INTEGRAL MEAN ESTIMATES FOR POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract. Let \( P(z) \) be a polynomial of degree \( n \) which does not vanish in the disk \( |z| < K \). For \( K = 1 \), it is known that for \( 0 < q < \infty \),

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(Re^{i\theta})|^q \, d\theta \right\}^{1/q} \leq B_q \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^q \, d\theta \right\}^{1/q},
\]

where

\[
B_q = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left| 1 + R^n e^{i\alpha} \right|^q \, d\alpha \right\}^{1/q} \left/ \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left| 1 + e^{i\alpha} \right|^q \, d\alpha \right\}^{1/q} \right..
\]

In this paper we present a generalization of this result by considering the case \( K \geq 1 \). We shall also prove a similar result for polynomials having all their zeros in \( |z| \leq K \), where \( K \geq 1 \).

1. Introduction and statement of results

Let \( P(z) \) be a polynomial of degree at most \( n \), then for each \( R \geq 1 \) and \( q > 0 \),

\[
\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| \tag{1}
\]

and

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(Re^{i\theta})|^q \, d\theta \right\}^{1/q} \leq R^n \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^q \, d\theta \right\}^{1/q}. \tag{2}
\]

Inequality (1) is a simple deduction from the maximum modulus principle (see [11, p. 346], or [8, Vol. 1, p. 137, prob. III 269]) and inequality (2) is a simple consequence of a classical result of Hardy [6] (see for example [9, Theorem 5.5]).

In both (1) and (2) equality holds only for \( P(z) = cz^n \), \( c \neq 0 \), i.e., when all the zeros of \( P(z) \) lie at the origin. Inequality (1) can be obtained by letting \( q \to \infty \) in inequality (2). The inequalities (1) and (2) can be sharpened if we restrict ourselves to...
a class of polynomials having no zeros in \(|z| < 1\). In fact, if \(P(z) \neq 0\) for \(|z| < 1\), then it was shown by Ankeny and Rivlin [1] that (1) can be replaced by

\[
\max_{|z|=R>1} |P(z)| \leq \frac{R^n + 1}{2}\max_{|z|=1} |P(z)|,
\]

(3)

where the corresponding refinement of (2) namely

\[
\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(Re^{i\theta})|^q d\theta \right\}^{1/q} \leq c_q \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q},
\]

(4)

where

\[
c_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + R^n e^{i\alpha} \right|^q d\alpha \right\}^{1/q} \bigg/ \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^q d\alpha \right\}^{1/q},
\]

was proved by Boas and Rahman [5] for \(1 < q < \infty\). Recently Rahman and Schmeisser [10] have shown that (4) remains true for \(0 < q < 1\) as well. It can be easily seen that if we let \(q \to \infty\) in (4), we get inequality (3).

Here we consider a class of polynomials having no zeros in \(|z| < K\), where \(K \geq 1\) and prove the following generalization of (4).

**THEOREM 1.** If \(P(z)\) is a polynomial of degree \(n\) having all its zeros in \(|z| \geq K\), then for every \(R > 1\) and \(q > 0\),

\[
\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(Re^{i\theta})|^q d\theta \right\}^{1/q} \leq B_q \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q},
\]

(5)

where

\[
B_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + R^n e^{i\alpha} \right|^q d\alpha \right\}^{1/q} \bigg/ \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + tKe^{i\alpha} \right|^q d\alpha \right\}^{1/q}
\]

with \(t_K = \left(\frac{1+RK}{R+K}\right)^n\).

Inequality (5) reduces to (4) for \(0 < q < \infty\) when \(K = 1\).

**REMARK 1.** Letting \(q \to \infty\) in (5), it follows that if \(P(z)\) is a polynomial of degree \(n\) having all its zeros in \(|z| \geq K\), then for \(R > 1\),

\[
\max_{|z|=R} |P(z)| \leq \frac{(R+K)^n(R^n+1)}{(1+RK)^n+(R+K)^n}\max_{|z|=1} |P(z)|.
\]

(6)

Inequality (6) is a generalization of a result of Ankeny and Rivlin [1], proved by Aziz [4].

If \(P(z)\) has all its zeros in \(|z| \leq 1\), then for each \(q > 0\),

\[
n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|.
\]

(7)
Inequality (7) is due to Malik [7]. As an extension of (7) Aziz [3] proved that if $P(z)$ has all zeros in $|z| \leq K$, where $K \geq 1$, then for each $q > 1$,

$$ n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + K^q e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \tag{8} $$

Since in the proof of the inequality (8), the inequality (4) proved by Boas and Rahman for $1 \leq q < \infty$ was used, it was not clear, whether the restriction on $q$ was indeed essential. Here we use Theorem 1 to show that the restriction on $q$ is not needed. In fact we establish the following generalization of (7) which shows that (8) remains true for $0 < q < 1$ also. We prove

**Theorem 2.** If $P(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq K$, where $K \geq 1$, then for each $q > 0$,

$$ n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + K^q e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \tag{9} $$

The result is best possible and equality holds for the polynomial $P(z) = \alpha z^n + \beta K^n$, where $|\alpha| = |\beta|$.

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2. Lemmas

The proof of Theorem 1 is based on a result of Arestov which we shall describe first.

For $\delta = (\delta_0, \delta_1, \ldots, \delta_n) \in \mathbb{C}^{n+1}$ and

$$ P(z) = \sum_{j=0}^{n} a_j z^j, $$

we define

$$ \Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j. $$

The operator $\Lambda_\delta$ is said to be admissible, if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$

(ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov may now be stated as follows:

**Lemma 1.** [2. Theorem 4]. Let $\phi(x) = \psi(\log x)$, where $\psi$ is a convex nondecreasing function on $\mathbb{R}$. Then for all polynomials $P(z)$ of degree at most $n$ and each admissible operator $\Lambda_\delta$

$$ \int_0^{2\pi} \phi\left( |\Lambda_\delta P(e^{i\theta})| \right) d\theta \leq \int_0^{2\pi} \phi(c(\delta, n)|P(e^{i\theta})|) d\theta, \tag{10} $$
where
\[ c(\delta, n) = \max(|\delta_0|, |\delta_n|). \]

In particular, Lemma 1 applies with \( \phi : x \to x^q \) for every \( q \in (0, \infty) \) and \( \phi : x \to \log x \) as well. Therefore, we have
\[
\left\{ \frac{1}{2\pi} \int_0^{2\pi} |\Lambda_0P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq c(\delta, n) \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q}, \quad 0 < q < \infty. \tag{11}
\]

We also need

**Lemma 2.** If \( P(z) \) is a polynomial of degree \( n \), which does not vanish for \( |z| \leq K \), \( K \geq 1 \). Then for all \( R \geq 1 \), \( r \leq 1 \), and for every \( \theta \), \( 0 \leq \theta < 2\pi \),
\[
|P(Rr e^{i\theta})| \leq \left( \frac{Rr + K}{r + RK} \right)^n |R^n P\left( \frac{re^{i\theta}}{R} \right)|. \tag{12}
\]

**Proof.** Since all the zeros of \( P(z) \) lie in \( |z| \geq K \), \( K \geq 1 \), we write
\[
P(z) = c \prod_{j=1}^{n} (z - R_j e^{i\theta_j}) \quad \text{where} \quad R_j \geq K, \quad j = 1, 2, \ldots, n.
\]

Therefore, for all \( R \geq 1 \), \( r \leq 1 \), and for every \( \theta \) with \( 0 \leq \theta < 2\pi \), we have
\[
\left| \frac{P(Rr e^{i\theta})}{R^n P\left( \frac{re^{i\theta}}{R} \right)} \right| = \prod_{j=1}^{n} \left| \frac{Rr e^{i\theta} - R_j e^{i\theta_j}}{re^{i\theta} - RR_j e^{i\theta_j}} \right|
\]
\[
= \prod_{j=1}^{n} \left| \frac{Rr e^{(\theta - \theta_j)} - R_j}{re^{(\theta - \theta_j)} - RR_j} \right|
\]
\[
= \prod_{j=1}^{n} \left( \frac{R^2 r^2 + R_j^2 - 2RR_j e^{i\theta} \cos(\theta - \theta_j)}{r^2 + R^2 R_j^2 - 2RR_j e^{i\theta} \cos(\theta - \theta_j)} \right)^{1/2}. \tag{13}
\]

Now, after a short calculation one can easily verify that for every \( r \leq 1 \) and \( R \geq 1 \),
\[
\frac{R^2 r^2 + R_j^2 - 2RR_j \cos(\theta - \theta_j)}{r^2 + R^2 R_j^2 - 2RR_j \cos(\theta - \theta_j)} \leq \left( \frac{Rr + R_j}{r + RR_j} \right)^2. \tag{14}
\]

Since \( R_j \geq K \), we see that
\[
\frac{Rr + R_j}{r + RR_j} \leq \frac{Rr + K}{r + KR}. \tag{15}
\]

From (13), (14) and (15), it follows that
\[
\left| \frac{P(Rr e^{i\theta})}{R^n P\left( \frac{re^{i\theta}}{R} \right)} \right| \leq \left( \frac{Rr + K}{r + KR} \right)^n,
\]
for all \( r \leq 1 \leq R \) and for every \( \theta \), \( 0 \leq \theta < 2\pi \), from which the desired result follows immediately.
3. Proofs of the theorems

Proof of Theorem 1. Since the polynomial $P(z)$ has all its zeros in $|z| \geq K \geq 1$, it follows from Lemma 2 that for every $R \geq 1$ and for $|z| = r < 1$

$$|P(Rz)| \leq \left( \frac{R|z| + K}{|z| + RK} \right)^n |R^n P(z/R)|. \quad (16)$$

If $R = 1$, then Theorem 1 is trivial, so we assume that $R > 1$. Now, it can be easily verified that

$$\frac{R|z| + K}{|z| + RK} < 1, \quad \text{for } |z| = r < 1 \text{ and } R > 1.$$  

Using this in (16), we get

$$|P(Rz)| < |R^n P(z/R)| \quad \text{for } |z| < 1 \text{ and } R > 1. \quad (17)$$

Let $F(z) = P(Rz) + e^{i\alpha R^n P(z/R)}$. We show for every $\alpha$, $0 \leq \alpha < 2\pi$ and $R > 1$, that polynomial $F(z)$ does not vanish in $|z| < 1$. If this is not true, then there is a point $z = z_0$ with $|z_0| < 1$, such that $F(z_0) = 0$. This gives

$$0 = F(z_0) = P(Rz_0) + e^{i\alpha R^n P(z_0/R)} \quad \text{where } |z_0| < 1.$$  

This implies

$$|P(Rz_0)| = |R^n P(z_0/R)| \quad \text{where } |z_0| < 1,$$

which clearly contradicts (17). Hence all zeros of $F(z) = P(Rz) + e^{i\alpha R^n P(z/R)}$ lie in $|z| \geq 1$, for every $\alpha$, $0 \leq \alpha < 2\pi$ and $R > 1$. This shows that the operator $\Lambda_{\delta}$ defined by

$$\Lambda_{\delta} P(z) = (1 + e^{i\alpha R^n})a_0 + (R + e^{i\alpha R^{n-1}})a_1 z + \cdots + (R^n + e^{i\alpha})a_n z^n = P(Rz) + e^{i\alpha R^n P(z/R)} \quad (18)$$

is an admissible operator. Applying (11), we obtain for $0 < q < \infty$

$$\int_0^{2\pi} |P(R e^{i\theta}) + e^{i\alpha R^n P(e^{i\theta}/R)}|^q d\theta \leq |R^n e^{i\alpha} + 1|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \quad (19)$$

Integrating both sides of (19) with respect to $\alpha$ from 0 to $2\pi$, we get for $0 < q < \infty$,

$$\int_0^{2\pi} \int_0^{2\pi} |P(R e^{i\theta}) + e^{i\alpha R^n P(e^{i\theta}/R)}|^q d\alpha d\theta \leq \int_0^{2\pi} |R^n e^{i\alpha} + 1|^q d\alpha \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \quad (20)$$

Now for every real $\alpha$ and $t \geq s \geq 1$, it can be easily verified that $|t + e^{i\alpha}| \geq |S + e^{i\alpha}|$, which implies for every $q > 0$,

$$\int_0^{2\pi} |t + e^{i\alpha}|^q d\alpha \geq \int_0^{2\pi} |S + e^{i\alpha}|^q d\alpha. \quad (21)$$
Taking \( r = 1 \) in Lemma 2, it follows from (12) that
\[
\left| \frac{R^n P(e^{i\theta}/R)}{P(Re^{i\theta})} \right| \geq \left( \frac{1 + RK}{R + K} \right)^n = t_K \geq 1,
\]  
for every \( \theta \), \( 0 \leq \theta < 2\pi \) and \( R > 1 \).

We take \( t = \frac{R^n P(e^{i\theta}/R)}{P(Re^{i\theta})} \) and \( S = t_K \), then from (22), \( t \geq t_K \geq 1 \) and we get with the help of (21),
\[
\int_0^{2\pi} |P(Re^{i\theta}) + e^{i\alpha} R^n P(e^{i\theta}/R)|^q d\alpha = |P(Re^{i\theta})|^q \int_0^{2\pi} \left| 1 + \frac{e^{i\alpha} R^n P(e^{i\theta}/R)}{P(Re^{i\theta})} \right|^q d\alpha \\
= |P(Re^{i\theta})|^q \int_0^{2\pi} e^{i\alpha} \left| R^n P(e^{i\theta}/R) \right| + 1|^q d\alpha \\
\geq |P(Re^{i\theta})|^q \int_0^{2\pi} t_K e^{i\alpha} + 1|^q d\alpha.
\]

Using this in (20), we conclude that for \( 0 < q < \infty \),
\[
\int_0^{2\pi} t_K e^{i\alpha} + 1|^q d\alpha \int_0^{2\pi} |P(Re^{i\theta})|^q d\theta \leq \int_0^{2\pi} R^n e^{i\alpha} + 1|^q d\alpha \int_0^{2\pi} |P(e^{i\theta})|^q d\theta,
\]
which immediately leads to (5) and this completes the proof of Theorem 1.

**Proof of Theorem 2.** The proof of Theorem 2 is identical with the proof of Theorem 1 of [3], except instead of using result of Boas and Rahman (inequality (4)) for \( 1 \leq q < \infty \), we use Theorem 1 with \( K = 1 \) for \( 0 < q < \infty \). We omit the details.

**REFERENCES**


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