Mathematical
Inequalities
& Applications
Volume 4, Number 4 (2001), 515–526

CLASSES OF NUMERICAL SEQUENCES

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(*communicated by J. Pečarić*)

Abstract. We generalize some known classes of numerical sequences and give sufficient conditions implying the identity of the new classes. Some further embedding relations are presented.

1. Introduction

Telyakovskiĭ [11] introduced the following very applicable definition of a class of sequences $\{a_n\}$ and denoted by S. A null-sequence $\{a_n\}$ belongs to the *class* S if there exists a monotonically decreasing sequence $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty, \quad \text{and} \quad |\Delta a_n| \leqslant A_n \quad \text{for all } n.$$

In [11] he proved, among others, that the classical result of Kolmogorov [3] concerning the L^1-convergence of the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

with a quasi-convex null-sequence $\{a_n\}$ ($\sum n|\Delta^2 a_n| < \infty$) can be extended to the class S.

Several authors have investigated similar problems and defined "wider" classes than S and proved that the class S emerging in the theorem of Telyakovskiĭ can be replaced by the "wider" classes.

In [6] we showed that some of these classes are identical with the class S, furthermore in [7] we proved that five other classes defined by different ways are truly wider than S, but they are identical among themselves.

The aim of the present paper is to generalize the classes considered in [7] and determine such conditions on the factors appearing in the generalizations which imply

Mathematics subject classification (2000): 26D15, 40G99.

Key words and phrases: Inequalities, embedding relations, sums, quasi β-power-monotone sequences.

This result was partially supported by the Hungarian National Foundation for Scientific Research under Grant ♯ T029080.

the identity of the new classes. Furthermore we shall consider some more embedding relations of the classes to be defined soon, the class S, and its analogues.

Next we establish five classes of sequences whose special cases $\rho_n = n^{-1/p}$ were considered in [7].

Here and later on $\rho := \{\rho_n\}$ denote always a positive monotonic sequence, $\mathbf{a} := \{a_n\}$ a null-sequence ($a_n \to 0$), K and K_i positive constants that are not necessarily the same of each occurrence, furthermore $p \geqslant 1$.

1. A sequence $\mathbf{a}$ belongs to the class $F_p(\rho)$ if

$$\sum_{n=1}^{\infty} \rho_n \left(\sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p} < \infty. \tag{1.1}$$

2. $\mathbf{a} \in F_p^*(\rho)$ if

$$\sum_{m=0}^{\infty} 2^m \rho_{2^m} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{1/p} < \infty. \tag{1.2}$$

3. $\mathbf{a} \in S_p(\rho)$ if there exists a monotonically decreasing sequence $\mathbf{A} := \{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad \sum_{n=1}^{m} \frac{|\Delta a_n|^p}{A_n^p} = \mathscr{O}\left(\rho_m^{-p}\right). \tag{1.3}$$

4. $\mathbf{a} \in S_p(A, \rho)$ if there exists a null-sequence $\mathbf{A}$ such that

$$\sum_{n=1}^{\infty} n|\Delta A_n| < \infty \tag{1.4}$$

and

$$\sum_{n=1}^{m} \frac{|\Delta a_n|^p}{A_n^p} = \mathscr{O}\left(\rho_m^{-p}\right) \tag{1.5}$$

hold.

5. $\mathbf{a} \in S_p(\delta, \rho)$ if there exists a δ-quasi-monotone sequence $\mathbf{A}$ (i.e. $A_n > 0$, $\Delta A_n \geqslant -\delta_n$ and $\delta_n > 0$) satisfying the assumptions (1.3) and $\sum_{n=1}^{\infty} n\delta_n < \infty$.

We mention that these classes in the special case $\rho_n = n^{-1/p}$, $p > 1$ were defined and investigated by the following authors. $F_p(\rho)$ and $F_p^*(\rho)$ by Fomin [1], $S_p(\rho)$ by C.V. Stanojevič and V.B. Stanojevič [10], $S_p(A, \rho)$ by Garrett-Rees-C.V. Stanojevič [2], and $S_p(\delta, \rho)$ by Mazhar [9] and Tomovski [12].

Before formulating our results we recall some definitions.

We shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi* β*-power-monotone increasing* (*decreasing*) if there exist a natural number $N := N(\beta, \gamma)$ and a constant $K := K(\beta, \gamma) \geqslant 1$ such that

$$Kn^\beta \gamma_n \geqslant m^\beta \gamma_m \quad (n^\beta \gamma_n \leqslant Km^\beta \gamma_m) \tag{1.6}$$

holds for any $n \geqslant m \geqslant N$.

If (1.6) holds with $\beta = 0$ then we omit the attribute "β-power" in the definition.

Furthermore, we shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi geometrically increasing* (*decreasing*) if there exist natural numbers $\mu := \mu(\gamma)$, $N := N(\gamma)$ and a constant $K := K(\gamma) \geqslant 1$ such that

$$\gamma_{n+\mu} \geqslant 2\gamma_n \quad \text{and} \quad \gamma_n \leqslant K\gamma_{n+1} \quad \left(\gamma_{n+\mu} \leqslant \frac{1}{2}\gamma_n \quad \text{and} \quad \gamma_{n+1} \leqslant K\gamma_n\right) \tag{1.7}$$

hold for all $n \geqslant N$.

Finally a sequence $\gamma := \{\gamma_n\}$ will be called bounded by blocks if the inequalities

$$\alpha_1 \Gamma_m^{(k)} \leqslant \gamma_n \leqslant \alpha_2 \Gamma_M^{(k)}, \quad 0 < \alpha_1 < \alpha_2 < \infty$$

hold for any $2^k \leqslant n \leqslant 2^{k+1}$, $k = 1, 2, \ldots,$ where

$$\Gamma_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad \text{and} \quad \Gamma_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

2. Results

We now proceed to formulate our new results.

THEOREM 1. *Assume that $p \geqslant 1$ and a given positive sequence $\rho := \{\rho_n\}$, for a certain positive β, is quasi β-power-monotone decreasing and simultaneously quasi $(1-\beta)$-power-monotone increasing. Then the following embedding relations*

$$F_p(\rho) \subseteq F_p^*(\rho) \subseteq S_p(\rho) \subseteq S_p(A, \rho) \subseteq S_p(\delta, \rho) \subseteq F_p(\rho) \tag{2.1}$$

hold, i.e. these classes are identical.

In the special case $\rho_n = n^{-1/p}$ and $p > 1$, Theorem 1 reduces to the theorem proved in [7]. It is clear that this sequence satisfies the assumptions of Theorem 1, moreover it is easy to see that *any sequence $\rho_n = n^\gamma$ with $-1 < \gamma < 0$ also fulfills the assumptions of Theorem 1.*

It is also easy to verify that in the special case $\rho_n = n^{-1/p}$, $p > 1$, the class $F_p^*(\rho) \equiv F_p^*(n^{-1/p})$, and consequently all of the others, are wider when p is closer to 1. However no class $F_p(n^{-1/p}) \equiv F_p^*(n^{-1/p})$ with $p > 1$ is embedded into the class $F_1(n^{-1})$ $(\rho_n = n^{-1}, p = 1)$, but

$$F_p(n^{-1/p}) \equiv S_p(n^{-1/p}) \subseteq S_1(n^{-1}), \quad p > 1, \tag{2.2}$$

always holds. Namely if $\Delta a_n = 1/n \log^2 n$ then $\mathbf{a} \in F_p(n^{-1/p})(p > 1)$, but $\mathbf{a} \notin F_1(n^{-1})$, and the statement (2.2) can be verified by Hölder inequality.

We shall verify that the class $S_1(n^{-1})$ is wider than $F_1(n^{-1})$.

THEOREM 2. *The embedding*

$$F_1(n^{-1}) \subseteq S_1(n^{-1}) \tag{2.3}$$

holds.

As it has been mentioned, we ([5]) proved that

$$S \subset S_p(n^{-1/p}), \quad p > 1 \tag{2.4}$$

holds, moreover the class S a strict subclass of $S_p(n^{-1/p})$. By the definition of the classes S and $S_p(1)$ $(\rho_n \equiv 1)$ it is obvious that

$$S_p(1) \subset S \tag{2.5}$$

also holds, thus, by (2.4) and (2.5) we have

$$S_p(1) \subset S \subset S_p(n^{-1/p}), \quad p > 1. \tag{2.6}$$

On the other hand, since

$$S_p(n^{-\gamma_1}) \subset S_p(n^{-\gamma_2}) \quad \text{if} \quad \gamma_1 < \gamma_2,$$

thus, by (2.6), it is natural to ask: Are there $0 < \gamma_1 < \gamma_2 < \frac{1}{p}$ such that

$$S_p(n^{-\gamma_1}) \subset S \subset S_p(n^{-\gamma_2})$$

hold?

The following theorem gives a negative answer to this problem.

THEOREM 3. *If* $0 < \gamma < \frac{1}{p}$ $(p \geqslant 1)$ *then neither*

$$S \subset S_p(n^{-\gamma}), \tag{2.7}$$

nor

$$S_p(n^{-\gamma}) \subset S \tag{2.8}$$

hold.

Surveying the results with $p > 1$, we see that $S_p(1)$ is the smallest class among the treated ones and

$$S_p(1) \subset S.$$

This raises the following problem: Can we modify the definition of S such that a certain subclass of S be embedded in $S_p(1)$? We shall give an affirmative answer.

Now let us define a new class of sequences.

6. Let $\alpha := \{\alpha_n\}$ be a positive monotone sequence tending to infinity. We shall say that a sequence $\mathbf{a} := \{a_n\}$ belongs to $S(\alpha)$, or $\mathbf{a} \in S(\alpha)$, if there exists a monotonically decreasing sequence $\mathbf{A} := \{A_n\}$ such that

$$\sum_{n=1}^{\infty} \alpha_n A_n < \infty, \quad \text{and} \quad |\Delta a_n| \leqslant A_n \quad \text{for all } n.$$

THEOREM 4. *If* $p > 1$ *and*

$$\sum_{n=1}^{\infty} \alpha_n^{-p} < \infty, \tag{2.9}$$

then

$$S(\alpha) \subset S_p(1). \tag{2.10}$$

Furthermore there exists a sequence $\overline{\alpha} := \{\overline{\alpha}_n\}$ *tending to infinity such that*

$$S_p(1) \subset S(\overline{\alpha}) \quad (\subset S) \tag{2.11}$$

holds.

3. Lemmas

The following result can be found in [13].

LEMMA 1. *Let* $\{c_n\}$ *be a* δ*-quasi-monotone sequence with*

$$\sum_{n=1}^{\infty} n\delta_n < \infty.$$

If

$$\sum_{n=1}^{\infty} c_n$$

converges, then

$$\sum_{n=1}^{\infty} (n+1)|\Delta c_n| < \infty.$$

LEMMA 2. ([4]). *For any positive sequence* $\gamma := \{\gamma_n\}$ *the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \leqslant K\gamma_m \quad (m = 1, 2, \ldots; K \geqslant 1),$$

or

$$\sum_{n=1}^{m} \gamma_n \leqslant K\gamma_m \quad (m = 1, 2, \ldots; K \geqslant 1),$$

hold if and only if the sequence γ *is quasi geometrically decreasing or increasing, respectively.*

LEMMA 3. ([8]). *A positive sequence* $\gamma := \{\gamma_n\}$ *bounded by blocks is quasi* β*-power monotone increasing (decreasing) with a certain negative (positive) exponent* β *if and only if the sequence* $\{\gamma_{2^n}\}$ *is quasi geometrically increasing (decreasing).*

The following lemma can be found in [5] implicitly.

LEMMA 4. *If $\{R_n\}$ is a monotone decreasing sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} R_n < \infty,$$

then there exists a monotone decreasing sequence $\{A_n\}$ such that for any $n \geqslant 1$

$$R_n \leqslant A_n$$

$$A_n \leqslant K A_{2n}$$

and

$$\sum_{n=1}^{\infty} A_n < \infty.$$

4. Proofs

Proof of Theorem 1. First we prove the relation $F_p(\rho) \subseteq F_p^*(\rho)$. Since the sequence ρ is quasi monotone decreasing thus an easy consideration shows that

$$\sum_{m=1}^{\infty} 2^m \rho_{2^m} \left\{ \sum_{k=2^m+1}^{2^{m+1}} |\Delta a_k|^p \right\}^{1/p} \leqslant K \sum_{m=1}^{\infty} \sum_{n=2^{m-1}+1}^{2^m} \rho_n \left\{ \sum_{k=n}^{\infty} |\Delta a_k|^p \right\}^{1/p}, \qquad (4.1)$$

and this was to be proved.

Next we prove the converse statement, $F_p^*(\rho) \subseteq F_p(\rho)$. The assumptions on ρ imply clearly that it is bounded by blocks, and consequently the sequence $\{n\rho_n\}$ has the same property; furthermore this latter sequence is β-power-monotone increasing with a negative β. Thus, at the end of the coming calculations we can use the Lemmas 2 and 3, therefore we have that

$$\begin{aligned} \sum_{n=2}^{\infty} \rho_n \left(\sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p} &\leqslant K \sum_{m=0}^{\infty} 2^m \rho_{2^m} \left\{ \sum_{k=2^m+1}^{\infty} |\Delta a_k|^p \right\}^{1/p} \\ &\leqslant K \sum_{m=0}^{\infty} 2^m \rho_{2^m} \sum_{n=m}^{\infty} \left\{ \sum_{k=2^n+1}^{2^{n+1}} |\Delta a_k|^p \right\}^{1/p} \\ &\leqslant K \sum_{n=0}^{\infty} \left\{ \sum_{k=2^n+1}^{2^{n+1}} |\Delta a_k|^p \right\}^{1/p} \sum_{m=0}^{n} 2^m \rho_{2^m} \\ &\leqslant K_1 \sum_{n=0}^{\infty} 2^n \rho_{2^n} \left\{ \sum_{k=2^n+1}^{2^n+1} |\Delta a_k|^p \right\}^{1/p}, \end{aligned} \qquad (4.2)$$

which gives the conclusion.

The inequalities (4.1) and (4.2) imply that

$$F_p(\rho) \equiv F_p^*(\rho). \tag{4.3}$$

In the following stage we prove that $F_p(\rho) \subseteq S_p(\rho)$.
First we set

$$R_n := \rho_n \left(\sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p}.$$

The assumptions of Theorem 1 imply that the sequence $\{R_n\}$ is monotone decreasing and

$$\sum_{n=1}^{\infty} R_n < \infty$$

if $\mathbf{a} \in F_p(\rho)$. Hence, by Lemma 4, we know that there exists a monotone decreasing sequence $\{A_n\}$ such that

$$R_n \leqslant A_n \tag{4.4}$$

$$A_n \leqslant KA_{2n} \tag{4.5}$$

and

$$\sum_{n=1}^{\infty} A_n < \infty. \tag{4.6}$$

hold.

To prove $\mathbf{a} \in S_p(\rho)$ we have only to show that

$$\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} \leqslant K\rho_n^{-p} \tag{4.7}$$

with this sequence $\{A_n\}$ fulfills if $\mathbf{a} \in F_p(\rho)$. Let $2^{i-1} < n \leqslant 2^i$. Then

$$\begin{aligned} \sum_{k=2}^{n} \frac{|\Delta a_k|^p}{A_k^p} &\leqslant \sum_{m=0}^{i} A_{2^{m+1}}^{-p} \sum_{k=2^m+1}^{2^{m+1}} |\Delta a_k|^p \\ &\leqslant \sum_{m=0}^{i} A_{2^{m+1}}^{-p} R_{2^m}^p \rho_{2^m}^{-p} \leqslant K \sum_{m=0}^{i} \rho_{2^m}^{-p} = \sigma_i. \end{aligned} \tag{4.8}$$

Since the sequence $\{\rho_n\}$ is β-power monotone decreasing with a positive β, thus the sequence $\{\rho_{2^n}\}$ is quasi geometrically decreasing (see Lemma 3), consequently the sequences $\{\rho_{2^n}^{-1}\}$ and $\{\rho_{2^n}^{-p}\}$ $(p \geqslant 1)$ are quasi geometrically increasing. Thus, by Lemma 2,

$$\sigma_i \leqslant K_1 \rho_{2^i}^{-p} \leqslant K_2 \rho_n^{-p},$$

this and (4.8) imply (4.7), that is, the embedding

$$F_p(\rho) \subseteq S_p(\rho) \tag{4.9}$$

is proved.

The embedding relation

$$S_p(\rho) \subseteq S_p(\delta,\rho) \tag{4.10}$$

always holds without any additional condition on ρ, it is enough to choose $\delta_n := n^{-3}$.

Our next aim is to prove the relation

$$S_p(\delta,\rho) \subseteq S_p(A,\rho). \tag{4.11}$$

If $\mathbf{a} \in S_p(\delta,\rho)$ then we can apply Lemma 1 with $c_n := A_n$, where $\mathbf{A} := \{A_n\}$ denotes the sequence appearing in the definition of the class $S_p(\delta,\rho)$, and this shows that this sequence $\mathbf{A}$ satisfies (1.4).

On the other hand the condition (1.5) is automatically satisfied by the assumption $\mathbf{a} \in S_p(\delta,\rho)$, see (1.3).

Thus (4.11) is verified, without conditions on ρ.

Finally we verify

$$S_p(A,\rho) \subseteq F_p^*(\rho). \tag{4.12}$$

Setting

$$D_m := \sum_{n=2^m}^{2^{m+1}} |\Delta A_n|,$$

by (1.4) we obtain that

$$\sum_{m=0}^{\infty} 2^m D_m < \infty. \tag{4.13}$$

Since $A_n \to 0$ thus

$$A_{2^m} = \sum_{n=2^m}^{\infty} \Delta A_n \leqslant \sum_{n=m}^{\infty} D_n.$$

This and (4.13) imply that

$$\sum_{m=1}^{\infty} 2^m A_{2^m} \leqslant \sum_{m=1}^{\infty} 2^m \sum_{n=m}^{\infty} D_n = \sum_{n=1}^{\infty} D_n \sum_{m=1}^{n} 2^m \leqslant 2 \sum_{n=1}^{\infty} 2^n D_n < \infty. \tag{4.14}$$

If $2^m < n \leqslant 2^{m+1}$ then

$$A_n = A_{2^m} - \sum_{k=2^m}^{n-1} \Delta A_k \leqslant A_{2^m} + D_m =: C_m.$$

Using this estimation we get that

$$\sum_{m=1}^{\infty} 2^m \rho_{2^m} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{1/p} \leqslant \sum_{m=1}^{\infty} 2^m C_m \left\{ \rho_{2^m}^p \sum_{n=2^m+1}^{2^{m+1}} \frac{|\Delta a_n|^p}{A_n^p} \right\}^{1/p}. \tag{4.15}$$

Here the sum in the bracket is $\mathscr{O}(1)$ by (1.5) taking into consideration the properties of ρ. On the other hand, by (4.13) and (4.14), the sum

$$\sum_{m=1}^{\infty} 2^m C_m < \infty.$$

Thus, by (4.15), the embedding relation (4.12) is also verified.

Summing up our partial results (4.3), (4.9), (4.10), (4.11) and (4.12), we obtain the assertion (2.1) of Theorem 1.

The proof is complete.

Proof of Theorem 2. Let

$$R_n := n^{-1} \sum_{k=n}^{\infty} |\Delta a_k|.$$

If $\mathbf{a} \in F_1(n^{-1})$ then the sum of these R_n is convergent. Therefore we can apply Lemma 4 and obtain a monotone decreasing sequence $\{A_n\}$ with the properties (4.4), (4.5) and (4.6). Using these properties we get that if $2^{i-1} < n \leqslant 2^i$ then

$$\sum_{k=2}^{n} \frac{|\Delta a_k|}{A_k} \leqslant \sum_{m=0}^{i-1} A_{2^{m+1}}^{-1} \sum_{k=2^m+1}^{2^{m+1}} |\Delta a_k| \leqslant \sum_{m=0}^{i} A_{2^{m+1}}^{-1} 2^m R_{2^m} \leqslant K \sum_{m=0}^{i} 2^m \leqslant K_1 n.$$

This clearly shows that $\mathbf{a} \in S_1(n^{-1})$ also holds, herewith Theorem 2 is proved.

Proof of Theorem 3. In order to prove that (2.7) does not hold, let us define the following sequence $\mathbf{a} := \{a_n\}$:

$$a_n := \frac{1}{n^2 \log^2(n+1)} \quad \text{for all } n \geqslant 1 . \tag{4.16}$$

Then the sequence

$$A_n := \frac{K}{n \log^2 n}$$

satisfies the conditions required in the definition of the class S, therefore $\mathbf{a} \in S$.

On the other hand if a monotonic sequence $\{\overline{A}_n\}$ satisfies the condition

$$\sum_{n=1}^{m} \frac{|\Delta a_n|^p}{\overline{A}_n^p} \leqslant K m^{\gamma p}$$

for the sequence $\mathbf{a}$ given in (4.16) then

$$2K m^{\gamma p} \geqslant \sum_{n=m}^{2m} \frac{|\Delta a_n|^p}{\overline{A}_n^p} \geqslant \overline{A}_m^{-p} \sum_{n=m}^{2m} |\Delta a_k|^p \geqslant K_1 \overline{A}_m^{-p} m^{1-p} \log^{-2p} m$$

holds, whence

$$\overline{A}_m \geqslant K_2 m^{-1+\frac{1}{p}-\gamma} \log^{-2} m$$

follows, thus

$$\sum_{m=1}^{\infty} \overline{A}_m = \infty \quad \left(\frac{1}{p} > \gamma\right).$$

Consequently this $\mathbf{a}$ does not belong to $S_p(n^{-\gamma})$, but $\mathbf{a} \in S,$ that is,

$$S \not\subset S_p(n^{-\gamma}).$$

To prove that (2.8) does not hold let us consider the sequence $\mathbf{a} := \{a_n\}$ given as follows:

$$a_1 := 1, \quad a_n := 2^{-m} \quad \text{if} \quad 2^{m-1} < n \leqslant 2^m, \quad m \geqslant 1. \tag{4.17}$$

Then

$$|\Delta a_{2^m}| = 2^{-m-1},$$

therefore, for this sequence $\mathbf{a}$ given in (4.17), no sequence $\{A_n\}$ satisfying the three conditions

$$A_n \geqslant A_{n+1}, \quad |\Delta a_n| \leqslant A_n \ \text{ and } \ \sum_{m=1}^{\infty} 2^m A_m < \infty$$

jointly, required to $\mathbf{a} \in S$, can be given. Thus this sequence $\mathbf{a} \notin S$.

Next we show that this $\mathbf{a} \in S_p(n^{-\gamma})$. Let

$$\overline{A}_n := \frac{1}{n \log^2(n+1)}, \quad n \geqslant 1.$$

Clearly

$$\sum_{n=1}^{\infty} \overline{A}_n < \infty, \quad \overline{A}_n \geqslant \overline{A}_{n+1},$$

furthermore if $2^{m-1} < n \leqslant 2^m$ then

$$\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{\overline{A}_k^p} \leqslant \sum_{k=1}^{m} \frac{|\Delta a_{2^k}|^p}{\overline{A}_{2^k}^p} \leqslant K \sum_{k=1}^{m} k^{2p} \leqslant K_1 m^{2p+1} \leqslant K_2 n^{\gamma p},$$

thus $\mathbf{a} \in S_p(n^{-\gamma})$ is proved.

Since $\mathbf{a} \subset S_p(n^{-\gamma})$ but $\mathbf{a} \notin S$, this shows that

$$S_p(n^{-\gamma}) \not\subset S.$$

Herewith the proof of Theorem 3 is complete.

Proof of Theorem 4. To prove (2.10) we assume that $\mathbf{a} \in S(\alpha)$. Then there exists a monotone decreasing sequence $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} \alpha_n A_n < \infty \ \text{ and } \ |\Delta a_n| \leqslant A_n. \tag{4.18}$$

Next we show that this $\mathbf{a}$ belongs to $S_p(1)$, too. If we can give a monotone sequence $\overline{\mathbf{A}} := \{\overline{A}_n\}$ such that

$$\sum_{n=1}^{\infty} \overline{A}_n < \infty \ \text{ and } \ \sum_{n=1}^{\infty} \frac{|\Delta a_n|^p}{\overline{A}_n^p} < \infty \tag{4.19}$$

then $\mathbf{a} \in S_p(1)$ will be proved.

Let

$$\overline{A}_n := A_n^{1-\frac{1}{p}} \alpha_n^{-\frac{1}{p}}.$$

This $\overline{\mathbf{A}}$ is clearly monotone $(p > 1, \quad \alpha_n \leqslant \alpha_{n+1})$. With $q := \frac{p}{p-1}$, by (2.9) and (4.18), we get that

$$\sum_{n=1}^{\infty} \overline{A}_n = \sum_{n=1}^{\infty} A_n^{1/q} \alpha_n^{1/q-1} \leqslant \left(\sum_{n=1}^{\infty} A_n \alpha_n \right)^{1/q} \left(\sum_{n=1}^{\infty} \alpha_n^{-p} \right)^{1/p} < \infty,$$

and, by (4.18),

$$\sum_{n=1}^{\infty} \frac{|\Delta a_n|^p}{\overline{A}_n^p} = \sum_{n=1}^{\infty} \frac{|\Delta a_n|^p}{A_n^{p-1}} \alpha_n \leqslant \sum_{n=1}^{\infty} A_n \alpha_n < \infty.$$

Thus (4.19), and hereby (2.10) is also proved.

The proof of (2.11) is very easy. Namely if $\mathbf{a} \in S_p(1)$ then there exists a monotone sequence $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} \leqslant K. \tag{4.20}$$

The second condition in (4.20) implies that $|\Delta a_n| \leqslant K_1 A_n$, and the first inequality manifests the existence of a monotone sequence $\overline{\alpha} := \{\overline{\alpha}_n\}$ tending to infinity such that

$$\sum_{n=1}^{\infty} \overline{\alpha}_n A_n < \infty$$

also holds. Thus, e.g. the sequence $\overline{\mathbf{A}} := \{\overline{A}_n\}$ with $\overline{A}_n := K_1 A_n$ satisfies all of the conditions required for $\mathbf{a} \in S(\overline{\alpha})$.

Herewith (2.11) is also verified, consequently the proof of Theorem 4 is complete.

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(Received August 2, 2000)

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