Mathematical
Inequalities
& Applications
Volume 4, Number 4 (2001), 507–513

AN INTEGRAL INEQUALITY

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(*communicated by A. M. Fink*)

Abstract. In this paper we consider a general integral inequality in measure spaces. It is motivated by some individual integral inequalities which have fundamental applications in the study of some Gronwall type integral inequalities and the corresponding integral equations. The treatment of the problem shows that the connection between these special integral inequalities is even closer than what their proofs indicates.

1. Introduction

Let $(X, \mathscr{A}, \mu)$ be a measure space. As in [4, 5], we say that the function $S : X \to \mathscr{A}$ satisfies the condition (C) if

(C1) $x \notin S(x),\ x \in X$,

(C2) if $y \in S(x)$, then $S(y) \subset S(x),\ x \in X$,

(C3) $\{(x_1, x_2) \in X^2 \mid x_2 \in S(x_1)\}$ is $\mu \times \mu$ measurable.

In [4, 5] we studied Gronwall type integral inequalities and the corresponding integral equations in measure spaces. The key to the treatment is the following integral inequality.

THEOREM 1.1. *(See [4]) Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $S : X \to \mathscr{A}$ satisfy the condition (C). If $f : X \to \mathbb{R}$ is a nonnegative and μ integrable function on X, then*

$$\int_X \left(\int_{S(x_1)} \left(\cdots \left(\int_{S(x_{n-1})} f(x_1) \ldots f(x_n) d\mu(x_n) \right) \cdots \right) d\mu(x_2) \right) d\mu(x_1)$$

$$\leqslant \frac{1}{n!} \left(\int_X f \, d\mu \right)^n, \qquad n = 2, 3, \ldots .$$

Mathematics subject classification (2000): 26D15, 28A25.

Key words and phrases: Integral inequalities, measure spaces, abstract Lebesgue integral.

A special case of the previous theorem was considered by Fink [3], and it was used to prove a Wendroff type integral inequality. Gronwall type integral inequalities in higher dimensions and systems of multidimensional Volterra integral equations was investigated by Beesack [1, 2]. The arguments depend on the next integral inequality.

THEOREM 1.2. *(See [1]) Let* $I = \{x \in \mathbb{R}^p \mid a \leqslant x \leqslant b\}$, *and let* $f : I \to \mathbb{R}$ *be a nonnegative and Lebesgue integrable function on* I. *Then*

$$\int_a^x \left(f(t) \left(\int_a^t f \right)^{n-1} \right) dt \leqslant \frac{1}{n} \left(\int_a^x f \right)^n, \quad x \in I, \quad n = 1, 2, \ldots .$$

These two integral inequalities which provide the starting point for much of the discussion in the above-mentioned papers, are particular cases of a general integral inequality. This inequality is the main result of the paper. It contains the essence of Theorem 1.1 and Theorem 1.2.

2. Preliminaries

Let $\mathbb{N}^+$ denote the set of positive integers.

Throughout this section, $(X, \mathscr{A}, \mu)$ will denote a measure space, and $S : X \to \mathscr{A}$ will be regarded as a function satisfying the condition (C).

In what follows, $\mathscr{A}$ is a σ algebra, and the μ integrable functions over $A \in \mathscr{A}$ are considered to be almost measurable on A. The n-fold product of $(X, \mathscr{A}, \mu)$ is denoted by $(X^n, \mathscr{A}^n, \mu^n)$, and it is interpreted as in [4, 5].

The results of this section are technical preliminaries to the main theorem.

DEFINITION 2.1. (a) For $A \subset X$ and $n \in \mathbb{N}^+$, let

$$H_n(A) = \{(x_1, \ldots, x_n) \in X^n \mid x_1 \in A \quad \textit{and} \quad x_k \in S(x_{k-1}), \quad k = 2, \ldots, n\}.$$

(b) For $A \subset X$ and $m, n \in \mathbb{N}^+$, let

$$B_{m,n}(A) = \big\{(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \in X^{m+n} \mid x_k \in A, \quad k = 1, \ldots, m \\ \textit{and} \quad x_{m+l} \in S(x_m), \quad l = 1, \ldots, n\big\}.$$

LEMMA 2.1. *(a) For each* $A \in \mathscr{A}$ *and for each* $n \in \mathbb{N}^+$, $H_n(A) \in \mathscr{A}^n$.
(b) For each $A \in \mathscr{A}$ *and for each* $m, n \in \mathbb{N}^+$, $B_{m,n}(A) \in \mathscr{A}^{m+n}$.

Proof. (a) See [4], p. 185.

(b) Suppose first that $m = 1$. Since $B_{1,1}(A) = H_2(A)$, the case $n = 1$ being part of (a). Let $n \in \mathbb{N}^+$ such that the result holds, and let

$$h : X^{n+2} \to X^{n+2}, \quad h(x_1, \ldots, x_{n+2}) = (x_1, x_{n+2}, x_3, \ldots, x_{n+1}, x_2).$$

Then

$$B_{1,n+1}(A) = (B_{1,n}(A) \times X) \cap \big\{(x_1, \ldots, x_{n+2}) \in X^{n+2} \mid x_{n+2} \in S(x_1)\big\} =$$

$$= (B_{1,n}(A) \times X) \cap h(H_2(x) \times X^n),$$

and hence, by the induction hypothesis, $B_{1,n+1}(A) \in \mathscr{A}^{1+(n+1)}$.

By combining this with

$$B_{m,n}(A) = A^{m-1} \times B_{1,n}(A), \quad m > 1, \quad n \in \mathbb{N}^+,$$

we obtain the result. □

We now introduce the important notion of the set $V_{l_1,\ldots,l_n}(A)$.

DEFINITION 2.2. (a) For a set A and $n \in \mathbb{N}^+$, let $D_n(A)$ be the n-fold Cartesian product of A.

(b) For $A \subset X$ and $n \in \mathbb{N}^+$ we use $C_n(A)$ to denote either $H_n(A)$ or $D_n(A)$.

Let $n \in \mathbb{N}^+$, $l_i \in \mathbb{N}^+$, $i = 1, \ldots, n$, and let $p = l_1 + \ldots + l_n$.

(c) For $(x_{(1,1)}, \ldots, x_{(1,l_1)}, \ldots, x_{(n,1)}, \ldots, x_{(n,l_n)}) \in X^p$, let $x_i = (x_{(i,1)}, \ldots, x_{(i,l_i)}) \in X^{l_i}$, $i = 1, \ldots, n$.

(d) For $A \subset X$, $V_{l_1,\ldots,l_n}(A)$ means one of the following sets

$$\Big\{(x_1, \ldots, x_n) \in X^p \mid x_1 \in C_{l_1}(A) \quad \text{and} \quad x_k \in C_{l_k}(S(x_{(k-1,l_{k-1})})), \quad k = 2, \ldots, n\Big\}.$$

The integer l_i is said to be H-type (D-type) if $C_{l_i}(\cdot) = H_{l_i}(\cdot)$ $(C_{l_i}(\cdot) = D_{l_i}(\cdot))$.

REMARK 2.1. If l_i and l_{i+1} are H-type, then

$$V_{l_1,\ldots,l_n}(A) = V_{l_1,\ldots,l_{i-1},l_i+l_{i+1},l_{i+2},\ldots,l_n}(A).$$

LEMMA 2.2. *For each $A \in \mathscr{A}$, $V_{l_1,\ldots,l_n}(A) \in \mathscr{A}^p$.*

Proof. We argue by induction on n. Since $V_{l_1}(A) = H_{l_1}(A)$ or $V_{l_1}(A) = D_{l_1}(A)$, the case $n = 1$ follows from Lemma 2.1(a). Suppose $n \in \mathbb{N}^+$ for which the result holds, and let $l_i \in \mathbb{N}^+$, $i = 1, \ldots, n+1$. If l_{n+1} is H-type, then

$$V_{l_1,\ldots,l_{n+1}}(A) = (V_{l_1,\ldots,l_n}(A) \times X^{l_{n+1}}) \cap (X^{p-1} \times H_{l_{n+1}+1}(X)),$$

while if l_{n+1} is D-type, then

$$V_{l_1,\ldots,l_{n+1}}(A) = (V_{l_1,\ldots,l_n}(A) \times X^{l_{n+1}}) \cap B_{p,l_{n+1}}(X),$$

and therefore the result follows from the induction hypothesis and Lemma 2.1. □

We now introduce some additional notions. Their signficance is given primarily by the main theorem.

DEFINITION 2.3. We consider a set $V_{l_1,\ldots,l_n}(A)$, where $A \subset X$.

(a) Let $I_{l_1,\ldots,l_n}$ denote the set of indices

$$\{(1,1), \ldots, (1,l_1), \ldots, (n,1), \ldots, (n,l_n)\},$$

with the total ordering relation $(i,j) < (k,l)$ defined to mean that either $i < k$ or $i = k$ and $j < l$.

The set of the permutations of $I_{l_1,\ldots,l_n}$ is denoted by P_p.

(b) We say that the permutations $\pi \in P_p$ and $\tau \in P_p$ are equivalent if for every l_i with type D,

$$f_i(\pi(i,1)) = \tau(i,1), \ldots, f_i(\pi(i,l_i-1)) = \tau(i,l_i-1),$$

where f_i is the unique order preserving function from $(\{\pi(i,1),\ldots,\pi(n,l_n)\},<)$ onto $(\{\tau(i,1),\ldots,\tau(n,l_n)\},<)$.

It is clear that this is an equivalence relation on P_p which is denoted by E.

(c) For each $\pi \in P_p$, let $h_\pi : X^p \to X^p$,

$$h_\pi(x_{(1,1)},\ldots,x_{(1,l_1)},\ldots,x_{(n,1)},\ldots,x_{(n,l_n)}) = (x_{\pi(1,1)},\ldots,x_{\pi(1,l_1)},\ldots,x_{\pi(n,1)},\ldots,x_{\pi(n,l_n)}).$$

LEMMA 2.3. *We consider a set $V_{l_1,\ldots,l_n}(A)$, where $A \subset X$.*

(a) Let $\pi,\tau \in P_p$, let π be equivalent to τ under E, and let

$$(i,j) = \min\{(k,l) \in I_{l_1,\ldots,l_n} \mid \pi(k,l) \neq \tau(k,l)\}.$$

Then either l_i is H-type or l_i is D-type and $j = l_i$.

(b) Each equivalence class corresponding to E has $\prod_{i=1}^{n} c_i$ elements, where

$$c_i = \frac{(l_i+\ldots+l_n)!}{(l_{i+1}+\ldots+l_n)!} \quad \text{if } l_i \text{ is } H-type \quad (for\ i=n,\ c_n = l_n!),$$

and

$$c_i = 1 + l_{i+1} + \ldots + l_n \quad \text{if } l_i \text{ is } D-type \quad (for\ i=n,\ c_n = 1).$$

Proof. These two results follow immediately from the definition of the equivalence relation E. Computation of the number in (b) is based upon easy combinatorial considerations. □

LEMMA 2.4. *We consider a set $V_{l_1,\ldots,l_n}(A)$, where $A \subset X$.*

Let $(x_1,\ldots,x_n) \in V_{l_1,\ldots,l_n}(A)$, and let (i,j), $(k,l) \in I_{l_1,\ldots,l_n}$ with $(i,j) < (k,l)$. If either l_i is H-type or l_i is D-type and $j = l_i$, then $S(x_{(k,l)}) \subset S(x_{(i,j)})$.

Proof. We first assume that $k = i$. Then $j < l$, so that l_i is H-type, and hence

$$x_{(i,l)} \in S(x_{(i,l-1)}),\ldots,x_{(i,j+1)} \in S(x_{(i,j)}).$$

It therefore follows from the condition (C2) that

$$S(x_{(i,l)}) \subset S(x_{(i,l-1)}) \subset \ldots \subset S(x_{(i,j+1)}) \subset S(x_{(i,j)}).$$

Suppose now that $i < k$ and l_i is H-type. Then we can show, as above (using the definition of $V_{l_1,\ldots,l_n}(A)$), that

i. if l_k is H-type then

$$S(x_{(k,l)}) \subset S(x_{(k,l-1)}) \subset \ldots \subset S(x_{(k,1)}) \subset S(x_{(k-1,l_{k-1})}) \subset \ldots \subset S(x_{(i,l_i)}) \subset \ldots \subset S(x_{(i,j)}),$$

ii. if l_k is D-type then

$$S(x_{(k,l)}) \subset S(x_{(k-1,l_{k-1})}) \subset \ldots \subset S(x_{(i,l_i)}) \subset \ldots \subset S(x_{(i,j)}).$$

Finally, let $i < k$, and let l_i be D-type and $j = l_i$. In this case we can use what we have already proved and the remark that $x_{(i+1,1)} \in S(x_{(i,l_i)})$ implies $S(x_{(i+1,1)}) \subset S(x_{(i,l_i)})$.

The proof is complete. □

3. Main results

By using the notions of the set $V_{l_1,\ldots,l_n}(X)$ and the equivalence relation E, we are now able to establish the following integral inequality.

THEOREM 3.1. *Let $n \in \mathbb{N}^+$, $l_i \in \mathbb{N}^+$, $i = 1,\ldots,n$, and let $p = l_1 + \ldots + l_n$. Let $f : X^p \to \mathbb{R}$ be a nonnegative and μ^p integrable function on X^p, and suppose that there exists an equivalence class Q corresponding to the equivalence relation E such that for each $\pi \in Q$, we have $f \circ h_\pi = f$. Then*

$$\int_{V_{l_1,\ldots,l_n}(X)} f\,d\mu^p \leqslant \frac{1}{|Q|} \int_{X^p} f\,d\mu^p,$$

where $|Q|$ means the number of elements in Q (see Lemma 2.3(b)).

Proof. First we show that if $\pi,\ \tau \in Q$ and $\pi \neq \tau$, then

$$h_\pi^{-1}(V_{l_1,\ldots,l_n}(X)) \cap h_\tau^{-1}(V_{l_1,\ldots,l_n}(X)) = \emptyset.$$

Suppose on the contrary that

$$\left(x_{\pi(1,1)},\ldots,x_{\pi(1,l_1)},\ldots,x_{\pi(n,1)},\ldots,x_{\pi(n,l_n)}\right) \in V_{l_1,\ldots,l_n}(X)$$

and

$$\left(x_{\tau(1,1)},\ldots,x_{\tau(1,l_1)},\ldots,x_{\tau(n,1)},\ldots,x_{\tau(n,l_n)}\right) \in V_{l_1,\ldots,l_n}(X).$$

Let

$$(i,j) = \min\left\{(k,l) \in I_{l_1,\ldots,l_n} \mid \pi(k,l) \neq \tau(k,l)\right\}.$$

By Lemma 2.3(a), either l_i is H-type or l_i is D-type, and in the latter case $j = l_i$. There exists $(k,l) \in I_{l_1,\ldots,l_n}$ for which $(i,j) < (k,l)$ and $\pi(i,j) = \tau(k,l)$. Let $(k,l)^-$ be the largest index $< (k,l)$ ($(k,l)^- = (k,l-1)$ or $(k-1,l_{k-1})$, depending on whether $1 < l \leqslant l_k$ or $l = 1$). We distinguish two cases by the type of l_k and the form of $(k,l)^-$.

Case 1. If l_k is D-type and $(k,l)^- = (k,l-1)$, then we set $\pi(s,t) = \tau(k-1,l_{k-1})$. Since $(i,j) \leqslant (k-1,l_{k-1})$, it follows that $(i,j) < (s,t)$, and therefore, by Lemma 2.4,

$$x_{\pi(i,j)} = x_{\tau(k,l)} \in S(x_{\tau(k-1,l_{k-1})}) = S(x_{\pi(s,t)}) \subset S(x_{\pi(i,j)}),$$

and this contradicts the condition (C1).

Case 2. For the remaining three possibilities, we set $\pi(s,t) = \tau((k,l)^-)$. Since $(i,j) \leqslant (k,l)^-$, it follows that $(i,j) < (s,t)$, hence, again by Lemma 2.4,

$$x_{\pi(i,j)} = x_{\tau(k,l)} \in S(x_{\tau((k,l)^-)}) = S(x_{\pi(s,t)}) \subset S(x_{\pi(i,j)}),$$

contrary to the condition (C1).

By the transformation theorem for integrals, we have

$$\int\limits_{V_{l_1,\dots,l_n}(X)} f\,d\mu^p = \int\limits_{h_\pi^{-1}(V_{l_1,\dots,l_n}(X))} f \circ h_\pi d\mu^p$$

for every $\pi \in P_p$. Further, since f is nonnegative and $f \circ h_\pi = f$ for each $\pi \in Q$, it follows from the first part of the proof that

$$|Q| \cdot \int\limits_{V_{l_1,\dots,l_n}(X)} f\,d\mu^p = \sum_{\pi \in Q} \int\limits_{h_\pi^{-1}(V_{l_1,\dots,l_n}(X))} f \circ h_\pi d\mu^p =$$

$$= \sum_{\pi \in Q} \int\limits_{h_\pi^{-1}(V_{l_1,\dots,l_n}(X))} f\,d\mu^p \leqslant \int\limits_{X^p} f\,d\mu^p,$$

and this is the required result. □

REMARK 3.1. (a) The constant $\frac{1}{|Q|}$ in the theorem above is the best possible. The following example illustrates this.

Let $X = [0,1]$, let $\mathscr{A}$ be the σ algebra of Lebesgue measurable sets in X, and let μ be the Lebesgue measure on $\mathscr{A}$. If $S : X \to \mathscr{A}$, $S(x) = [0,x[$, then the function S satisfies the condition (C). It now follows by an easy induction argument on n that

$$\int\limits_{V_{l_1,\dots,l_n}(X)} 1 d\mu^p = \frac{1}{|Q|} \int\limits_{X^p} 1 d\mu^p$$

for arbitrarily chosen $V_{l_1,\dots,l_n}(X)$.

(b) Let $(X,\mathscr{A},\mu)$ be the measure space described in (a), and let $S : X \to \mathscr{A}$ be the function of (a). We consider the set $V_{l_1,l_2}(X)$, where $l_1 = 2$, $l_2 = 1$, l_1 is D-type and l_2 is H-type. Let $f : X^3 \to \mathbb{R}$, $f(x,y,z) = x^2 + y^2 z^2$. Then the quotient class of P_3 by E has three elements, and

$$Q = \{((1,1),(1,2),(2,1)),\ ((1,1),(2,1),(1,2))\}$$

is the only equivalence class such that $f \circ h_\pi = f$ for each $\pi \in Q$.

We note explicitly the case of Theorem 3.1 in which the function f has a special form.

THEOREM 3.2. *Let $n \in \mathbb{N}^+$, $l_i \in \mathbb{N}^+$, $i = 1, \dots, n$, and let $p = l_1 + \dots + l_n$. If $g : X \to \mathbb{R}$ is a nonnegative and μ integrable function on X, then*

$$\int\limits_{V_{l_1,\dots,l_n}(X)} g \times \dots \times g d\mu^p \leqslant \frac{1}{|Q|} \left(\int\limits_X g d\mu \right)^p,$$

where Q is an equivalence class corresponding to the equivalence relation E.

REMARK 3.2. Suppose that the hypotheses of the previous theorem are satisfied. When $n = 1$ and l_1 is H-type, we can use Theorem 3.2 to obtain Theorem 1.1.

Similarly, when $n = 2$, $l_1 = 1$, l_1 is H-type and l_2 is D-type, Theorem 3.2 gives the following inequality

$$\int\limits_X \left(g(x) \left(\int\limits_{S(x)} g d\mu \right)^{l_2} \right) d\mu(x) \leqslant \frac{1}{l_2 + 1} \left(\int\limits_X g d\mu \right)^{l_2+1},$$

which includes Theorem 1.2.

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(Received September 26, 2000)

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