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INEQUALITIES FOR CERTAIN FINITE DIFFERENCE AND SUM–DIFERENCE EQUATIONS

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Abstract. The aim of the present paper is to establish some basic finite difference inequalities which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in applications in the theory of certain finite difference and sum–difference equations.

1. Introduction

During the past few years the study of finite difference equations has acquired a new signifiance. As a consequence, the study of various classes of finite difference equations has led to the discovery of a number of new finite difference inequalities. In 1993. Pachpatte [6] proved the following useful finite difference inequality.

LEMMA. *Let* $u(n)$ *and* $f(n)$ *be real–valued nonnegative functions defined on* $N_0 = \{0, 1, 2, \dots\}$ *and* $c \geqslant 0$ *is a real constant. If*

$$u^2(n) \leqslant c + \sum_{s=n+1}^{\infty} f(s)u(s),$$

for $n \in N_0$, *then*

$$u(n) \leqslant \sqrt{c} + \frac{1}{2}\sum_{s=n+1}^{\infty} f(s),$$

for $n \in N_0$.

In many cases when studying the behaviour of solutions of certain classes of finite difference and sum–difference equations, the bounds provided by the earlier inequalities are inadequate in applications and we need some new and specific type of finite difference inequalities. In this paper, we offer some basic finite difference inequalities in one and two independent variables which claim their origin to the inequality given in Lemma and can be used more conveniently in the study of certain classes of finite difference and sum–difference equations.

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2. Main results

In what follows R denotes the set of real numbers and $R_+ = [0,\infty)$, $N_0 = \{0,1,2,\dots\}$ are the given subsets of R. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. We assume that all the functions which appear in the inequalities are real–valued and all the sums and products involved exist on the respective domains of their definations.

An interesting and useful inequality is established in the following theorem.

THEOREM 1. *Let $u(n), a(n), b(n), f(n), g(n)$ be nonnegative functions defined for $n \in N_0$ and $p > 1$ is a real constant. If*

$$u^p(n) \leqslant a(n) + b(n) \sum_{s=n+1}^{\infty} [f(s)u(s) + g(s)], \tag{2.1}$$

for $n \in N_0$, then

$$u(n) \leqslant \left[a(n) + b(n)A(n) \prod_{s=n+1}^{\infty} \left(1 + \frac{b(s)}{p} f(s)\right)\right]^{1/p}, \tag{2.2}$$

for $n \in N_0$, where

$$A(n) = \sum_{s=n+1}^{\infty} \left[f(s)\left(\frac{p-1}{p} + \frac{a(s)}{p}\right) + g(s)\right], \tag{2.3}$$

for $n \in N_0$.

Proof. Define a function $z(n)$ by

$$z(n) = \sum_{s=n+1}^{\infty} [f(s)u(s) + g(s)]. \tag{2.4}$$

Then (2.1) can be written as

$$u^p(n) \leqslant a(n) + b(n)z(n). \tag{2.5}$$

From (2.5) and using the elementary inequality (see [4,p.30])

$$x^{1/p}y^{1/q} \leqslant \frac{x}{p} + \frac{y}{q},$$

where $x \geqslant 0$, $y \geqslant 0$ and $1/p + 1/q = 1$ with $p > 1$, we observe that

$$\begin{aligned} u(n) &\leqslant [a(n) + b(n)z(n)]^{1/p}[1]^{1/p/(p-1)} \\ &\leqslant \frac{p-1}{p} + \frac{a(n)}{p} + \frac{b(n)}{p} z(n). \end{aligned} \tag{2.6}$$

From (2.4) and (2.6) we have

$$z(n) \leqslant \sum_{s=n+1}^{\infty}\Big[f(s)\Big(\frac{p-1}{p}+\frac{a(s)}{p}+\frac{b(s)}{p}z(s)\Big)+g(s)\Big]$$
$$= A(n)+\sum_{s=n+1}^{\infty} f(s)\frac{b(s)}{p}z(s), \tag{2.7}$$

where $A(n)$ is defined by (2.3). Clearly $A(n)$ is nonnegative and nonincreasing function for $n \in N_0$. First we assume that $A(n) > 0$ for $n \in N_0$. From (2.7) we observe that

$$\frac{z(n)}{A(n)} \leqslant 1+\sum_{s=n+1}^{\infty} f(s)\frac{b(s)}{p}\frac{z(s)}{A(s)}. \tag{2.8}$$

Define a function $v(n)$ by the right side of (2.8), then $\dfrac{z(n)}{A(n)} \leqslant v(n)$ and

$$v(n)-v(n+1) = f(n+1)\frac{b(n+1)}{p}\frac{z(n+1)}{A(n+1)}$$
$$\leqslant f(n+1)\frac{b(n+1)}{p}v(n+1),$$

i.e.

$$v(n) \leqslant [1+f(n+1)\frac{b(n+1)}{p}]v(n+1). \tag{2.9}$$

By setting $n = s$ and substituting $s = n, n+1, \ldots, m-1$ ($m \geqslant n+1$ is arbitrary in N_0), successively, we get

$$v(n) \leqslant v(m)\prod_{s=n+1}^{m}\Big[1+f(s)\frac{b(s)}{p}\Big]. \tag{2.10}$$

Noting that $\lim_{m\to\infty} v(m) = 1$ and letting $m \to \infty$ in (2.10) we get

$$v(n) \leqslant \prod_{s=n+1}^{\infty}\Big[1+f(s)\frac{b(s)}{p}\Big]. \tag{2.11}$$

Using (2.11) in $\dfrac{z(n)}{A(n)} \leqslant v(n)$, we get

$$z(n) \leqslant A(n)\prod_{s=n+1}^{\infty}\Big[1+f(s)\frac{b(s)}{p}\Big]. \tag{2.12}$$

If $A(n)$ is nonnegative, then we carry out the above procedure with $A(n)+\epsilon$ instead of $A(n)$, where $\epsilon > 0$ is arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (2.12). The desired inequality in (2.2) follows from (2.5) and (2.12).

We next establish the following inequality which can be used in certain situations.

THEOREM 2. *Let* $u(n)$, $a(n)$, $b(n)$ *be noneqative functions defined for* $n \in N_0$, $F : N_0 \times R_+ \to R_+$ *be a function which satisfies the condition*

$$0 \leqslant F(n,u) - F(n,v) \leqslant G(n,v)(u-v),$$

for $u \geqslant v \geqslant 0$, *where* $G(n,v)$ *is a nonnegative function defined for* $n \in N_0$, $v \in R_+$ *and* $p > 1$ *is real constant. If*

$$u^p(n) \leqslant a(n) + b(n) \sum_{s=n+1}^{\infty} F(s,u(s)), \tag{2.13}$$

for $n \in N_0$, *then*

$$u(n) \leqslant \left[a(n) + b(n)B(n) \prod_{s=n+1}^{\infty} \left[1 + G\Big(s, \frac{p-1}{p} + \frac{a(s)}{p}\Big)\frac{b(s)}{p}\right]\right]^{1/p}, \tag{2.14}$$

for $n \in N_0$, *where*

$$B(n) = \sum_{s=n+1}^{\infty} F\Big(s, \frac{p-1}{p} + \frac{a(s)}{p}\Big), \tag{2.15}$$

for $n \in N_0$.

Proof. Define a function $z(n)$ by

$$z(n) = \sum_{s=n+1}^{\infty} F(s,u(s)). \tag{2.16}$$

Then as in the proof of Theorem 1, from (2.13) we see that the inequalities (2.5) and (2.6) hold. From (2.16), (2.6) and the conditions on F it follows that

$$\begin{aligned} z(n) &\leqslant \sum_{s=n+1}^{\infty} \Big[F\Big(s, \frac{p-1}{p} + \frac{a(s)}{p} + \frac{b(s)}{p}z(s)\Big) - F\Big(s, \frac{p-1}{p} + \frac{a(s)}{p}\Big) \\ &\qquad + F\Big(s, \frac{p-1}{p} + \frac{a(s)}{p}\Big)\Big] \\ &\leqslant B(n) + \sum_{s=n+1}^{\infty} G\Big(s, \frac{p-1}{p} + \frac{a(s)}{p}\Big)\frac{b(s)}{p}z(s), \end{aligned} \tag{2.17}$$

where $B(n)$ is defined by (2.15). The rest of the proof can be completed by closely looking at the proof of Theorem 1 given above. Here we omit the further details.

3. Two independent variable generalizations

In this section, we establish two independent variable versions of Theorems 1 and 2 which can be used in the study of certain partial finite difference and sum–difference equations.

THEOREM 3. *Let* $u(m,n), a(m,n), b(m,n), f(m,n), g(m,n)$ *be nonnegative functions defined for* $m,n \in N_0$ *and* $p > 1$ *is a real constant. If*

$$u^p(m,n) \leqslant a(m,n) + b(m,n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [f(s,t)u(s,t) + g(s,t)], \tag{3.1}$$

for $m,n \in N_0$, *then*

$$u(m,n) \leqslant \left[a(m,n) + b(m,n)e(m,n) \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} f(s,t) \frac{b(s,t)}{p} \right] \right]^{1/p}, \tag{3.2}$$

for $m,n \in N_0$, *where*

$$e(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f(s,t) \left(\frac{p-1}{p} + \frac{a(s,t)}{p} \right) + g(s,t) \right], \tag{3.3}$$

for $m,n \in N_0$.

Proof. Define a function $z(m,n)$ by

$$z(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [f(s,t)u(s,t) + g(s,t)]. \tag{3.4}$$

Then (3.1) can be written as

$$u^p(m,n) \leqslant a(m,n) + b(m,n)z(m,n). \tag{3.5}$$

As in the proof of Theorem 1, from (3.5) we get

$$u(m,n) \leqslant \frac{p-1}{p} + \frac{a(m,n)}{p} + \frac{b(m,n)}{p} z(m,n). \tag{3.6}$$

From (3.4) and (3.6) we have

$$\begin{aligned} z(m,n) &\leqslant \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f(s,t) \left(\frac{p-1}{p} + \frac{a(s,t)}{p} + \frac{b(s,t)}{p} z(s,t) \right) + g(s,t) \right] \\ &= e(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s,t) \frac{b(s,t)}{p} z(s,t), \end{aligned} \tag{3.7}$$

where $e(m,n)$ is defined by (3.3). Clearly $e(m,n)$ is a nonnegative and nonincreasing function for $m,n \in N_0$. First we assume that $e(m,n) > 0$ for $m,n \in N_0$. From (3.7) we observe that

$$\frac{z(m,n)}{e(m,n)} \leqslant 1 + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s,t) \frac{b(s,t)}{p} \frac{z(s,t)}{e(s,t)}.$$

Define a function $v(m,n)$ by

$$v(m,n) = 1 + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s,t) \frac{b(s,t)}{p} \frac{z(s,t)}{e(s,t)}, \tag{3.8}$$

then $\frac{z(m,n)}{e(m,n)} \leqslant v(m,n)$ and

$$\begin{aligned}[v(m,n)-v(m+1,n)]-[v(m,n+1)-v(m+1,n+1)]\\ = f(m+1,n+1)\frac{b(m+1,n+1)}{p}\frac{z(m+1,n+1)}{e(m+1,n+1)}\\ \leqslant f(m+1,n+1)\frac{b(m+1,n+1)}{p}v(m+1,n+1). \qquad (3.9)\end{aligned}$$

From (3.9) and using the facts that $v(m,n) > 0$, $v(m+1,n+1) \leqslant v(m+1,n)$ for $m,n \in N_0$, we observe that

$$\begin{aligned}\frac{[v(m,n)-v(m+1,n)]}{v(m+1,n)} - \frac{[v(m,n+1)-v(m+1,n+1)]}{v(m+1,n+1)}\\ \leqslant f(m+1,n+1)\frac{b(m+1,n+1)}{p} \qquad (3.10)\end{aligned}$$

keeping m fixed in (3.10), set $n = t$ and sum over $t = n, n+1, \ldots, q-1$ ($q \geqslant n+1$ is arbitrary in N_0) to obtain

$$\begin{aligned}\frac{[v(m,n)-v(m+1,n)]}{v(m+1,n)} - \frac{[v(m,q)-v(m+1,q)]}{v(m+1,q)}\\ \leqslant \sum_{t=n+1}^{q} f(m+1,t)\frac{b(m+1,t)}{p}. \qquad (3.11)\end{aligned}$$

Noting that $\lim_{q\to\infty} v(m,q) = \lim_{q\to\infty} v(m+1,q) = 1$ and letting $q \to \infty$ in (3.11) we get

$$\frac{[v(m,n)-v(m+1,n)]}{v(m+1,n)} \leqslant \sum_{t=n+1}^{\infty} f(m+1,t)\frac{b(m+1,t)}{p},$$

i.e.

$$v(m,n) \leqslant \left[1 + \sum_{t=n+1}^{\infty} f(m+1,t)\frac{b(m+1,t)}{p}\right] v(m+1,n). \qquad (3.12)$$

Now by keeping n fixed in (3.12), setting $m = s$ and substituting $s = m, m+1, \ldots, r-1$ ($r \geqslant m+1$ is arbitrary in N_0) successively and noting that $\lim_{r\to\infty} v(r,n) = 1$, we obtain the estimate

$$v(m,n) \leqslant \prod_{s=m=1}^{\infty}\left[1 + \sum_{t=n+1}^{\infty} f(s,t)\frac{b(s,t)}{p}\right]. \qquad (3.13)$$

Using (3.13) in $\frac{z(m,n)}{e(m,n)} \leqslant v(m,n)$ we get

$$z(m,n) \leqslant e(m,n) \prod_{s=m+1}^{\infty}\left[1 + \sum_{t=n+1}^{\infty} f(s,t)\frac{b(s,t)}{p}\right]. \qquad (3.14)$$

If $e(m,n)$ is nonnegative, then we carry out the above procedure with $e(m,n)+\epsilon$ instead of $e(m,n)$, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (3.14). The required inequality in (3.2) follows from (3.5) and (3.14).

THEOREM 4. *Let* $u(m,n), a(m,n), b(m,n)$ *be nonnegative functions defined for* $m,n \in N_0$, $L : N_0^2 \times R_+ \to R_+$ *be a function which satisfies the condition*

$$0 \leqslant L(m,n,u) - L(m,n,v) \leqslant M(m,n,v)(u-v),$$

for $u \geqslant v \geqslant 0$, *where* $M(m,n,v)$ *is a nonnegative function defined for* $m,n \in N_0$, $v \in R_+$ *and let* $p > 1$ *be a real constant. If*

$$u^p(m,n) \leqslant a(m,n) + b(m,n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L(s,t,u(s,t)), \tag{3.15}$$

for $m,n \in N_0$, *then*

$$u(m,n) \leqslant \Big[a(m,n) + b(m,n)E(m,n) \prod_{s=m+1}^{\infty} \Big[1 + \sum_{t=n+1}^{\infty} \times M\Big(s,t,\frac{p-1}{p} + \frac{a(s,t)}{p}\Big)\frac{b(s,t)}{p}\Big]\Big]^{1/p}, \tag{3.16}$$

for $m,n \in N_0$, *where*

$$E(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L\Big(s,t,\frac{p-1}{p} + \frac{a(s,t)}{p}\Big), \tag{3.17}$$

for $m,n \in N_0$.

The proof of this theorem follows by closely looking at the proofs of Theorems 2 and 3 given above. Here we omit the details.

4. An application

In this section, we present an immediate application of Theorem 1 to obtain the bound on the solution of a nonlinear sum–difference equation of the form

$$u^p(n) = g(n) + \sum_{s=n+1}^{\infty} h(n,s,u(s)), \tag{4.1}$$

where $p > 1$ is a real constant, $u, g : N_0 \to R$, $h : N_0^2 \times R \to R$ and

$$|g(n)| \leqslant a(n), \tag{4.2}$$

$$|h(n,s,u(s))| \leqslant b(n)f(s)|u(s)|, \quad 0 \leqslant s \leqslant n, \tag{4.3}$$

for $n, s \in N_0$, where a, b, f are as defined in Theorem 1. Let $u(n)$ be a solution of (4.1) for $n \in N_0$. From (4.1)–(4.3) it is easy to observe that

$$|u(n)|^p \leqslant a(n) + b(n) \sum_{s=n+1}^{\infty} f(s)|u(s)|. \tag{4.4}$$

Now a suitable application of Theorem 1 to (4.4) yields

$$|u(n)| \leqslant \left[a(n) + b(n)\overline{A}(n) \prod_{s=n+1}^{\infty} \left[1 + f(s)\frac{b(s)}{p} \right] \right]^{1/p}, \tag{4.5}$$

where

$$\overline{A}(n) = \sum_{s=n+1}^{\infty} f(s)\left(\frac{p-1}{p} + \frac{a(s)}{p} \right),$$

for $n \in N_0$. The righ hand side of (4.5) gives the bound on the solution of (4.1) in terms of the known functions.

In concluding, we note that bounds obtained in Theorems 1–4 are independent of the unknown functions and will have many applications to boundedness, uniqueness, continuous dependence and other properties of the solutions of certain classes of finite difference and sum–difference equations. However, various applications of these inequalities will be given elsewhere.

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