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TWO–POINT BOUNDARY VALUE PROBLEMS ASSOCIATED WITH NON–LINEAR FUZZY DIFFERENTIAL EQUATIONS

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Abstract. This paper presents a criteria for the existence and uniqueness of solutions to two point boundary value problems associated with a second order non-linear fuzzy differential equations. The main tools employed are estimates on Green's function, Ascoli's Lemma and a fixed point theorem of Banach.

1. Introduction

Let X be a finite dimensional Banach Space and $f : R \times X \to X$ is continuous. Then the classical Peano existence theorem states that the initial value problem

$$y' = f(t,y), \quad y(t_0) = y_0 \tag{1.1}$$

has at least one solution on any real interval containing t_0, more specifically, let $a, b > 0$ and

$$D = \{(t,y) \in \mathbf{R} \times X : t \in [t_0, t_0 + a],\ ||y - y_0|| \leqslant b\}$$

and

$$M = \sup_D \{||f(t,y)||\}.$$

Then the initial value problem (1.1) possesses a solution defined on $I = [t_0, t_0 + \alpha]$ where $\alpha = \min\{a, b/M\}$.

Although, continuity of f will take care of existence of at least one solution on I, it does not guarantee the uniqueness. Hence to ensure uniqueness, some additional conditions on f are necessary. In fact, Picard's theorem ensures that if f is uniformly Lipschitz continuous with respect to the second variable y on D, then (1.1) has a unique solution on $[t_0, t_0 + \alpha]$.

However, if X is not locally compact, then it is possible to construct a continuous function f such that (1.1) has no solution. In fact, some additional conditions are to be satisfied by the non-linear function f. The initial value problems for first order fuzzy differential equations are studied by many authors in recent yart [2,4,5] on the metric

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space (E^n, D) of normal fuzzy convex sets with the distance D given by the supremum of the Hausdorff distance between the corresponding level set. J. J. Nieto [5] proved a version of the classical Peano existence theorem for initial value problems for a fuzzy differential equation in the metric space of normal fuzzy convex sets with the distance given by the maximum of the Hausdorff distance between level sets. The results of Nieto [5] complements the existence and uniqueness result of Kaleva [4].

In this paper we give a set of sufficient conditions under which a two-point non-linear Fuzzy differential equation has a unique solution.

2. Preliminaries

Let $\mathscr{P}_k(\mathbf{R}^n)$ denote the family of non-empty compact, convex subsets of $\mathbf{R}^n$. If α, $\beta \in \mathbf{R}$ and A, $B \in \mathscr{P}_k(\mathbf{R}^n)$

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1 \cdot A = A$$

and if α, $\beta \geqslant 0$, then $(\alpha+\beta)A = \alpha A + \beta A$.

For A, $B \in \mathscr{P}_k(\mathbf{R}^n)$ the Hausdorff metric is defined as

$$D(A,B) = \inf\{\epsilon : A \subset N(B,\epsilon), B \subset N(A,\epsilon)\}$$

where $N(A,\epsilon) = \{x \in \mathbf{R}^n : ||x-y|| < \epsilon \text{ for some } y \in A\}$.

Let $I = [t_0, t_0+\alpha]$, $t \geqslant 0$ and $\alpha > 0$ and denote by $E^n = \{u : \mathbf{R}^n \to [0,1]\}$. Then the α-level set is

$$[u]^\alpha = \{x \in \mathbf{R}^n : u(x) \geqslant \alpha\} \qquad \alpha \in (0,1];$$
$$[u]^0 = \{x \in \mathbf{R}^n : u(x) > 0\} \quad \text{is compact.}$$

We define E^n as the fuzzy sets $u : \mathbf{R}^n \to [0,1]$ that are normal, fuzzy convex, upper semicontinuous, and such that $[u]^0$ is compact. Thus, if $u \in E^n$ we have $[u]^\alpha \in \mathscr{P}_k(\mathbf{R}^n)$ for every $\alpha \in [0,1]$. Further if

$$D : E^n \times E^n \to [0,\infty) \quad \text{then}$$
$$D(u,v) = \sup\{d([u]^\alpha), d([v]^\alpha) : \alpha \in [0,1]\}.$$

It may be noted that D is a metric in E^n and that (E^n, D) is a complete metric space, but it is not locally compact. Moreover, D has a linear structure in the sense that if u, v, $w \in E^n$ and $\lambda \in \mathbf{R}$, then

$$D(u+w, v+w) = D(u,v) \quad \text{and} \quad D(\lambda u, \lambda v) = |\lambda| D(u,v).$$

Note that (E^n, D) is not a vector space but it can be embedded isomorphically as a cone in a Banach space [6].

3. First order fuzzy differential equations

Let $T=[a,b]$ be a closed subinterval of $\mathbf{R}$ and assume that $f:T\times E^n\to E^n$ is continuous. A mapping $\phi:T\to E^n$ is a solution of the initial value problem

$$y'=f(t,y),\quad y(a)=y_0 \tag{3.1}$$

if and only if ϕ is a solution of the integral equation

$$y(t)=y_0+\int_a^t f(s,y(s))ds. \tag{3.2}$$

Here we use the concept of *H-differentiability* adopted by Kaleva [4].

If we denote by $C(T,E^n)$ the set of all continuous mappings from T to E^n. Define for any ϕ, $\psi\in C(T,E^n)$,

$$H(\phi,\psi)=\sup_{t\in T}\{D(\phi(t),\psi(t))\}.$$

Thus $(C(T,E^n),H)$ is a complete metric space. For any $\psi\in C(T,E^n)$ define $G\psi$ as

$$[G\psi](t)=y_0+\int_a^t f(s,\psi(s))dx,\quad t\in T \tag{3.3}$$

We note that $y\in C(T,E^n)$ is a solution of (3.1) if and only if $Gy=y$, that is, y is a fixed point of G. For the proof of the next theorem we refer to Nieto [5].

THEOREM 3.1. *Suppose that $f:T\times E^n\to E^n$ is continuous and bounded and satisfies*

$$D(f(t,y),0)\leqslant r,\quad t\in T,\quad y\in E^n.$$

Then the initial value problem (3.1) posesses at least one solution on the interval I.

4. Two-point non-linear boundary value problems

In this section we consider the non-linear fuzzy differential equation of second order

$$y''=f(t,y,y'),\quad a\leqslant t\leqslant b \tag{4.1}$$

satisfying

$$y(a)=y_1,\quad y(b)=y_2 \tag{4.2}$$

where $f:T\times E^n\times E^n\to E^n$ is continuous. Denote by $C^1(T,E^n)$ the set of all continuously differentiable mappings from T to E^n. We define ϕ, $\psi\in C^1(T,E^n)$ by

$$H(\phi,\psi)=K\cdot\max_{t\in T}D(\phi(t),\psi(t))+L\cdot\max_{t\in T}D(\phi'(t),\psi'(t)).$$

Then $(C^1(T,E^n),H)$ is a complete metric space. For any $\phi\in C^1(T,E^n)$ define $G\phi\in C^1(T,E^n)$ by

$$[G\phi(t)]=\int_a^b G(t,s)f(s,\phi(s),\phi'(s))ds,\quad t\in T$$

where $G(t,s)$ is the Green's function for the homogeneous boundary value problem.

We note that $\phi \in C^1(T,E^n)$ is a solution of (4.1) satisfying (4.2) if and only if $G\phi = \phi$, that is, ϕ is a fixed point of G.

LEMMA 4.1. *Suppose there exists an $M > 0$ such that*

$$D(f(t,y,y'),0) \leqslant M \quad \textit{for} \quad t \in T, y \in E^n, y' \in E^n \tag{4.3}$$

and $D(G(t,s),G(t_1,s)) \leqslant K$ for all t, $t_1 \in T$. Then G is compact i. e., G transforms bounded sets into relatively compact sets.

Proof. Let B be a bounded set in $C^1(T,E^n)$. The set $GB = \{Gy : y \in B\}$ is totally bounded if and only if it is equicontinuous and for every $t \in T$ the set

$$[GB](t) = \{[Gx](t) : t \in T\}$$

is a totally bounded subset of E^n. Now for any $t_0 \leqslant t_1$ and $\phi \in B$ we have

$$\begin{aligned} D([G\phi](t_0),[G\phi](t_1)) &\leqslant |t_1 - t_0| \cdot \sup\{D(f(t,\phi(t),\phi'(t)),0)\} \\ &\leqslant |t_1 - t_0| M \quad \text{for any} \quad t \in T. \end{aligned}$$

Thus GB is equicontionuous and to prove that GB is totally bounded in E^n we have for any fixed $t \in T$,

$$D([G\phi](t),[G\phi](t_1)) \leqslant MK|t - t_1| \qquad \text{for every} \quad t \in T \quad \text{and} \quad \phi \in B.$$

Hence we see $\{[G\phi](t) : t \in T, \phi \in B\}$ is totally bounded in E^n.

By Ascoli's Lemma we conclude that GB is a relatively compact subset of $C^1(T,E^n)$. Hence the proof of the Lemma is complete.

THEOREM 4.1. *Suppose that $f : T \times E^n \times E^n \to E^n$ is continuous and suppose there exists an $M > 0$ such that $D(f(t,y,y'),0) \leqslant M$. Then the following initial value problem*

$$y''(t) = f(t,y(t),y'(t)) \tag{4.4}$$

$$y(a) = y_1, \quad y'(a) = m \tag{4.5}$$

possesses at least one solution on the interval T.

Proof. We first note that $\phi \in C^1(T,E^n)$ is a solution of (4.4), (4.5) if and only if ϕ is a solution of the integral equation

$$y(t) = y_1 + m(t-a) + \int_a^t (t-s)f(s,y(s),y'(s))ds. \tag{4.6}$$

On the other hand if we set

$$y' = z \qquad \text{then}$$

$$z' = f(t,y,z) \tag{4.7}$$

$$y(a) = y_1, \quad z(a) = m \tag{4.8}$$

ϕ is a solution of the initial value problem (4.7), (4.8) if and only if ϕ is a solution of the integral equation

$$z(t) = m + \int_a^t f(s, y(s), z(s))ds. \tag{4.9}$$

Now in the metric space $(C^1(T, E^n), H)$ consider the ball

$$B\{\phi \in C^1(T, E^n) : H(\phi, 0) \leqslant M_1\} \quad \text{where} \quad M_1 = M(b-a)$$

then $GB \subset B$. For $\phi \in C^1(T, E^n)$,

$$\begin{aligned} D([G\phi](t), [G\phi](a)) &\leqslant |t-a| \cdot M \\ &\leqslant |b-a| \cdot M. \end{aligned}$$

Define $\hat{0} : T \to E^n$, $\hat{0}(t) = \hat{0}$, $t \in T$. Then

$$H(Gx, G\hat{0}) = \sup_{t \in T}\{D([Gx](t), [G\hat{0}](t))\}.$$

Therefore, G is compact and hence it has a fixed point and this fixed point is a solution of the initial value problem (4.7), (4.8).

THEOREM 4.2. *Let* $f \in C[T \times E^n \times E^n, E^n]$ *and satisfy*

$$H[f(t,u,u'), f(t,v,v')] \leqslant K \cdot D[u,v] + L \cdot D[u',v']$$

and assume that

$$\alpha = K\frac{(b-a)^2}{8} + L\frac{(b-a)}{2} < 1.$$

Then the two-point fuzzy boundary value problem

$$y'' = f(t, y, y') \tag{4.10}$$

$$y(a) = y_1, \quad y(b) = y_2 \tag{4.11}$$

has one and only one solution.

Proof. Consider the boundary value problem

$$y'' = 0 \tag{4.12}$$

$$y(a) = 0, \quad y(b) = 0. \tag{4.13}$$

This problem has no nontrivial solution. Therfore if h is any continuous function on $[a,b]$, the equation $y''(t) = h(t)$ has a unique solution satisfying the boundary condition (4.13) give by

$$y(t) = \int_a^b G(t,s)h(s)ds$$

where

$$G(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{(b-a)}, & a \leqslant s \leqslant t \leqslant b \\[2ex] \dfrac{(b-s)(s-a)}{(b-a)}, & a \leqslant t \leqslant s \leqslant b. \end{cases}$$

It can be shown by elementary methods that

$$\max_{a\leqslant t\leqslant b}\int_a^b |G(t,s)|ds \leqslant \frac{(b-a)^2}{8} \tag{4.14}$$

and

$$\max_{a\leqslant t\leqslant b}\int_a^b |G_t(t,s)|ds \leqslant \frac{(b-a)}{2}. \tag{4.15}$$

Let $T = [a,b]$ and denote $C^1[T,E^n]$ as the set of all continuously differentiable mappings from T to E^n. Define the metric

$$H(u,v) = \max_{a\leqslant t\leqslant b} KD(u(t),v(t)) + \max_{a\leqslant t\leqslant b} LD(u'(t),v'(t)).$$

Then $(C^1(T\times E^n,H))$ is a complete metric space. Now for any $Tu \in C^1[T,E^n]$ define $T: C^1 \to C^1$ by

$$[Tu](t) = \int_a^b G(t,s)f(s,u(s),u'(s))ds \tag{4.16}$$

Using the bounds on G and G_t given by (4.14) and (4.15) and the definition of Tu we have

$$\begin{aligned} D(Tu(t),Tv(t)) &\leqslant \int_a^b |G(t,s)|[KD(u(s),v(s)) + LD(u'(s),v'(s))]ds \\ &\leqslant H(u,v)\int_a^b |G(t,s)|ds \\ &\leqslant \frac{(b-a)^2}{8}H(u,v) \end{aligned}$$

and

$$\begin{aligned} D(Tu',Tv') &\leqslant \int_a^b |G_t(t,s)|[KD(u,v) + LD(u',v')]ds \\ &\leqslant H(u,v)\int_a^b |G_t(t,s)ds \\ &\leqslant \frac{(b-a)}{2}H(u,v). \end{aligned}$$

Together these imply that

$$\begin{aligned} H[Tu,Tv] &\leqslant \Big[K\frac{(b-a)^2}{8} + L\frac{(b-a)}{2}\Big]H(u,v) \\ &\leqslant \alpha\cdot H(u,v) \end{aligned}$$

where $\alpha = K\cdot\dfrac{(b-a)^2}{8} + L\cdot\dfrac{(b-a)}{2} < 1$.

We note that $u \in C^1[T,E^n]$ is a solution of the boundary value problem (4.10) and (4.12) if and only if $Tu = u$. Thus u is a fixed point and this fixed point is the unique solution of (4.10), (4.12). However, it is desired to find a solution satisfying $y(a) = y_1$, $y(b) = y_2$. By applying the above procedure to the boundary value problem

$$y'' = f(t, y + p(t), y' + p'(t))$$
$$y(a) = 0, \quad y(b) = 0$$

where p is a polynomial of degree one such that $p(a) = y_1$, $p(b) = y_2$, we get a unique solution, say, $y_1(t)$. Let $y(t) = y_1(t) + p(t)$.

Then $y''(t) = y_1''(t) = f(t, y_1)(t) + p(t), (y_1(t) + p(t)') = f(t, y, y')$ and $y(a) = y_1$, $y(b) = y_2$. This completes the proof of the theorem.

Similar results hold for n th order non-linear two-point boundary value problems. In order to avoid monotony, we omit stating those results.

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