_w^\infty$ and h_w is a w-outer function in $\mathbf{H}_w^p$ such that $|h_w|^p$ satisfies (A_p)-condition for w. The same method implies the statement about z.

COROLLARY 6. *Suppose $1 < p < \infty$ and ϕ is a function in L^∞. Then*

$$\sigma(T_\phi) \supseteq \sigma(\mathbf{T}_\phi^w) \cup \sigma(\mathbf{T}_\phi^z).$$

THEOREM 5. *Suppose $1 < p < \infty$.*

(1) Suppose $\phi = \bar{h}/h$ for some nonzero function h in H^p. If T_ϕ is left invertible on H^p, then $|h|^p$ satisfies (A_p)-condition.

(2) Suppose k is an invertible function in H^∞, h is an outer function in H^∞ and $|h|^p$ satisfies (A_p)-condition. If $\phi = k\bar{h}/h$, then T_ϕ is invertible on H^p.

Proof. (1) If T_ϕ is invertible on H^p, then there exists a positive constant γ such that

$$\gamma \|\phi f + \bar{g}\|_p \geqslant \|f\|_p \quad (f \in H,\ g \in K).$$

As in the proof of Theorem 3, for any nonnegative integer n,

$$\gamma^p \int |\phi \bar{z}^n f + \bar{z}^n \bar{g}|^p dm \geqslant \int |\bar{z}^n f|^p dm$$

where $f \in H$ an $g \in (w\mathbf{H}_w^\infty) \cap L$, and so we can show that

$$\gamma\|\phi f + \bar{g}\|_p \geqslant \|f\|_p \quad (f \in \mathbf{H}_w^\infty,\ g \in w\mathbf{H}_w^\infty).$$

and

$$\gamma^p \int |h^{-1}f + \bar{h}^{-1}\bar{g}|^p |h|^p dm \geqslant \int |h^{-1}f|^p |h|^p dm$$

where $f \in \mathbf{H}_w^\infty$ and $g \in w\mathbf{H}_w^\infty$. For any $F \in \mathbf{H}_w^\infty$ and any $G \in w\mathbf{H}_w^\infty$,

$$\inf_{f \in \mathbf{H}_w^\infty} \int |h^{-1}f - F|^p |h|^p dm = \inf_{f \in \mathbf{H}_w^\infty} \int |f - hF|^p dm = 0$$

and

$$\inf_{g \in w\mathbf{H}_w^\infty} \int |\bar{h}^{-1}\bar{g} - \bar{G}|^p |h|^p dm = \inf_{g \in w\mathbf{H}_w^\infty} \int |\bar{g} - \bar{h}\bar{G}|^p dm = 0.$$

Hence

$$\gamma^p \int |F + \bar{G}|^p |h|^p dm \geqslant \int |F|^p |h|^p dm \quad (F \in \mathbf{H}_w^\infty,\ G \in w\mathbf{H}_w^\infty).$$

By the same argument, we can give the above inequality for $\mathbf{H}_z^\infty + \bar{z}\bar{\mathbf{H}}_z^\infty$ instead of $\mathbf{H}_w^\infty + \bar{w}\bar{\mathbf{H}}_w^\infty$. By Lemma 3, $|h|^p$ satisfies (A_p)-condition.

(2) Since $T_\phi = T_{\frac{\bar{h}}{h}} T_k$ and T_k is invertible on H^p, we may assume that $\phi = \bar{h}/h$. If $|h|^p$ satisfies (A_p)-condition, by Theorem 3

$$\int |f|^p |h|^p dm \leqslant \gamma_p \int |f + \bar{g}|^p |h|^p dm \quad (f \in H,\ g \in K)$$

and so

$$\int |hf|^p dm \leqslant \gamma_p \int |\phi hf + \bar{h}\bar{g}|^p dm \quad (f \in H,\ g \in K).$$

Since h is outer, h^{-1} belongs to N_*. Since $|h|^p$ satisfies (A_p)-condition, h^{-1} belongs to $N_* \cap L^p = H^p$. This implies that hH^p is dense in H^p because $h \in H^\infty$. Thus

$$\int |F|^p dm \leqslant \gamma_p \int |\phi F + \bar{G}|^p dm \quad (F \in H,\ G \in K).$$

This implies that T_ϕ is left invertible because $L^p/[\bar{K}]_p \cong H^p$. If we can prove that $[T_\phi(H)]_p = [P(\bar{h}H)]_p = H^p$, then the invertibity of T_ϕ follows.

Let $h = \sum_{j=0}^{\infty} h_j$ be a homogeneous expansion of h where h_j is a homogeneous polynomial of degree j. Since h is outer, h_0 is a nonzero constant and $1 \in [P(\bar{h}H)]_p$. $P(z\bar{h}) = z\bar{h}_0 + P(z\bar{h}_1) = z\bar{h}_0 + c$ for some constant c because

$$z\bar{h} = z\bar{h}_0 + z\bar{h}_1 + z\sum_{j=2}^{\infty} \bar{h}_j.$$

Hence $z \in [P(\bar{h}H)]_p$ because $1 \in [P(\bar{h}H)]_p$. Similarly $w \in [P(\bar{h}H)]_p$. $P(z^2\bar{h}) = z^2\bar{h}_0 + P(z^2\bar{h}_1 + z^2\bar{h}_2) = z^2\bar{h}_0 + cz + d$ for some constant c and d because

$$z^2\bar{h} = z^2\bar{h}_0 + z^2(\bar{h}_1 + \bar{h}_2) + z^2 \sum_{j=3}^{\infty} \bar{h}_j$$

Hence $z^2 \in [P(\bar{h}H)]_p$ because $1,\ z \in [P(\bar{h}H)]_p$. Similarly w^2 and zw belong to $[P(\bar{h}H)]_p$. By repeating this method, we can prove that $H \subset [P(\bar{h}H)]_p$.

COROLLARY 7. *Suppose $\phi = \phi_w\phi_z$ is a function in L^∞ where $\phi_w \in L^\infty(T_w, m_w)$ and $\phi_z \in L^\infty(T_z, m_z)$. T_ϕ is invertible on H^p if and only if T_{ϕ_w} is invertible on $H^p(T_w, m_w)$ and T_{ϕ_z} is invertible on $H^p(T_z, m_z)$.*

Proof. If T_ϕ is invertible on H^p, then both T_{ϕ_z} and T_{ϕ_w} are invertible on H^p, and there exists a positive constant ε such that

$$\int |\phi f + \bar{g}|^p dm \geqslant \varepsilon \int |f|^p dm$$

for $f \in H^p$ and $g \in w\mathbf{H}_w^\infty$. This implies that there exists a positive constant ε' such that

$$\int |\phi_w f + \bar{g}|^p dm \geqslant \varepsilon' \int |f|^p dm$$

for $f \in H^p$ and $g \in w\mathbf{H}_w^\infty$ because $\phi_z^{-1} w\mathbf{H}_w^\infty \subseteq w\mathbf{H}_w^\infty$. Hence T_{ϕ_w} is left invertible on $H^p(T_w, m_w)$. It is easy to see that $T_{\phi_w}H^p(T_w, m_w)$ is dense in $H^p(T_w, m_w)$. Thus T_{ϕ_w} is invertible on $H^p(T_w, m_w)$. Similarly T_{ϕ_z} is also invertible on $H^p(T_z, m_z)$.

Conversely if both T_{ϕ_w} and T_{ϕ_z} are invertible on $H^p(T_w, m_w)$ and $H^p(T_z, m_z)$ respectively, then by a theorem of R. Rochberg [13] $\phi = \phi_w\phi_z$ satisfies the condition in (2) of Theorem 5. Hence T_ϕ is invertible on H^p.

REMARK.

(1) Suppose $\phi = (2 - \bar{z}w)/(2 - z\bar{w})$. By Theorem 4, $\mathbf{T}_\phi^w$ and $\mathbf{T}_\phi^z$ are invertible on $\mathbf{H}_w^p$ and $\mathbf{H}_z^p$, respectively.

(2) If ϕ is a unimodular function and $\|\phi + \mathbf{H}_w^\infty\| + \|\phi + \mathbf{H}_z^\infty\| < 1$, then T_ϕ is left invertible on H^2. For by Theorem 1 $\|H_\phi\| < 1$ and so $\|1 - T_\phi^* T_\phi\| < 1$ because $T_\phi^* T_\phi + H_\phi^* H_\phi = I$. Suppose $\phi_a = (a - \bar{z}w)/(a - z\bar{w})$ and $|a| \geqslant 2$. Then $\|\phi_a + \mathbf{H}_w^\infty\| + \|\phi_a + \mathbf{H}_z^\infty\| < 1$ for some a and then $\|\bar{\phi_a} + \mathbf{H}_w^\infty\| + \|\bar{\phi_a} + \mathbf{H}_z^\infty\| < 1$. This implies that T_{ϕ_a} is invertible on H^2.

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