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DECAY RESULTS FOR THE HEAT EQUATION UNDER RADIATION BOUNDARY CONDITIONS

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Abstract. The authors derive exponential decay bounds for the spatial derivatives of the solutions of some initial-boundary value problems for the heat equation in one and two space dimensions when linear radiation (Robin) conditions are prescribed on the boundary. Maximum principles for solutions of second order parabolic equations are used in deriving the results.

1. Introduction

There are a number of papers in the literature on decay bounds, both spatial and temporal, for solutions of parabolic initial-boundary value problems and their derivatives. We refer the reader to Horgan and Wheeler [5], Horgan, Payne, and Wheeler [6], Payne and Philippin [9], [10], Philippin and Vernier-Piro [12], [13], Ames, Payne, and Schaefer [1], Payne, Schaefer, and Song [11], Shenker and Roseman [14], Ewer [3], Sperb [15], and the papers cited therein.

In a recent paper, Payne and Philippin [9] obtained sharp exponential temporal decay bounds for the gradient of the solutions of Dirichlet and Neumann initial-boundary value problems for a semi-linear heat equation (for the case of Neumann conditions, see the erratum [9]). As the authors mentioned in their paper, their method does not carry over directly to the case of linear radiation (Robin) boundary conditions. In this paper we indicate a method for obtaining these bounds for problems in one and two space dimensions when Robin boundary conditions are prescribed.

In Section 2. we derive decay bounds for the solution of the linear heat equation (in $\mathbb{R}^2$) and its gradient with radiation boundary conditions. In Section 3. we consider a mixed initial-boundary value problem for the heat equation when a Robin condition is imposed on part of the boundary and a homogeneous Dirichlet condition on the remainder of the boundary. Finally, in Section 4. we study a nonlinear heat equation for a rod of length L when radiation conditions are prescribed at the ends of the rod. Our results are obtained by means of the parabolic maximum principles [4], [8] applied to a suitable combination of the solution and its derivatives.

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2. Robin problem

Let Ω be a bounded convex domain in $\mathbb{R}^2$ with $C^{2+\epsilon}$ boundary $\partial\Omega$. We consider the initial-boundary value problem

$$\begin{aligned} &\Delta u - \frac{\partial u}{\partial t} = 0 && \text{in } \Omega \times (0,\infty), \\ &\frac{\partial u}{\partial n} + \alpha u = 0 && \text{on } \partial\Omega \times (0,\infty), \\ &u(x_1,x_2,0) = g(x_1,x_2) && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where Δ is the Laplace operator, $\partial/\partial n$ is the outward normal derivative operator on the boundary, α is a positive constant, and g is a nonnegative C^1 function ($g \not\equiv 0$) which satisfies

$$\frac{\partial g}{\partial n} + \alpha g = 0 \qquad \text{on } \partial\Omega. \tag{2.2}$$

We know that the solution u is nonnegative and decays exponentially in time, but in this paper we derive explicit decay bounds for both u and its gradient.

We define

$$\Phi(x_1,x_2,t) = \left[u_{,i}u_{,i} + \beta u^2\right] e^{2\beta t}, \tag{2.3}$$

where β is a positive constant to be determined, and we use the comma notation with the summation convention, i. e., the comma denotes partial differentiation and the repeated index indicates summation over $i = 1,2$. Thus, for example, $u_{,i}u_{,i} = (\partial u/\partial x_1)^2 + (\partial u/\partial x_2)^2 = |\nabla u|^2$. A straightforward calculation (see [9]) results in the differential inequality

$$\Delta\Phi + \frac{W_k\Phi_{,k}}{|\nabla u|^2} - \frac{\partial\Phi}{\partial t} \geqslant 0,$$

where

$$W_k = 2\beta u u_{,k} - \frac{1}{2}e^{-2\beta t}\Phi_{,k}.$$

Consequently, Φ takes its maximum value either at an interior point P^* where $\nabla u = 0$, at a point Q on $\partial\Omega$ for $t > 0$, or initially. We now determine conditions which exclude the first two possibilities.

We first suppose that the maximum value of Φ is taken at a point $P^* \in \Omega$ at time $t = t^*$. Then

$$u_{,i}u_{,i}(P,t^*) + \beta u^2(P,t^*) \leqslant \beta u^2(P^*,t^*) = \beta u_M^2, \tag{2.4}$$

where P denotes an arbitrary point in Ω. Let P_1 denote any point on $\partial\Omega$. Then evaluating (2.4) at P_1, we have

$$\left(\alpha^2 + \beta\right) u^2\left(P_1,t^*\right) + \left[\frac{\partial u}{\partial s}\left(P_1,t^*\right)\right]^2 \leqslant \beta u_M^2,$$

where $\partial u/\partial s$ denotes the tangential derivative of u. Thus

$$u^2\left(P_1,t^*\right) \leqslant \frac{\beta}{\alpha^2+\beta} u_M^2.$$

Now integrating (2.4) from P^* to P_1, we obtain

$$\int_{[\beta/(\alpha^2+\beta)]^{1/2}u_M}^{u_M} \frac{d\eta}{\sqrt{u_M^2-\eta^2}} \leqslant \int_{u(P_1,t^*)}^{u_M} \frac{d\eta}{\sqrt{u_M^2-\eta^2}} \leqslant \sqrt{\beta}\delta$$

or

$$\cos^{-1}\left(\frac{\beta}{\alpha^2+\beta}\right)^{1/2} \leqslant \sqrt{\beta}\delta,$$

where δ is the distance from P^* to P_1. The latter inequality implies that, provided $\sqrt{\beta}\delta < \pi/2$,

$$\sqrt{\beta}\tan\sqrt{\beta}\delta \geqslant \alpha.$$

Thus, if β is chosen so that

$$\sqrt{\beta}\tan\sqrt{\beta}\delta < \alpha, \qquad \sqrt{\beta}\delta < \pi/2,$$

then the maximum of Φ can not occur at the interior point P^*. Since δ is less than the radius d of the largest inscribed disk, we may choose β to satisfy

$$\sqrt{\beta}\tan\sqrt{\beta}d < \alpha, \qquad \sqrt{\beta}d < \pi/2, \tag{2.5}$$

to insure that the maximum of Φ does not occur in the interior of Ω. We note there is no restriction on the number of space variables in going from (2.3) to (2.5) except that one replaces $\partial u/\partial s$ by $\operatorname{grad}_s u$, the tangential component of the gradient.

We now assume that Φ takes its maximum value at a point $Q \in \partial\Omega$ for some $t > 0$. We note that if Φ is constant, then

$$u_{,i}u_{,i} + \beta u^2 \leqslant \max_{\overline{\Omega}} \left\{g_{,i}g_{,i} + \beta g^2\right\} e^{-2\beta t}$$

follows immediately, which is the bound we seek. Thus we assume Φ is not constant. Then by the boundary maximum principle [4], we must have

$$\frac{\partial\Phi}{\partial n}(Q) > 0, \tag{2.6}$$

where we have suppressed the t dependence from the argument of Φ. We know that at Q, we have by (2.3)

$$\frac{1}{2}e^{-2\beta t}\frac{\partial\Phi}{\partial s}(Q) = \frac{\partial u}{\partial s}\left[\frac{\partial^2 u}{\partial s^2} + \left(\alpha^2+\beta\right)u\right] = 0 \tag{2.7}$$

and

$$\frac{1}{2}e^{-2\beta t}\frac{\partial\Phi}{\partial t}(Q) = \left(\alpha^2+\beta\right)\left(u\frac{\partial u}{\partial t} + \beta u^2\right) + \frac{\partial u}{\partial s}\frac{\partial^2 u}{\partial s\partial t} + \beta\left(\frac{\partial u}{\partial s}\right)^2 \geqslant 0. \tag{2.8}$$

It follows from (2.7) that at Q, either

$$\frac{\partial u}{\partial s} = 0 \tag{2.9}$$

or

$$\frac{\partial^2 u}{\partial s^2} + \left(\alpha^2 + \beta\right) u = 0. \tag{2.10}$$

If (2.9) holds, then we also have

$$\frac{\partial^2 u}{\partial s^2}(Q) \leqslant 0, \tag{2.11}$$

since $\partial^2 \Phi(Q) \big/ \partial s^2 \leqslant 0$.

We now consider $\partial \Phi / \partial n$. Since on $\partial \Omega$,

$$\frac{\partial^2 u}{\partial n \partial s} = \frac{\partial^2 u}{\partial s \partial n} - \kappa \frac{\partial u}{\partial s},$$

where κ denotes the curvature of $\partial \Omega$, by (2.1) and (2.3) we compute

$$\frac{1}{2} e^{-2\beta t} \frac{\partial \Phi}{\partial n} = -\alpha u \frac{\partial^2 u}{\partial n^2} - (\alpha + \kappa) \left(\frac{\partial u}{\partial s}\right)^2 - \alpha \beta u^2.$$

Since $\partial \Omega$ is $C^{2+\epsilon}$ curve, we have in normal coordinates on $\partial \Omega$

$$\Delta u = \frac{\partial^2 u}{\partial n^2} + \frac{\partial^2 u}{\partial s^2} + \kappa \frac{\partial u}{\partial n},$$

and hence

$$\frac{1}{2} e^{-2\beta t} \frac{\partial \Phi}{\partial n} = -\alpha u \frac{\partial u}{\partial t} - \kappa \alpha^2 u^2 + \alpha u \frac{\partial^2 u}{\partial s^2} - (\alpha + \kappa) \left(\frac{\partial u}{\partial s}\right)^2 - \alpha \beta u^2. \tag{2.12}$$

Now, if (2.9) holds, it follows by (2.11) that

$$\frac{1}{2} e^{-2\beta t} \frac{\partial \Phi}{\partial n} \leqslant -\alpha u \left(\frac{\partial u}{\partial t} + \beta u\right),$$

and consequently by means of (2.8) that

$$\frac{\partial \Phi}{\partial n}(Q) \leqslant 0,$$

which contradicts (2.6). Thus, if (2.9) holds, then the maximum of Φ can not occur at $Q \in \partial \Omega$ for some $t > 0$.

We now suppose that (2.10) holds at Q. In this case, we find from (2.12) that

$$\frac{1}{2} e^{-2\beta t} \frac{\partial \Phi}{\partial n} \leqslant -\alpha u \left[\frac{\partial u}{\partial t} + \left(\alpha^2 + 2\beta\right) u\right].$$

However, by the maximum principle, we know that

$$\frac{\partial u}{\partial t} + \left(\alpha^2 + 2\beta\right) u$$

takes its positive maximum and/or its negative minimum at $t = 0$. Thus, if this expression is nonnegative initially, then it will remain nonnegative for all $t > 0$ and we can conclude that $\partial\Phi/\partial n \leqslant 0$ in contradiction to (2.6).

Now at points of Ω for $t = 0$, we have

$$\frac{\partial u}{\partial t} = \Delta g$$

provided g has bounded second derivatives. Thus, if we ask that

$$\Delta g + \left(\alpha^2 + 2\beta\right) g \geqslant 0, \tag{2.13}$$

then

$$\frac{\partial u}{\partial t} + \left(\alpha^2 + 2\beta\right) u \geqslant 0$$

for all $t > 0$ and we conclude that $\partial\Phi/\partial n \leqslant 0$ at Q.

We summarize our results in the following theorem.

THEOREM 1. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ with $C^{2+\epsilon}$ boundary $\partial\Omega$. If β satisfies (2.5) and a nonnegative g satisfies (2.2) and (2.13), then the gradient of the solution of (2.1) decays exponentially, i. e.,*

$$u_{,i}u_{,i} + \beta u^2 \leqslant \max_{\overline{\Omega}} \left\{g_{,i}g_{,i} + \beta g^2\right\} e^{-2\beta t}.$$

3. Mixed problem

In this section we assume that Ω is a bounded convex domain in $\mathbb{R}^2$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where for definiteness we assume Γ_2 lies on the x-axis in the interval $(0, L)$ and Γ_1 lies in the lower half plane. An example of the type of region we envision is the lower half of a disk. We consider the problem

$$\begin{aligned} \Delta u - \frac{\partial u}{\partial t} &= 0 && \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} + \alpha u &= 0 && \text{on } \Gamma_2 \times (0,\infty), \\ u &= 0 && \text{on } \Gamma_1 \times (0,\infty), \\ u(x,y,0) &= g(x,y) && \text{in } \Omega, \end{aligned} \tag{3.1}$$

and assume u to be a classical solution which vanishes with its y derivative at the points $(0,0)$ and $(L,0)$. Here g is to be a nonnegative function with bounded gradient which vanishes at $(0,0)$ and $(L,0)$ and satisfies

$$\frac{\partial g}{\partial y} + \alpha g = 0 \qquad \text{on } \overline{\Gamma}_2. \tag{3.2}$$

We shall show that $u_{,y}$ decays exponentially in time.

We define

$$\Phi(x,y,t) = \left[u_{,y}^2 + \gamma u^2\right] e^{2\gamma t}, \tag{3.3}$$

where γ is a constant to be determined, and compute

$$\Phi_{,i} = 2\left(u_{,y}u_{,yi} + \gamma u u_{,i}\right)e^{2\gamma t}, \tag{3.4}$$
$$\Delta\Phi - \frac{\partial\Phi}{\partial t} = 2\left(u_{,xy}^2 + u_{,yy}^2 + \gamma u_{,x}^2 - \gamma^2 u^2\right)e^{2\gamma t}.$$

From (3.4) we have

$$2u_{,yy}^2 e^{2\gamma t} = \frac{e^{-2\gamma t}}{2u_{,y}^2}\left(\Phi_{,y} - 2\gamma u u_{,y} e^{2\gamma t}\right)^2$$

so that

$$\Delta\Phi - \frac{W\Phi_{,y}}{u_{,y}^2} - \frac{\partial\Phi}{\partial t} = 2\left(u_{,xy}^2 + \gamma u_{,x}^2\right)e^{2\gamma t} \geqslant 0,$$

where

$$W = \frac{1}{2}e^{-2\gamma t}\Phi_{,y} - 2\gamma u u_{,y}.$$

Consequently, Φ takes its maximum value either at an interior point P^* where $u_{,y} = 0$, at a point Q on $\partial\Omega$ for some $t > 0$, or initially. We shall again rule out the first two possibilities.

First we assume that Φ takes its maximum at the interior point $P^* = (x^*, y^*)$ at time t^*. Then

$$\left[u_{,y}(P,t^*)\right]^2 + \gamma\left[u(P,t^*)\right]^2 \leqslant \gamma\left[u(P^*,t^*)\right]^2 = \gamma u_M^2, \tag{3.5}$$

where P denotes an arbitrary point in Ω. Let P_1 be the point on Γ_1 nearest to P^*. Then integrating (3.5) from P^* to P_1 (see [9]), we obtain

$$\frac{\pi}{2} = \int_{u_M}^{0} \frac{|u_{,y}|}{\sqrt{u_M^2 - u^2}}\,ds \leqslant \sqrt{\gamma}\rho,$$

where ρ is the distance from P^* to P_1. Now let P_2 be the point on Γ_2 nearest to P^*. Then, as in the derivation of (2.5), we have

$$\sqrt{\gamma}\tan\sqrt{\gamma}d \geqslant \alpha, \qquad \sqrt{\gamma}d < \pi/2,$$

where d is the distance from P^* to P_2. Thus, if γ satisfies either of the inequalities

$$\gamma < \pi^2/4\rho^2, \quad \sqrt{\gamma}\tan\sqrt{\gamma}d < \alpha \text{ with } \gamma < \pi^2/4d^2, \tag{3.6}$$

then Φ can not attain its maximum value at P^* in Ω.

Since the location of P^* is unknown, neither ρ nor d is known so that the above requirements on γ are not easy to check. We can make the result more explicit in the special case that every vertical line that intersects $\partial\Omega$ contains a point of Γ_2 and a point of Γ_1, such as when Ω is the lower half of a disk. We let h be the maximum distance along the line $x =$ const. from Γ_1 to Γ_2 and observe that

$$\rho + d \leqslant h. \tag{3.7}$$

Suppose we choose $\bar{\gamma}$ such that

$$\sqrt{\bar{\gamma}}\tan\sqrt{\bar{\gamma}}(h-\rho)<\alpha,\qquad \bar{\gamma}<\pi^2\big/4(h-\rho)^2.$$

Then using (3.7) it follows that

$$\sqrt{\bar{\gamma}}\tan\sqrt{\bar{\gamma}}d<\alpha,\qquad \bar{\gamma}<\pi^2\big/4d^2.$$

Now let ρ^* be a solution of

$$\frac{\pi}{2\rho^*}\tan\frac{\pi}{2\rho^*}(h-\rho^*)=\alpha,\qquad \rho^*>h/2. \tag{3.8}$$

If γ^* is chosen to satisfy

$$\gamma^*<\pi^2\big/4\rho^{*2}, \tag{3.9}$$

then it follows that

$$\sqrt{\gamma^*}\tan\sqrt{\gamma^*}d<\alpha,\qquad \gamma^*<\pi^2\big/4d^2,$$

since $\rho^*>h/2$ implies that $d<h/2$ and hence that $d<\rho^*$.

Thus, if ρ^* is a solution of (3.8), then the choice of γ^* in (3.9) insures that the maximum of Φ does not occur at P^* in Ω. For a more general convex domain it, of course, suffices to take $\gamma<\pi^2\big/4h^2$, a very conservative choice for γ.

We now suppose that Φ takes its maximum value at the point Q on $\partial\Omega$ for some $t>0$. We assume that Φ is not constant since as in the previous section the decay bound follows immediately if Φ is constant. First we consider the case when $Q\in\Gamma_1$, i. e., where $u=0$. From (3.3), we compute

$$\begin{aligned}\frac{\partial\Phi}{\partial n}&=2e^{2\gamma t}u_{,y}u_{,yi}n_i\\&=2e^{2\gamma t}u_{,y}\left[\left(n_x\frac{\partial}{\partial y}-n_y\frac{\partial}{\partial x}\right)u_{,x}\right]\\&=2e^{2\gamma t}u_{,y}\frac{\partial}{\partial s}(u_{,x})\\&=2e^{2\gamma t}\left[\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s\partial n}n_xn_y+\left(\frac{\partial u}{\partial n}\right)^2n_y\frac{\partial}{\partial s}n_x\right],\end{aligned} \tag{3.10}$$

where n_x and n_y are the x and y components of the unit outer normal vector n. Furthermore, we have

$$\begin{aligned}\frac{\partial\Phi}{\partial s}&=2e^{2\gamma t}u_{,y}u_{,ys}\\&=2e^{2\gamma t}\left(\frac{\partial u}{\partial n}n_y\right)\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial n}n_y\right)\\&=2e^{2\gamma t}\left[\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s\partial n}n_y^2+\left(\frac{\partial u}{\partial n}\right)^2n_y\frac{\partial}{\partial s}n_y\right].\end{aligned}$$

Since

$$\frac{\partial}{\partial s}n_x = -n_y(n_{x,x}+n_{y,y}) = -\kappa n_y, \qquad \frac{\partial}{\partial s}n_y = \kappa n_x,$$

where, as before, κ is the curvature on $\partial\Omega$, we have at $Q \in \Gamma_1$,

$$\frac{\partial \Phi}{\partial s} = 2e^{2\gamma t}n_y\left[\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s\partial n}n_y + \left(\frac{\partial u}{\partial n}\right)^2 n_x\kappa\right] = 0. \tag{3.11}$$

From (3.10) it follows that when Φ is not identically equal to a constant,

$$\frac{\partial \Phi}{\partial n} = 2e^{2\gamma t}n_y\left[\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s\partial n}n_x - \left(\frac{\partial u}{\partial n}\right)^2 n_y\kappa\right] = -2e^{2\gamma t}\left(\frac{\partial u}{\partial n}\right)^2\kappa \leqslant 0,$$

where we have used (3.11). As this contradicts the boundary maximum principle in this case, Φ can not take its maximum on Γ_1.

We now consider the case that $Q \in \Gamma_2$, i. e., where $\partial u/\partial n + \alpha u = 0$. Here we have by (3.1) and (3.3),

$$\frac{\partial \Phi}{\partial n} = \frac{\partial \Phi}{\partial y} = 2e^{2\gamma t}(-\alpha u)(-u_{,xx}+u_{,t}+\gamma u).$$

Moreover, at Q

$$\frac{\partial \Phi}{\partial x} = 2e^{2\gamma t}(u_{,y}u_{,yx}+\gamma u u_{,x}) = 2e^{2\gamma t}\left(\alpha^2+\gamma\right)uu_{,x} = 0$$

and

$$\frac{\partial^2 \Phi}{\partial x^2} = 2e^{2\gamma t}\left(\alpha^2+\gamma\right)\left(u_{,x}^2+uu_{,xx}\right) \leqslant 0,$$

where the latter implies that $u_{,xx} \leqslant 0$. In addition

$$\frac{\partial \Phi}{\partial t} = 2e^{2\gamma t}\left(u_{,y}u_{,yt}+\gamma uu_{,t}+\gamma u_{,y}^2+\gamma^2u^2\right) = 2e^{2\gamma t}\left(\alpha^2+\gamma\right)\left(uu_{,t}+\gamma u^2\right) \geqslant 0,$$

so that

$$\frac{\partial \Phi}{\partial y} = -2e^{2\gamma t}\left(-uu_{,xx}+uu_{,t}+\gamma u^2\right) \leqslant 0$$

at Q on Γ_2. We conclude that a nonconstant Φ can not attain its maximum at a point Q on Γ_2 for $t > 0$. We note that under our assumptions Φ vanishes at $(0,0)$ and $(L,0)$ so that the maximum value of Φ can not occur at either of these points at any time t. Thus we conclude that Φ can not take its maximum on $\partial\Omega$.

We formulate our result in the following theorem.

THEOREM 2. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where Γ_2 lies on the x-axis. If γ satisfies (3.6) and g is a nonnegative function satisfying (3.2), then the classical solution u of (3.1) satisfies*

$$u_{,y}^2+\gamma u^2 \leqslant \sup_{\Omega}\left\{g_{,y}^2+\gamma g^2\right\}e^{-2\gamma t}.$$

4. Nonlinear problem

We now consider the one space dimension problem

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + f(u) - \frac{\partial u}{\partial t} &= 0, && 0 < x < L,\ t > 0, \\ \frac{\partial u}{\partial x} + \alpha u &= 0, && x = L,\ t > 0, \\ \frac{\partial u}{\partial x} - \alpha u &= 0, && x = 0,\ t > 0, \\ u(x,0) &= g(x), && 0 < x < L, \end{aligned} \tag{4.1}$$

where f is a C^1 function for which

$$sf'(s) \geqslant f(s) > 0,\ s > 0, \qquad f(0) = 0, \tag{4.2}$$

α is a positive constant, and g is a nonnegative function ($g \not\equiv 0$) with bounded gradient. A typical form for f is $f(u) = u^p$, $p > 1$. We note that condition (4.2) implies that

$$uf(u) \geqslant 2F(u), \qquad F(u) = \int_0^u f(s)\,ds. \tag{4.3}$$

We first impose restrictions on g that will guarantee that the solution remains bounded and then establish the exponential decay of u_x (using subscript notation for partial differentiation) by means of the parabolic maximum principle.

Since the nonnegative solution of (4.1) may blow up at some time T, we consider the time interval $0 < t \leqslant T_1 < T$ (for the existence of solution prior to blow up time see [2], [7]). We set

$$\Phi(x,t) = \left[u_x^2 + \sigma u^2 + 2F(u)\right] e^{2\sigma t}, \tag{4.4}$$

where σ is a positive constant to be determined, and compute

$$\Phi_x = 2\left[u_x u_{xx} + \sigma u u_x + f(u) u_x\right] e^{2\sigma t} \tag{4.5}$$

and

$$\Phi_{xx} - \Phi_t = 2\left[u_{xx}^2 + \sigma u_x^2 - \sigma u f(u) - (f(u))^2 - \sigma\left(u_x^2 + \sigma u^2 + 2F(u)\right)\right] e^{2\sigma t}.$$

From (4.5), we have

$$u_{xx}^2 = \frac{1}{u_x^2} W \Phi_x + [f(u) + \sigma u]^2,$$

where

$$W = \frac{1}{4} e^{-4\sigma t} \Phi_x - e^{-2\sigma t} u_x [f(u) + \sigma u].$$

Thus it follows that

$$\Phi_{xx} - \Phi_t = \frac{1}{u_x^2} W \Phi_x + 2\sigma[uf(u) - 2F(u)] e^{2\sigma t}$$

and by (4.3) that

$$\Phi_{xx} - \frac{W \Phi_x}{u_x^2} - \Phi_t \geqslant 0.$$

By the parabolic maximum principle, we conclude that Φ takes its maximum either at an interior point P where $u_x = 0$, at a point Q on $x = 0$ or $x = L$ for $t > 0$, or initially.

We begin by assuming that Φ takes its maximum at x^* in $(0,L)$ for some t^* where $0 < t^* \leqslant T_1$. Then

$$u_x^2(x,t^*) + \sigma u^2(x,t^*) + 2F(u(x,t^*)) \leqslant \sigma u_M^2 + 2F(u_M) \tag{4.6}$$

where $u_M = u(x^*,t^*)$. Evaluating (4.6) at the endpoints, in particular at $x = L$, we have

$$\left(\alpha^2 + \sigma\right) u^2(L) \leqslant \sigma u_M^2 + \left[\frac{f(u)}{u}\right]_M \left(u_M^2 - u^2(L)\right),$$

where we have suppressed the time argument and used the property that

$$2F(u_M) - 2F(u) \leqslant \left[\frac{f(u)}{u}\right]_M \left(u_M^2 - u^2(L)\right).$$

Moreover, since $f(u)/u$ is nondecreasing by (4.2), we have

$$\left[\frac{f(u)}{u}\right]_M = \frac{f(u_M)}{u_M} = K.$$

Thus

$$u^2(L) \leqslant \frac{K+\sigma}{K+\alpha^2+\sigma} u_M^2$$

with the same bound for $u^2(0)$. It now follows by integration of (4.6), as in the derivation of (2.5) from (2.4), that

$$\sqrt{K+\sigma}\tan\sqrt{K+\sigma}\, d \geqslant \alpha, \qquad \sqrt{K+\sigma}\, d < \pi/2,$$

where d is the distance from x^* to the nearer boundary point. Consequently, since $d \leqslant L/2$, if σ satisfies

$$\sqrt{K+\sigma}\tan\sqrt{K+\sigma}\frac{L}{2} < \alpha, \qquad \sqrt{K+\sigma}L < \pi, \tag{4.7}$$

then the maximum of Φ can not occur at an interior point.

We now suppose that Φ takes its maximum at Q on $x = 0$ or $x = L$. For definiteness, we assume it is on $x = L$. The argument is similar if it is on $x = 0$. As in the previous sections, we assume that Φ is not constant as then the decay estimate follows immediately. Using (4.1), we compute

$$\Phi_t = 2\left[\left(\alpha^2+\sigma\right) u u_t + \sigma\left(\alpha^2+\sigma\right) u^2 + f(u) u_t + 2\sigma F(u)\right] e^{2\sigma t},$$

which is zero at Q. By (4.3), we then conclude that

$$\left[\left(\alpha^2+\sigma\right) u + f(u)\right]\left[u_t + \sigma u\right] \geqslant 0$$

which implies that

$$u_t + \sigma u \geqslant 0.$$

But then the normal derivative at Q is

$$\begin{aligned}\Phi_x &= 2\left[-\alpha u u_{xx} - \alpha\sigma u^2 - \alpha u f(u)\right] e^{2\sigma t}\\ &= -2\alpha u\left[u_t + \sigma u\right] e^{2\sigma t} \leqslant 0,\end{aligned}$$

which violates the boundary maximum principle since Φ is not constant. Thus a nonconstant Φ can not attain its maximum on $x = 0$ or $x = L$ for $t > 0$.

We conclude therefore that for $0 \leqslant t \leqslant T_1$, provided (4.7) holds,

$$u_x^2 + \sigma u^2 + 2F(u) \leqslant \max\left\{\sup_\Omega\left\{g_x^2 + \sigma g^2 + 2F(g)\right\},\right.$$
$$\left.\left(\alpha^2+\sigma\right) g^2(L) + 2F(g(L)), \left(\alpha^2+\sigma\right) g^2(0) + 2F(g(0))\right\} e^{-2\sigma t},$$

where the last two possibilities result from the fact that u_x may be discontinuous at $(0,0)$ and $(L,0)$, the endpoints of the bar at time $t = 0$, and

$$\lim_{t\to 0}\left[u_x^2 + \sigma u^2 + 2F(u)\right]\Big|_{x=L} = \left(\alpha^2+\sigma\right) g(L) + 2F(g(L)),$$

with a similar limit at the left end.

Now suppose that the data function g is small enough so that

$$\sqrt{\frac{f(g_M)}{g_M} + \sigma}\ \tan\sqrt{\frac{f(g_M)}{g_M} + \sigma}\,\frac{L}{2} < \alpha$$

has a positive solution σ. If the solution u is to blow up at time T, then there must be a first time T_2 in (T_1, T) for which

$$\sqrt{\frac{f(u_M)}{u_M}}\ \tan\sqrt{\frac{f(u_M)}{u_M}}\,\frac{L}{2} = \alpha.$$

Thus for $0 < t < T_2$, we have

$$u_x^2 + \sigma u^2 + 2F(u) \leqslant Q^2 e^{-2\sigma t},$$

where

$$Q^2 = \max\left\{\sup_\Omega\left\{g_x^2 + \sigma g^2 + 2F(g)\right\},\right.$$
$$\left.\left(\alpha^2+\sigma\right) g^2(L) + 2F(g(L)), \left(\alpha^2+\sigma\right) g^2(0) + 2F(g(0))\right\}, \quad (4.8)$$

so that

$$u^2 \leqslant \sigma^{-1} Q^2 e^{-2\sigma t}.$$

But if the data is such that Q is sufficiently small to insure that

$$\sqrt{\frac{f\left(\frac{Q}{\sqrt{\sigma}}\right)}{\frac{Q}{\sqrt{\sigma}}} + \sigma}\tan\sqrt{\frac{f\left(\frac{Q}{\sqrt{\sigma}}\right)}{\frac{Q}{\sqrt{\sigma}}} + \sigma}\,\frac{L}{2} = \alpha \qquad (4.9)$$

has a positive solution for σ, then we arrive at a contradiction. Thus there is no such T_2 and no blow up of the solution.

We state our result in the following theorem.

THEOREM 3. *If f is a C^1 function satisfying (4.2) and g is a nonnegative function with bounded gradient satisfying (4.9) for some positive constant σ, then the solution u of (4.1) satisfies*

$$u_x^2 + \sigma u^2 + 2F(u) \leqslant Q^2 e^{-2\sigma t},$$

where Q^2 is given by (4.8).

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