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AN EXTENSION OF A THEOREM BY C. MIRANDA IN WEIGHTED SPACES

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Dedicated to Mario Troisi on the occasion of his 65^{th} birthday

(*communicated by B. Opic*)

Abstract. This paper is a continuation of [4] and extends the well-known theorem of C. Miranda (*Ann. Mat. Pura Appl.* (4) 61 (1963)) concerning a priori estimates of weak solutions of linear elliptic equations with discontinuous coefficients in a bounded open subset of R^n. In [4] we derive such estimates in weighted spaces considered over an arbitrary open subset of R^n. In this paper we obtain the same result replacing a condition on the coefficients by a considerably weaker one.

1. Introduction

Let Ω be an open subset of R^n. In Ω we consider the Dirichlet problem

$$\begin{cases} Lu = f, \qquad f \in W^{-1,2}(\Omega), \\ u \in \overset{\circ}{W}{}^{1,2}(\Omega), \end{cases} \tag{1.1}$$

where L is the second order linear differential operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}\, u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i\, u_{x_i} + c\, u \tag{1.2}$$

with real and measurable coefficients.

It is well-known (see [10]) that, if $n \geqslant 3$, Ω is bounded, L is uniformly elliptic, $a_{ij} \in L^\infty(\Omega)$, $b_i \in L^n(\Omega)$, $c \in L^{n/2}(\Omega)$, $c \geqslant c_0$, $c_0 \in R_+$, then the problem (1.1) is uniquely solvable. Moreover, if $f \in L^\infty(\Omega)$, the solution u belongs to $L^\infty(\Omega)$ and verifies the estimate

$$\|u\|_{L^\infty(\Omega)} \leqslant c_0^{-1}\, \|f\|_{L^\infty(\Omega)}. \tag{1.3}$$

In a recent paper (see [4]) an a priori estimate of type (1.3) is given in weighted spaces for $n \geqslant 2$ and Ω not necessarily bounded.

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More precisely, denoting by ρ a function in the class $\mathscr{A}(\Omega)$ defined in Section 2 and by $\overset{\circ}{W}{}^1_s(\Omega)$, $W^{-1}_{-s}(\Omega)$, $s \in R$, some weighted Sobolev spaces, where the weight functions are suitable powers of ρ (see Section 2), in [4] we study the Dirichlet problem

$$\begin{cases} Lu = f\,, \qquad f \in W^{-1}_{-s}(\Omega), \\ u \in \overset{\circ}{W}{}^1_s(\Omega) \end{cases} \tag{1.4}$$

under the following assumptions:

0) ρ is a function in the class $\mathscr{A}(\Omega)$ which goes to zero near to a subset of $\partial\Omega$ and at infinity;

1) there exists an open subset Ω^* of R^n with the *uniform* C^1*-regularity property* such that $\Omega \subset \Omega^*$ and $\partial\Omega \backslash S_\rho \subset \partial\Omega^*$, where S_ρ is the subset of $\partial\Omega$ on which ρ goes to zero;

2) $$a_{ij} \in L^\infty_{-2s}(\Omega)\,, \qquad (a_{ij})_{x_h} \in L^\infty_{-2s+1}(\Omega)\,, \qquad i,j,h = 1,...,n\,,$$

$$\exists\, \nu \in R_+ \quad : \quad \sum_{i,j=1}^n a_{ij}\,\xi_i\,\xi_j \geqslant \nu\rho^{2s}|\xi|^2 \qquad \forall\, \xi \in R^n\,, \quad \text{a.e. in} \quad \Omega\,,$$

$$b_i \in \overset{\circ}{K}{}^q_{-2s+1}(\Omega) \cap L^\infty_{-2s+1}(\Omega)\,, \qquad i = 1,...,n\,, \qquad c \in K^{q/2}_{-2s+2}(\Omega)\,,$$
$$\exists\, c_0 \in R_+ \quad : \quad c \geqslant c_0\,\rho^{2s-2} \qquad \text{a.e.} \quad \text{in} \quad \Omega\,,$$

where the previous weighted spaces are defined in Section 2, $s \in R$ and

$$q > 2 \quad \text{if} \quad n = 2\,, \qquad q = n \quad \text{if} \quad n \geqslant 3;$$

3) ρ is such that its regularization verifies a suitable condition (see hypothesis h_3)).

If $f \in L^2_{-s}(\Omega) \cap L^\infty_{-s}(\Omega)$, then it is shown that any solution of problem (1.4) belongs to $L^\infty_s(\Omega)$ and there exists $p_0 \geqslant 2$ such that for any $p \geqslant p_0$,

$$\|u\|_{L^p_{s-\frac{2}{p}}(\Omega)} \leqslant c\,\|f\|_{L^p_{-s}(\Omega)}\,, \tag{1.5}$$

where the constant c depends only on p, ρ, s, c_0.

In this paper we still consider problem (1.4) with the hypotheses 0), 1), 3) unchanged, but condition 2) is weakened in a considerable way. More precisely, we remove the requirement of the existence of derivatives $(a_{ij})_{x_h}$ and the assumption $b_i \in L^\infty_{-2s+1}(\Omega)$.

We obtain the following result: the problem (1.4) is uniquely solvable and the solution u verifies the estimate

$$\|u\|_{W^1_s(\Omega)} \leqslant c_1\|f\|_{W^{-1}_{-s}(\Omega)}\,, \tag{1.6}$$

where the variables on which c_1 depends are stated. Furthermore, if $f \in L^2_{-s}(\Omega) \cap L^\infty_{-s}(\Omega)$, then there exists $p_0 \geqslant 2$ such that for any $p \geqslant p_0$ the solution $u \in L^p_{s-\frac{2}{p}}(\Omega)$ and verifies the bound (1.5). We also state the variables on which p_0 depends.

To prove the result, we will need to extend methods introduced in [10], and we will use the theorem stated in [4]. In particular, it is necessary to introduce a variant of a modulus of continuity of a function $g \in \overset{\circ}{K}{}^{q}_{-s+1}(\Omega)$ and to study the multiplication operator

$$u \to g\,u$$

from $W^1_s(\Omega)$ into $L^2(\Omega)$ (see Lemmas 3.1 and 3.2).

2. Notations

If A is a Lebesgue measurable subset of R^n, we denote by $\mathscr{D}(A)$ the class of restrictions to A of functions $\zeta \in C^\infty_0(R^n)$ with $\operatorname{supp} \zeta \cap \overline{A} \subset A$.

If $p \in [1, +\infty]$, we denote by $L^p_{loc}(A)$ the class of functions $f : A \to C$ such that $\zeta f \in L^p(A)$ for all $\zeta \in \mathscr{D}(A)$. We set

$$|f|_{p,A} = \|f\|_{L^p(A)} .$$

Let $B(x,r)$, $x \in R^n$, $r \in R_+$, be the open ball centered in x of radius r.

Let Ω be an open subset of R^n, $n \geqslant 2$. We denote by $\mathscr{A}(\Omega)$ the class of measurable functions $\rho : \Omega \to R_+$ such that

$$\gamma^{-1}\rho(y) \leqslant \rho(x) \leqslant \gamma\,\rho(y) \qquad \forall\, y \in \Omega, \quad \forall\, x \in \Omega \cap B(y, \rho(y)), \tag{2.1}$$

where $\gamma \in R_+$ is independent of x and y.

If $\rho \in \mathscr{A}(\Omega)$, we put

$$S_\rho = \{z \in \partial\Omega : \lim_{x \to z} \rho(x) = 0\} . \tag{2.2}$$

It is known that S_ρ is a closed subset in $\partial\Omega$ (see [6]) and that if $S_\rho \neq \emptyset$ it results

$$\rho(x) \leqslant \operatorname{dist}(x, S_\rho) \qquad \forall\, x \in \Omega$$

(see [14]).

It is well-known (see, *e.g.*, Theor. 2, Chap. VI in [12] and Lemma 3.6.1 in [16]) that there exist $\alpha \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$, $c_1, c_2 \in R_+$ such that

$$c_1 \operatorname{dist}(x, S_\rho) \leqslant \alpha(x) \leqslant c_2 \operatorname{dist}(x, S_\rho) \qquad \forall\, x \in \Omega .$$

We put

$$\Omega_k = \{x \in \Omega : \ |x| < k, \ \alpha(x) > 1/k\} \qquad \forall\, k \in N .$$

If $f \in \mathscr{D}(\overline{R}_+)$ is a fixed function satisfying

$$0 \leqslant f \leqslant 1 , \qquad f(t) = 1 \quad \text{if} \quad t \leqslant 1/2 , \qquad f(t) = 0 \quad \text{if} \quad t \geqslant 1 ,$$

we define the functions

$$\psi_k : \ x \in \overline{\Omega} \longrightarrow \Big(1 - f(k\alpha(x))\Big) f\left(|x|/2k\right) , \qquad k \in N .$$

Note that, for any $k \in N$, ψ_k belongs to $\mathscr{D}(\overline{\Omega} \setminus S_\rho)$ and

$$0 \leqslant \psi_k \leqslant 1 , \qquad \psi_{k|_{\overline{\Omega}_k}} = 1 , \qquad \operatorname{supp}\, \psi_k \subset \overline{\Omega}_{2k} .$$

If $\rho \in \mathscr{A}(\Omega)$, then

$$\rho \in L^{\infty}_{loc}(\overline{\Omega}), \qquad \rho^{-1} \in L^{\infty}_{loc}(\overline{\Omega}\setminus S_\rho) \tag{2.3}$$

(see [14], [6]).

For some examples and properties of functions of $\mathscr{A}(\Omega)$ we refer to [14], [2], [6].

If $r \in N$, $p \in [1,+\infty]$, $s \in R$ and $\rho \in \mathscr{A}(\Omega)$, then $W^{r,p}_s(\Omega)$ is the space of distributions u on Ω such that $\rho^{s+|\alpha|-r}\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leqslant r$, equipped with the norm

$$\|u\|_{W^{r,p}_s(\Omega)} = \sum_{|\alpha|\leqslant r} |\rho^{s+|\alpha|-r}\,\partial^\alpha u|_{p,\Omega}\,.$$

We denote by $\overset{\circ}{W}{}^{r,p}_s(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{r,p}_s(\Omega)$ and set

$$L^p_s(\Omega) = W^{0,p}_s(\Omega)\,, \qquad W^r_s(\Omega) = W^{r,2}_s(\Omega)\,, \qquad \overset{\circ}{W}{}^r_s(\Omega) = \overset{\circ}{W}{}^{r,2}_s(\Omega)\,.$$

Moreover, $W^{-1}_{-s}(\Omega)$ means the space of distributions f on Ω, which can be represented in the form

$$f = f_0 - \sum_{i=1}^n (f_i)_{x_i}\,, \qquad f_0 \in L^2_{-s+1}(\Omega)\,, \quad f_i \in L^2_{-s}(\Omega)\,, \qquad i = 1,\dots,n, \tag{2.4}$$

equipped with the norm

$$\|f\|_{W^{-1}_{-s}(\Omega)} = \inf\left(\|f_0\|_{L^2_{-s+1}(\Omega)} + \sum_{i=1}^n \|f_i\|_{L^2_{-s}(\Omega)}\right),$$

where the "inf" is taken over all possible representations of f of type (2.4). We observe that, by the well-known results (see, *e.g.*, [15]), the space $W^{-1}_{-s}(\Omega)$ can be identified with the dual space $(\overset{\circ}{W}{}^1_s(\Omega))'$ of $\overset{\circ}{W}{}^1_s(\Omega)$.

For some properties of weighted Sobolev spaces, where the weight functions are powers of a function $\rho \in \mathscr{A}(\Omega)$, see, *e.g.*, [1], [8], [11], [9], [13], [15], [2], [3], [6].

If $p \in [1,+\infty[$, $s \in R$ and $\rho \in \mathscr{A}(\Omega)$, we put

$$\Omega(x) = \Omega \cap B\big(x,\rho(x)\big) \qquad \forall\, x \in \Omega\,, \tag{2.5}$$

and consider the spaces $K^p_s(\Omega)$, $\overset{\circ}{K}{}^p_s(\Omega)$ defined in [2] in correspondence with the family of open sets (2.5) as follows.

By $K^p_s(\Omega)$ we denote the space of functions $g \in L^p_{loc}(\overline{\Omega}\setminus S_\rho)$ such that

$$\|g\|_{K^p_s(\Omega)} = \sup_{x\in\Omega}\left(\rho^{s-n/p}(x)\ |g|_{p,\Omega(x)}\right) < +\infty\,, \tag{2.6}$$

with the norm defined by (2.6),

$\overset{\circ}{K}{}^p_s(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $K^p_s(\Omega)$.

Some properties of the spaces $K^p_s(\Omega)$ and $\overset{\circ}{K}{}^p_s(\Omega)$ are recalled in [2], [3], [6].

3. Preliminary results

Let us fix $\rho \in \mathscr{A}(\Omega)$ such that $S = S_\rho \neq \emptyset$ and $\lim\limits_{|x|\to+\infty} \rho(x) = 0$.

We consider the following conditions:

$h_1)$ there exists an open subset Ω^* of R^n with uniform C^1 - regularity property satisfying

$$\Omega \subset \Omega^*, \qquad \partial\Omega \backslash S \subset \partial\Omega^*; \tag{3.1}$$

$i_1)$ $s, q \in R$ and q is such that

$$q > 2 \quad \text{if} \quad n = 2, \qquad q = n \quad \text{if} \quad n \geqslant 3. \tag{3.2}$$

REMARK 3.1. By hypothesis $h_1)$ (see remark 3.1 of [6]) there exists $\theta \in]0, \frac{\pi}{2}[$ such that

$$\forall x \in \Omega \qquad \exists\, C_\theta(x) \; : \qquad \overline{C_\theta(x, \rho(x))} \subset \Omega\,,$$

where $C_\theta(x)$ is an open infinite cone (with the vertex in x, the angle θ) and $C_\theta(x,r)$, $r \in R_+$, is the intersection of $C_\theta(x)$ and $B(x,r)$. □

We have the following

LEMMA 3.1. *If the hypotheses $h_1)$ and $i_1)$ are verified, then $g\,u \in L^2(\Omega)$ for any $g \in K^q_{-s+1}(\Omega)$ and any $u \in W^1_s(\Omega)$, and*

$$|g\,u|_{2,\Omega} \leqslant H\, \|g\|_{K^q_{-s+1}(\Omega)}\, \|u\|_{W^1_s(\Omega)}\,, \tag{3.3}$$

where $H = H(n,\theta,\rho,s,q)$ is a positive constant. □

REMARK 3.2. The proof of Lemma 3.1 can be found in [2] and in [6] but here we explicitly give the dependence of H. □

Now we introduce some notations.

Let $\Sigma(\Omega)$ be the σ-algebra of Lebesgue measurable subsets of Ω. If $p \in [1,+\infty[$, $s \in R$ and $g \in K^p_s(\Omega)$, we set

$$\omega^p_s[g](k) = \|(1-\psi_k)\,g\|_{K^p_s(\Omega)}\,, \qquad k \in N\,,$$

$$\delta^p_s[g](t) = \sup_{|E|\leqslant t} \|g\,\chi_E\|_{K^p_s(\Omega)}\,, \qquad t \in R_+\,,$$

where χ_E is the characteristic function of $E \in \Sigma(\Omega)$.

It is known (see [2]) that $g \in \overset{\circ}{K}{}^p_s(\Omega)$ if and only if $g \in K^p_s(\Omega)$ and

$$\lim_{k\to+\infty} \omega^p_s[g](k) = 0\,.$$

We define the modulus of continuity of $g \in \overset{\circ}{K}{}^p_s(\Omega)$ as a function $\omega[g] : N \to R_+$ satisfying

$$\omega^p_s[g](k) + \delta^p_s[g]\Big(\frac{1}{k}\Big) \leqslant \omega[g](k)\,, \qquad k \in N\,, \qquad \lim_{k\to+\infty} \omega[g](k) = 0\,. \tag{3.4}$$

For any $r \in R_+$ we put

$$[t]_r = \begin{cases} r & \text{if} \quad t > r \\ t & \text{if} \quad |t| \leqslant r \\ -r & \text{if} \quad t < -r \end{cases} \qquad \tau_r(t) = t - [t]_r\,, \qquad t \in R\,. \tag{3.5}$$

Clearly, we have

$$|\tau_r(t)| \leqslant |t|\,, \qquad |t| \leqslant |\tau_r(t)| + r\,. \tag{3.6}$$

We note that, if $f : \Omega \to R$ and

$$A_r(f) = \{x \in \Omega : |f(x)| \geqslant r\}\,, \qquad r \in R_+\,, \tag{3.7}$$

then

$$\operatorname{supp} \tau_r(f) \subset \overline{A}_r(f)\,, \qquad r \in R_+\,.$$

Moreover, if $f \in L^p(\Omega)$, $p \in [1, +\infty[$, then

$$\lim_{r \to +\infty} |A_r(f)| = 0\,. \tag{3.8}$$

Finally, if $g \in L^p_{loc}(\overline{\Omega})$, we denote by $r_k = r_k(g)$, $k \in N$, a real number such that

$$|A_{r_k}(\psi_k\, g)| \leqslant \frac{1}{k} \tag{3.9}$$

and by $r[g]$ the function

$$r[g] : k \in N \to r[g](k) = r_k \in R_+\,. \tag{3.10}$$

Using Lemma 3.1, we can prove the following assertion which we will use later (see Corollary 2 in [2], too).

LEMMA 3.2. *If the hypotheses $h_1)$ and $i_1)$ are verified, $g \in \overset{\circ}{K}{}^q_{-s+1}(\Omega)$ and $k \in N$, then*

$$|g\, u|_{2,\Omega} \leqslant H\, \omega[g](k)\, \|u\|_{W^1_s(\Omega)} + r[g](k)\, |u|_{2,\Omega_{2k}} \qquad \forall\, u \in W^1_s(\Omega)\,, \tag{3.11}$$

where H is the constant from Lemma 3.1, $\omega[g]$ is a modulus of continuity of g in $\overset{\circ}{K}{}^q_{-s+1}(\Omega)$ and $r[g]$ is the function defined by (3.10).

Proof. Fix $u \in W^1_s(\Omega)$ and put $g_k = [\psi_k\, g]_{r_k}$, where $r_k = r[g](k)$. Clearly,

$$|g_k| \leqslant r_k\,, \qquad \operatorname{supp} g_k \subset \overline{\Omega}_{2k}\,.$$

Thus, by Lemma 3.1,

$$\begin{aligned} |g\, u|_{2,\Omega} &\leqslant \left|(g - g_k)\, u\right|_{2,\Omega} + \left|g_k\, u\right|_{2,\Omega} \\ &\leqslant H\, \|g - g_k\|_{K^q_{-s+1}(\Omega)}\, \|u\|_{W^1_s(\Omega)} + r_k\, |u|_{2,\Omega_{2k}}\,, \end{aligned} \tag{3.12}$$

where H is the constant from Lemma 3.1.

On the other hand, by (3.5), (3.6) and (3.9),

$$\begin{aligned} \|g - g_k\|_{K^q_{-s+1}(\Omega)} &\leqslant \|(1 - \psi_k)\, g\|_{K^q_{-s+1}(\Omega)} + \|\tau_{r_k}(\psi_k\, g)\|_{K^q_{-s+1}(\Omega)} \\ &\leqslant \|(1 - \psi_k)\, g\|_{K^q_{-s+1}(\Omega)} + \|\psi_k\, g_{|A_{r_k}(\psi_k\, g)}\|_{K^q_{-s+1}(\Omega)} \leqslant \omega[g](k)\,, \end{aligned} \tag{3.13}$$

and the result follows from (3.12) and (3.13). $\square$

4. Hypotheses

In Ω we consider the second order linear differential operator in divergence form defined by (1.2).

Fix $s \in R$ and suppose that the following hypothesis holds:

$h_2)$
$$a_{ij} \in L^{\infty}_{-2s}(\Omega)\,, \qquad i,j = 1,...,n\,,$$

$$\exists\, \nu \in R_+ \quad : \quad \sum_{i,j=1}^{n} a_{ij}\,\xi_i\,\xi_j \geqslant \nu\rho^{2s}|\xi|^2 \qquad \forall\, \xi \in R^n\,, \quad \text{a.e. in} \quad \Omega\,,$$

$$b_i \in \overset{\circ}{K}{}^{q}_{-2s+1}(\Omega)\,, \qquad i = 1,...,n\,, \qquad c \in K^{q'}_{-2s+2}(\Omega)\,,$$
$$\exists\, c_0 \in R_+ \quad : \quad c \geqslant c_0\,\rho^{2s-2} \qquad \text{a.e. in} \quad \Omega\,,$$

where q satisfies (3.2) and

$$q' = 2 \quad \text{if} \quad 2 \leqslant n < 4\,, \qquad q' > 2 \quad \text{if} \quad n = 4\,, \qquad q' = \frac{n}{2} \quad \text{if} \quad n > 4\,.$$

Let us fix $\mu, b_0 \in R_+$ such that

$$\sum_{i,j=1}^{n} \|a_{ij}\|_{L^{\infty}_{-2s}(\Omega)} \leqslant \mu\,, \qquad \sum_{i=1}^{n} \|b_i\|_{K^{q}_{-2s+1}(\Omega)} \leqslant b_0\,. \tag{4.1}$$

REMARK 4.1. We observe that, under hypotheses $h_1)$ and $h_2)$, Lemma 3.1 implies that the operator

$$u \in \overset{\circ}{W}{}^{1}_{s}(\Omega) \;\rightarrow\; Lu \in W^{-1}_{-s}(\Omega)$$

is bounded.

In fact, if $\varphi \in \overset{\circ}{W}{}^{1}_{s}(\Omega)$, then

$$\int_{\Omega} |a_{ij}\,u_{x_i}\,\varphi_{x_j}|\,dx \leqslant \|a_{ij}\|_{L^{\infty}_{-2s}(\Omega)}\,\|u_{x_i}\|_{L^2_s(\Omega)}\,\|\varphi_{x_j}\|_{L^2_s(\Omega)}\,,$$

$$\int_{\Omega} |b_i\,u_{x_i}\,\varphi|\,dx \leqslant \|u_{x_i}\|_{L^2_s(\Omega)}\,\|b_i\varphi\|_{L^2_{-s}(\Omega)} \leqslant H\,\|u_{x_i}\|_{L^2_s(\Omega)}\,\|\rho^{-s}\,b_i\|_{K^{q}_{-s+1}(\Omega)}\,\|\varphi\|_{W^1_s(\Omega)}\,,$$

$$\int_{\Omega} |c\,u\,\varphi|\,dx \leqslant \|\sqrt{c}u\|_{L^2(\Omega)}\,\|\sqrt{c}\varphi\|_{L^2(\Omega)} \leqslant H^2\,\|\sqrt{c}\|^2_{K^{2q'}_{-s+1}(\Omega)}\,\|u\|_{W^1_s(\Omega)}\,\|\varphi\|_{W^1_s(\Omega)}\,,$$

and so

$$\Big|\int_{\Omega}\Big(\sum_{i,j=1}^{n} a_{ij}\,u_{x_i}\,\varphi_{x_j} + \sum_{i=1}^{n} b_i\,u_{x_i}\,\varphi + c\,u\,\varphi\Big)\,dx\Big| \leqslant c(H)\,\|u\|_{W^1_s(\Omega)}\,\|\varphi\|_{W^1_s(\Omega)}\,,$$

where $c(H)$ is a positive constant. $\square$

We use the following notations:

$$f_x = \Big(\sum_{i=1}^{n} f^2_{x_i}\Big)^{1/2}, \qquad f_{xx} = \Big(\sum_{i,j=1}^{n} f^2_{x_i x_j}\Big)^{1/2}.$$

REMARK 4.2. By Theorem 3.2 of [14] and hypothesis $h_1)$ there exists $\sigma \in \mathscr{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ such that

$$c_1\, \rho(x) \leqslant \sigma(x) \leqslant c_2\, \rho(x) \qquad \forall\, x \in \Omega\,, \tag{4.2}$$

$$\sigma_x\,,\ \sigma\, \sigma_{xx} \in L^\infty(\Omega)\,, \tag{4.3}$$

where the constants $c_1\,,\ c_2 \in R_+$ are independent of x. □

Now, consider the sequence of functions $(\eta_h)_{h\in N}$ defined in [4] by

$$\eta_h(x) = \zeta(h\sigma(x))\,\frac{1}{h} + \Big(1 - \zeta(h\sigma(x))\Big)\,\sigma(x)\,, \qquad h \in N\,, \quad x \in \Omega\,,$$

where $\zeta \in C^\infty([0,+\infty[)$ is a function such that $0 \leqslant \zeta \leqslant 1$, $\zeta(t) = 1$ if $t \geqslant 1$ and $\zeta(t) = 0$ if $t \leqslant \frac{1}{2}$.

Let us recall that, if we put

$$\Omega'_h = \left\{x \in \Omega\ :\ \sigma(x) > \frac{1}{h}\right\}, \qquad h \in N\,,$$

we have

$$c_3\, \sigma(x) \leqslant \eta_h(x) \leqslant c_4\, \sigma(x)\,, \qquad x \in \Omega \backslash \Omega'_h\,, \tag{4.4}$$

$$(\eta_h)_x(x) = 0\,, \qquad x \in \Omega'_h\,, \qquad (\eta_h)_x(x) \leqslant c_5\, \sigma_x(x)\,, \qquad x \in \Omega\,, \tag{4.5}$$

where the constants $c_3,\ c_4,\ c_5 \in R_+$ are independent of h and x.

Observe that (4.4) is verified also in Ω'_h, but the constant c_3 depends on h.

Now we suppose that the function ρ is such that the function σ from (4.2) satisfies:

$h_3)$
$$\lim_{x \to x_0} \sigma_x(x) = \lim_{x \to x_0} \sigma(x)\, \sigma_{xx}(x) = 0 \qquad \forall\, x_0 \in S\,,$$

$$\lim_{|x| \to +\infty} \sigma_x(x) = \lim_{|x| \to +\infty} \sigma(x)\, \sigma_{xx}(x) = 0\,.$$

An example of such a function ρ is given in [7].

5. Main result

We consider the problem

$$\begin{cases} Lu = f\,, \qquad f \in W^{-1}_{-s}(\Omega), \\ u \in \mathring{W}^1_s(\Omega) \end{cases} \tag{5.1}$$

and state the following

THEOREM 5.1. *If the hypotheses $h_1) - h_3)$ hold, then the problem (5.1) is uniquely solvable and the solution u verifies*

$$\|u\|_{W^1_s(\Omega)} \leqslant c_1 \, \|f\|_{W^{-1}_{-s}(\Omega)}\,, \tag{5.2}$$

where $c_1 = c_1\big(n, \theta, \rho, s, \nu, q, b_0, c_0\big)$ is a positive constant.

Moreover, if $f \in L^2_{-s}(\Omega) \cap L^\infty_{-s}(\Omega)$ and $\omega[b^i]$ is a modulus of continuity of $b^i = \rho^{-s} b_i$ in $\overset{\circ}{K}{}^q_{-s+1}(\Omega)$, $i = 1, ..., n$, then there exists $p_0 \in [2, +\infty[$, $p_0 = p_0\big(n, \theta, \rho, s, \nu, \mu, q, \omega[b^i], r[b^i], c_0\big)$, such that for any $p \in [p_0, +\infty[$ the solution u belongs to $L^p_{s-\frac{2}{p}}(\Omega)$ and

$$\|u\|_{L^p_{s-\frac{2}{p}}(\Omega)} \leqslant c(p) \, \|f\|_{L^p_{-s}(\Omega)}\,, \tag{5.3}$$

where $c(p) = c(p, \rho, s, c_0)$.

Proof. The existence and uniqueness of the solution u of problem (5.1) and inequality (5.2) follow from Theorem 2 in [3]. To prove the rest, we adapt to our case some classical methods (see [10] and [5], too).

Let us fix $f \in L^2_{-s}(\Omega) \cap L^\infty_{-s}(\Omega)$ and consider the solution u of problem (5.1).

Firstly, let us suppose that

$$(a_{ij})_{x_h}\,,\ b_i \in L^\infty_{-2s+1}(\Omega)\,, \qquad i, j, h = 1, ..., n\,. \tag{5.4}$$

Then, by Theorem 3.1 in [4], $u \in L^\infty_s(\Omega)$. So, by Lemma 4.2 in [4], $|u|^p\, u\, \eta_h^{sp} \in \overset{\circ}{W}{}^1_s(\Omega)$ for $p \geqslant 1$ and for any $h \in N$.

As in the proof of Theorem 3.1 in [4], we get from hypothesis $h_2)$ that

$$\begin{aligned}\int_\Omega f\, |u|^p\, u\, \eta_h^{sp}\, dx \geqslant \int_\Omega \Big[(p+1)\, \nu\, \rho^{2s}\, \eta_h^{sp}\, |u|^p\, u_x^2 + c_0\, \rho^{2s-2}\, \eta_h^{sp}\, |u|^p\, u^2 \\ + s\,p \sum_{i,j=1}^n a_{ij}\, \eta_h^{sp-1}\, (\eta_h)_{x_j}\, |u|^p\, u\, u_{x_i} + \sum_{i=1}^n b_i\, \eta_h^{sp}\, |u|^p\, u\, u_{x_i}\Big]\, dx\,.\end{aligned} \tag{5.5}$$

Fix $\epsilon > 0$. It is easy to see that there exists $c_1(\epsilon) = c_1(\epsilon, \rho, s, \mu)$ such that

$$\begin{aligned}&\Big|\int_\Omega s\,p \sum_{i,j=1}^n a_{ij}\, \eta_h^{sp-1}\, (\eta_h)_{x_j}\, |u|^p\, u\, u_{x_i}\, dx\Big| \\ &\leqslant \frac{\epsilon}{3} \int_\Omega p^2\, \rho^{2s-2}\, \eta_h^{sp}\, (\eta_h)_x^2\, |u|^p\, u^2\, dx + c_1(\epsilon) \int_\Omega \rho^{2s}\, \eta_h^{sp}\, |u|^p\, u_x^2\, dx\,.\end{aligned} \tag{5.6}$$

If $\omega[b^i]$ is a modulus of continuity of function $b^i = \rho^{-s} b_i \in \overset{\circ}{K}{}^q_{-s+1}(\Omega)$, $i = 1, ..., n$, it follows from the Hölder inequality and Lemma 3.2 that

$$\begin{aligned}\Big|\int \sum_{i=1}^n b_i\, \eta_h^{sp}\, |u|^p\, u\, u_{x_i}\, dx\Big| \leqslant \Big[H \sum_{i=1}^n \omega[b^i](k)\, \|\eta_h^{sp/2}\, |u|^{p/2}\, u\|_{W^1_s(\Omega)} \\ + \sum_{i=1}^n r^i_k\, |\eta_h^{sp/2}\, |u|^{p/2}\, u|_{2,\Omega_{2k}}\Big]\, \big|\rho^s\, \eta_h^{sp/2}\, |u|^{p/2}\, u_x\big|_{2,\Omega} \qquad \forall\, k \in N\,,\end{aligned} \tag{5.7}$$

where $r^i_k = r[b^i](k)$ are given by (3.10) and H is from Lemma 3.2.

By (3.4) there exists $k_0 \in N$ such that

$$\sum_{i=1}^{n} \omega[b^i](k_0) \leqslant \frac{\nu}{n\,H}\,.$$

So, we deduce from (5.7) and (2.3), (4.2), (4.4), (4.5) that there exist $c_2(\epsilon) = c_2\big(\epsilon, n, \rho, s, \nu\big)$ and $c_3(\epsilon) = c_3\big(\epsilon, n, \theta, \rho, s, \nu, q, \omega[b^i]\big)$ such that

$$\begin{aligned}
&\Big|\int \sum_{i=1}^{n} b_i\, \eta_h^{sp}\, |u|^p\, u\, u_{x_i}\, dx\Big| \\
&\leqslant \Big[\frac{\nu}{n}\, \big\|\eta_h^{sp/2}\, |u|^{p/2}\, u\big\|_{W^1_s(\Omega)} + \sum_{i=1}^{n} r^i_{k_0}\, \big|\eta_h^{sp/2}\, |u|^{p/2}\, u\big|_{2,\Omega_{2k_0}}\Big]\, \big|\rho^s\, \eta_h^{sp/2}\, |u|^{p/2}\, u_x\big|_{2,\Omega} \\
&= \Big[\frac{\nu}{n}\, \big|\rho^{s-1}\, \eta_h^{sp/2}\, |u|^{p/2}\, u\big|_{2,\Omega} + \frac{\nu}{n} \sum_{i=1}^{n} \big|\rho^s\, \big(\eta_h^{sp/2}\, |u|^{p/2}\, u\big)_{x_i}\big|_{2,\Omega} \\
&+ \sum_{i=1}^{n} r^i_{k_0}\, \big|\eta_h^{sp/2}\, |u|^{p/2}\, u\big|_{2,\Omega_{2k_0}}\Big]\, \big|\rho^s\, \eta_h^{sp/2}\, |u|^{p/2}\, u_x\big|_{2,\Omega} \\
&\leqslant \Big[\frac{\nu}{n}\, \big|\rho^{s-1}\, \eta_h^{sp/2}\, |u|^{p/2}\, u\big|_{2,\Omega} + \nu\, \frac{|s|}{2}\, \big|p\, \rho^s\, \eta_h^{sp/2-1}\, (\eta_h)_x\, |u|^{p/2}\, u\big|_{2,\Omega} \\
&+ \nu\, \big(\frac{p}{2} + 1\big)\, \big|\rho^s\, \eta_h^{sp/2}\, |u|^{p/2}\, u_x\big|_{2,\Omega} + \sum_{i=1}^{n} r^i_{k_0}\, \big|\eta_h^{sp/2}\, |u|^{p/2}\, u\big|_{2,\Omega_{2k_0}}\Big]\, \big|\rho^s\, \eta_h^{sp/2}\, |u|^{p/2}\, u_x\big|_{2,\Omega} \\
&\leqslant \frac{\epsilon}{3} \int_\Omega \rho^{2s-2}\, \eta_h^{sp}\, |u|^p\, u^2\, dx + \frac{\epsilon}{3} \int_\Omega p^2\, \rho^{2s-2}\, \eta_h^{sp}\, (\eta_h)^2_x\, |u|^p\, u^2\, dx \\
&+ \Big[c_2(\epsilon) + c_3(\epsilon)\, \big(\sum_{i=1}^{n} r^i_{k_0}\big)^2 + \nu\, \big(\frac{p}{2} + 1\big)\Big] \int_\Omega \rho^{2s}\, \eta_h^{sp}\, |u|^p\, u^2_x\, dx\,.
\end{aligned} \tag{5.8}$$

We obtain from (5.5), (5.6) and (5.8) that

$$\begin{aligned}
\int_\Omega f\, |u|^p\, u\, \eta_h^{sp}\, dx &\geqslant \int_\Omega \Big[c_0 - \frac{\epsilon}{3}\big(1 + 2\,p^2\, (\eta_h)^2_x\big)\Big]\, \rho^{2s-2}\, \eta_h^{sp}\, |u|^p\, u^2\, dx \\
&+ \int_\Omega \Big[\frac{p}{2}\, \nu - c_1(\epsilon) - c_2(\epsilon) - c_3(\epsilon)\, \big(\sum_{i=1}^{n} r^i_{k_0}\big)^2\Big]\, \rho^{2s}\, \eta_h^{sp}\, |u|^p\, u^2_x\, dx\,.
\end{aligned} \tag{5.9}$$

Let us choose $\epsilon = \frac{c_0}{2}$ and $p \geqslant 2\left[\frac{c_1(\epsilon)+c_2(\epsilon)+c_3(\epsilon)\,\big(\sum_{i=1}^{n} r^i_{k_0}\big)^2}{\nu}\right]$. Note that there exists $h_0 \in N$ such that

$$(\eta_{h_0})_x \leqslant \frac{1}{p}\,. \tag{5.10}$$

Indeed, given $\epsilon' > 0$, hypothesis $h_3)$ implies that there exists a bounded open subset $E_{\epsilon'}$ of Ω such that $\overline{E}_{\epsilon'} \subset \overline{\Omega}\backslash S$ and

$$\sigma_x < \epsilon' \qquad\qquad \text{in} \quad \Omega\backslash\overline{E}_{\epsilon'}\,.$$

Furthermore, we can found $h_{\epsilon'} > 0$ so that

$$\overline{E}_{\epsilon'} \cap \Omega \subset \Omega'_{h_{\epsilon'}} .$$

Then we can deduce (5.10) from (4.5).

We obtain from (5.9) and (5.10) that

$$\int_\Omega f\, |u|^p\, u\, \eta_{h_0}^{sp}\, dx \geqslant \frac{c_0}{2} \int_\Omega \rho^{2s-2}\, \eta_{h_0}^{sp}\, |u|^{p+2}\, dx ,$$

and so it follows from (4.2) and (4.4) that

$$\int_\Omega f\, |u|^p\, u\, \rho^{sp}\, dx \geqslant c_1(p) \int_\Omega \rho^{s(p+2)-2}\, |u|^{p+2}\, dx , \tag{5.11}$$

where $c_1(p) = c_1(p,\rho,s,c_0)$.

Now, using the Hölder inequality, we get

$$\begin{aligned} &\int_\Omega f\, |u|^p\, u\, \rho^{sp}\, dx \leqslant \Big(\int_\Omega \rho^{-s(p+2)}\, |f|^{p+2}\, dx\Big)^{\frac{1}{p+2}} \Big(\int_\Omega \rho^{s(p+2)}\, |u|^{p+2}\, dx\Big)^{\frac{p+1}{p+2}} \\ &\leqslant c_2(p) \Big(\int_\Omega \rho^{-s(p+2)}\, |f|^{p+2}\, dx\Big)^{\frac{1}{p+2}} \Big(\int_\Omega \rho^{s(p+2)-2}\, |u|^{p+2}\, dx\Big)^{\frac{p+1}{p+2}} , \end{aligned} \tag{5.12}$$

with $c_2(p) = c_2(p,\rho)$.

Then (5.11) and (5.12) give (5.3) under the assumption (5.4).

If the hypothesis (5.4) is not verified, we argue as follows.

Let us set, for any $i,j = 1,\dots,n$,

$$\tilde{a}_{ij} = \begin{cases} \rho^{-2s}\, a_{ij} & \text{in} \quad \Omega \\ \nu\, \delta_{ij} & \text{in} \quad R^n \backslash \Omega, \end{cases}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

Let $(J_k)_{k\in N}$ be a sequence of mollifiers. For any $k \in N$, we define

$$a_{ij}^k = \sigma^{2s}\, J_k * \tilde{a}_{ij} .$$

It is easy to prove that

$a_1)$ $$\sum_{i,j=1}^n a_{ij}^k\, \xi_i \xi_j \geqslant \tilde{\nu}\, \rho^{2s}\, |\xi|^2 \qquad \forall\, \xi \in R^n , \quad \text{a.e. in} \quad \Omega ,$$

where $\tilde{\nu} = \tilde{\nu}(\rho,s,\nu)$,

$a_2)$ $$a_{ij}^k \in L^\infty_{-2s}(\Omega) , \qquad i,j = 1,\dots,n , \qquad \sum_{i,j=1}^n \|a_{ij}^k\|_{L^\infty_{-2s}(\Omega)} \leqslant \tilde{\mu} ,$$

where $\tilde{\mu} = \tilde{\mu}(\rho,s,\nu,\mu)$,

$a_3)$ $$(a_{ij}^k)_{x_h} \in L^\infty_{-2s+1}(\Omega) , \qquad i,j,h = 1,\dots,n .$$

Moreover, if we put

$$b_k^i = [\psi_k\, b^i]_{r_k^i} , \qquad b_i^k = \rho^s\, b_k^i , \qquad i = 1,...,n , \qquad k \in N ,$$

where the functions $\psi_k, k \in N$, are defined in Section 2, we get

$b_1)$ $$b_i^k \in \overset{\circ}{K}{}^q_{-2s+1}(\Omega) \cap L^\infty_{-2s+1}(\Omega)\,,$$

$b_2)$ $$\|b_i^k\|_{K^q_{-2s+1}(\Omega)} \leqslant \|b_i\|_{K^q_{-2s+1}(\Omega)}\,,$$

$b_3)$ we can choose

$$\omega[b_k^i] = \omega[b^i]\,, \qquad r[b_k^i] = r[b^i]\,,$$

$b_4)$ $$b_i^k \to b_i \quad \text{in} \quad K^q_{-2s+1}(\Omega) \quad \text{for} \quad k \to +\infty\,.$$

In fact, since

$$|b_k^i| \leqslant r_k^i\,, \qquad \text{supp}\, b_k^i \subset \overline{\Omega}_{2k}\,,$$

we deduce that $b_k^i \in \overset{\circ}{K}{}^q_{-s+1}(\Omega) \cap L^\infty_{-s+1}(\Omega)$ and so we obtain $b_1)$. Observing that

$$|b_k^i| \leqslant |b^i|\,,$$

we easily get $b_2)$ and $b_3)$. The condition $b_4)$ follows from the inequality (see (3.13))

$$\|b_k^i - b^i\|_{K^q_{-s+1}(\Omega)} \leqslant \omega[b^i]\,(k)$$

and from the convergence $\omega[b^i]\,(k) \to 0$ as $k \to +\infty$.

Let us set

$$L_k u = -\sum_{i,j=1}^n (a_{ij}^k\, u_{x_i})_{x_j} + \sum_{i=1}^n b_i^k\, u_{x_i} + c\, u\,, \qquad k \in N\,.$$

The solution u_k of the weak problem

$$\begin{cases} L_k\, u = f, & f \in W^{-1}_{-s}(\Omega), \\ u \in \overset{\circ}{W}{}^1_s(\Omega) \end{cases}$$

satisfies the inequality

$$\|u_k\|_{W^1_s(\Omega)} \leqslant c_2\, \|f\|_{W^{-1}_{-s}(\Omega)}\,, \tag{5.13}$$

where $c_2 = c_2(n, \theta, \rho, s, \nu, q, b_0, c_0)$.

In a standard way we get, from (5.13), that there exists a subsequence $(u'_k)_{k\in N}$ of $(u_k)_{k\in N}$ weakly convergent in $W^1_s(\Omega)$ to a solution u' of (5.1). The uniqueness of the solution implies that $u' = u$.

On the other hand, since $a_1) - a_3)$ and $b_1) - b_4)$ hold, the coefficients of L_k satisfy the hypothesis (5.4). In such a case we have proved that there exists $p_0 \in [2, +\infty[$, $p_0 = p_0\big(n, \theta, \rho, s, \nu, \mu, q, \omega[b^i], r[b^i], c_0\big)$, such that for any $p \geqslant p_0$,

$$\|u'_k\|_{L^p_{s-\frac{2}{p}}(\Omega)} \leqslant c_3(p)\, \|f\|_{L^p_{-s}(\Omega)}\,, \tag{5.14}$$

where $c_3(p) = c_3(p, \rho, s, c_0)$. Then there exists a subsequence $(u''_k)_{k\in N}$ of $(u'_k)_{k\in N}$ weakly convergent in $L^p_{s-\frac{2}{p}}(\Omega)$ to $u'' \in L^p_{s-\frac{2}{p}}(\Omega)$.

Observing that $u'' = u'$, inequality (5.14) gives (5.3). □

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