Mathematical
Inequalities
& Applications
Volume 4, Number 4 (2001), 599–608

THE PROPERTIES OF FOUR ELEMENTS IN ORLICZ–MUSIELAK SPACES

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(*communicated by L.-E. Persson*)

Abstract. "The property of four elements" (PFE), closely related to the isotonicity of the metric projection operator, has been recently introduced and proved in ordered Hilbert spaces, L_p-spaces and Orlicz-Musielak spaces (see [5], [6], [12]). Moreover, a dual inequality named "the upper property of four elements" $(UPFE)$ for norms in L_p-spaces has been discussed in [13].

In this paper we prove that the inequality $(UPFE)$ holds in all Orlicz-Musielak spaces generated by a convex modular. It is also shown that both properties of four elements can be reversed (with some other constants) if the function generating the modular satisfies the condition (Δ_2). This generalizes Theorems 3.1, 3.4 from [13].

1. Introduction

The metric projection operator onto a closed convex subset of a Banach space is widely used as an important tool in many different areas of mathematics. This operator has been studied by several authors from various points of view, for example differentiability [2], [3], smoothness [4] and uniform continuity [1].

Since 1986, the projection operator has been investigated from the point of view of isotonicity with respect to an ordering given by a pointed convex cone [5], [6], [7], [8], [9], [10], [11], [12]. This is an important property which can be used in the study of many problems such as Variational Inequalities, Complementarity Problems, Optimization and Numerical Analysis (see [5] and the references given there).

In 1995, a special inequality related to the isotonicity, named "the property of four elements" (PFE) was introduced in [5]. Since then, this property has been considered for Hilbert spaces, L_p-spaces and Lyapunov functionals in the papers [12], [13] and for modular spaces in [6]. Recently, Isac and Persson have examined another interesting latticial property, named "the upper property of four elements" $(UPFE)$ (see [13]). This property is important for the study of the antiprojection operator in L_p-spaces and Hilbert lattices.

The aim of this paper is to generalize some results from [13] to the case of Orlicz-Musielak spaces. Section 2 contains a brief summary of the basic facts concerning modulars. In Section 3 we prove that the property $(UPFE)$ is valid in Musielak-Orlicz

Mathematics subject classification (2000): 46E30, 41A65.

Key words and phrases: Inequalities, modular spaces, Orlicz-Musielak spaces, antiprojection operators, antiisotone operators.

Paper MIA-04-52

spaces generated by a convex modular. We also point out a connection between the condition $(UPFE)$ and the antiisotonicity of the antiprojection operator. Section 4 is devoted to the study of some inequalities related with the properties of four elements. We show that the inequalities (PFE) and $(UPFE)$ for Orlicz-Musielak modulars both can be partially reversed if the generating function satisfies the property (Δ_2).

2. Preliminaries

Let X be a vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

DEFINITION 2.1. (see e.g. [15, p. 1]) A function $\rho : X \to [0, +\infty]$ is called a *modular* if the following properties are satisfied:

(1) $\rho(x) = 0$ if and only if $x = 0$,

(2) $\rho(\alpha x) = \rho(x)$ if $\alpha \in \mathbb{K}$ and $|\alpha| = 1$,

(3) $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ if $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$.

The set $X_\rho = \left\{ x \in X : \lim_{\alpha \to 0} \rho(\alpha x) = 0 \right\}$ is called a *modular space*. It is easy to show that X_ρ is a vector subspace of X.

If (3) is replaced by

(3') $\rho(\alpha x + \beta y) \leqslant \alpha\rho(x) + \beta\rho(y)$ if $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$,

ρ is said to be a *convex modular*.

DEFINITION 2.2. (see [15, p. 33]) Let (Ω, Σ, μ) be a measurable space and let $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a φ-function with a parameter, i.e. Φ satisfies the following properties:

$(\varphi 1)$ for every $t \in \Omega$, $\Phi(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing, continuous function such that $\Phi(t, 0) = 0$ and $\Phi(t, x) > 0$ for $x > 0$.

$(\varphi 2)$ for every $x \in \mathbb{R}_+$, $\Phi(\cdot, x) : \Omega \to \mathbb{R}_+$ is a Σ-measurable function. Here X denotes the set of all real Σ-measurable functions defined on Ω, with equality μ-almost everywhere. For $f \in X$, set

$$\rho_\Phi(f) = \int_\Omega \Phi(t, |f(t)|)\, d\mu(t).$$

ρ_Φ is the *Orlicz-Musielak modular* given by Φ and the corresponding modular space X_{ρ_Φ} will be called *Orlicz-Musielak space*. If the function Φ is independent of t, the Orlicz-Musielak space X_{ρ_Φ} is said to be the *Orlicz space*. If

$(\varphi 3)$ $\Phi(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is convex for every $t \in \Omega$,

then ρ_Φ is a convex modular.

Suppose that X_ρ is a modular space and $K \subset X_\rho$ a pointed convex cone, i.e. $K + K \subset K$, $\alpha K \subset K$ for all $\alpha \in \mathbb{R}_+$ and $K \cap (-K) = \{0\}$. Then we can define an ordering on X_ρ given by K ($x \geqslant y$ if and only if $x - y \in K$). Denote $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$ (with respect to the ordering given by K). We will always suppose that X_ρ is a lattice, i.e. $x \wedge y$ and $x \vee y$ exist for any $x, y \in X_\rho$.

DEFINITION 2.3. A subset D of X_ρ is called *latticially closed* if $x \wedge y \in D$ and $x \vee y \in D$ for all $x, y \in D$.

EXAMPLE 2.4. Every Orlicz-Musielak space X_{ρ_Φ} can be ordered by the pointed convex cone $K = \left\{x \in X_{\rho_\Phi} : x \geqslant 0\right\}$. Then the following subsets of X_{ρ_Φ} are latticially closed:

a) $D = \{x \in X_{\rho_\Phi} : f_2 \geqslant x \geqslant f_1\}$, where f_1, $f_2 \in X_{\rho_\Phi}$ and $f_2 \geqslant f_1$,

b) $D = \{x \in X_{\rho_\Phi} : x$ is a simple function defined on $\Omega\}$,

c) $D = \bigcap\limits_{\alpha \in A} D_\alpha$, where $\{D_\alpha\}_{\alpha \in A}$ is a family of latticially closed subsets of X_{ρ_Φ},

d) $D = \bigcup\limits_{\alpha \in A} D_\alpha$, where $\{D_\alpha\}_{\alpha \in A}$ is an oriented family of latticially closed subsets of X_{ρ_Φ}, i.e. for any α_1, $\alpha_2 \in A$ there exists $\alpha_3 \in A$ such that $D_{\alpha_1} \cup D_{\alpha_2} \subset D_{\alpha_3}$.

If $\Omega \subset \mathbb{R}$, then the subsets of all non-decreasing and all continuous elements of X_{ρ_Φ} are also latticially closed.

3. The properties $(LPFE)$ and $(UPFE)$ in Orlicz-Musielak spaces

Now we are ready to define the properties of four elements in modular spaces. Assume that X_ρ is a modular space ordered by a pointed convex cone K and $D \subset X_\rho$ is a latticially closed subset.

DEFINITION 3.1. (see [6]) We say that ρ satisfies *the lower property of four elements* $(LPFE)$ with respect to D and K if for any $x, y \in X_\rho$ such that $x \geqslant y$ and for any $w, z \in D$, we have

$$\rho(x-w) + \rho(y-z) \geqslant \rho(x - w \vee z) + \rho(y - w \wedge z). \tag{1}$$

DEFINITION 3.2. We say that ρ satisfies *the upper property of four elements* $(UPFE)$ with respect to D and K if for any $x, y \in X_\rho$ such that $x \geqslant y$ and for any $w, z \in D$, we have

$$\rho(x-w) + \rho(y-z) \leqslant \rho(x - w \wedge z) + \rho(y - w \vee z). \tag{2}$$

REMARK 3.3. Suppose that $\rho_1, \rho_2, \ldots, \rho_n$ are modulars, each satisfying the property $(LPFE)$ (resp. $(UPFE)$) with respect to D and K and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative numbers. Then the modular $\rho = \sum\limits_{k=1}^{n} \alpha_k \rho_k$ also has the property $(LPFE)$ (resp. $(UPFE)$) with respect to D and K. However, the modular $\tilde{\rho} = \max\{\rho_1, \ldots, \rho_n\}$ may not satisfy these conditions (see Example 3.9).

The property $(LPFE)$ in modular spaces has been recently studied in [6]. The main result of that paper was the following

THEOREM 3.4. *Suppose that X_{ρ_Φ} is an Orlicz-Musielak space given by a function Φ which satisfies the condition (φ 3), $K = \left\{x \in X_{\rho_\Phi} : x \geqslant 0\right\}$ and D is a latticially closed subset of X_{ρ_Φ}. Then ρ_Φ has the property $(LPFE)$ with respect to D and K.*

Now we prove a version of Theorem 3.4 for the property $(UPFE)$. We will need the following technical lemmas.

LEMMA 3.5. *(*Lim's inequality (see e.g. [14], p. 194) *If* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a convex function such that* $\varphi(0) = 0$ *and* $a, b, c \geqslant 0$ *with* $c \geqslant a$*, then*

$$\varphi(a) + \varphi(b+c) \geqslant \varphi(a+b) + \varphi(c).$$

COROLLARY 3.6. *If* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a convex function such that* $\varphi(0) = 0$*, then for all* $a, b, c \geqslant 0$*, we have*

$$\varphi(a+b) + \varphi(b+c) \leqslant \varphi(b) + \varphi(a+b+c) \tag{3}$$

and

$$\varphi(a) + \varphi(b) \leqslant \varphi(a+b) \leqslant \varphi(a+b+c). \tag{4}$$

Moreover, φ *must be a non-decreasing function on* $\mathbb{R}_+$*.*

LEMMA 3.7. *Assume that* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a convex function such that* $\varphi(0) = 0$ *and* x_1, x_2, x_3, x_4 *are real numbers with* $x_1 \geqslant x_3$*. Then*

$$\varphi(|x_1 - x_2|) + \varphi(|x_3 - x_4|) \leqslant \varphi(|x_1 - x_2 \wedge x_4|) + \varphi(|x_3 - x_2 \vee x_4|). \tag{5}$$

Proof. Let us first observe that the inequality (5) reduces to an equality if $x_4 \geqslant x_2$. So it's sufficient to prove our lemma for $x_2 \geqslant x_4$. We need to consider 6 cases.

i) $x_2 \geqslant x_4 \geqslant x_1 \geqslant x_3$. Put $a = x_1 - x_3$, $b = x_4 - x_1$, $c = x_2 - x_4$, then $a, b, c \geqslant 0$. By (3), we have

$$\begin{aligned} \varphi(x_2 - x_1) + \varphi(x_4 - x_3) &= \varphi(a+b) + \varphi(b+c) \\ &\leqslant \varphi(b) + \varphi(a+b+c) = \varphi(x_4 - x_1) + \varphi(x_2 - x_3), \end{aligned}$$

and (5) is proved.

ii) $x_1 \geqslant x_3 \geqslant x_2 \geqslant x_4$. We argue as in part (i), taking $a = x_2 - x_4$, $b = x_3 - x_2$, $c = x_1 - x_3$.

iii) $x_2 \geqslant x_1 \geqslant x_4 \geqslant x_3$. Set $a = x_2 - x_1$, $b = x_4 - x_3$, $c = x_1 - x_4$. From (4) it follows that

$$\begin{aligned} \varphi(x_2 - x_1) + \varphi(x_4 - x_3) &= \varphi(a) + \varphi(b) \\ &\leqslant \varphi(a+b+c) = \varphi(x_2 - x_3) \leqslant \varphi(x_1 - x_4) + \varphi(x_2 - x_3), \end{aligned}$$

which is the desired conclusion.

iv) $x_1 \geqslant x_2 \geqslant x_3 \geqslant x_4$. Put $a = x_1 - x_2$, $b = x_3 - x_4$, $c = x_2 - x_3$. Reasoning as in the previous case we get (5).

v) $x_2 \geqslant x_1 \geqslant x_3 \geqslant x_4$. Since φ is non-decreasing, we see that

$$\varphi(x_2 - x_1) + \varphi(x_3 - x_4) \leqslant \varphi(x_1 - x_4) + \varphi(x_2 - x_3),$$

and (5) is proved.

vi) $x_1 \geqslant x_2 \geqslant x_4 \geqslant x_3$. Then, for the same reasons as in case v),

$$\varphi(x_1 - x_2) + \varphi(x_4 - x_3) \leqslant \varphi(x_1 - x_4) + \varphi(x_2 - x_3),$$

which completes the proof.

We can now formulate our result which shows that $(UPFE)$ is valid in Orlicz-Musielak spaces.

THEOREM 3.8. *Suppose that X_{ρ_Φ} is an Orlicz-Musielak space given by a function $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies the condition (φ 3), D is a latticially closed subset of X_{ρ_Φ} and $K = \{x \in X_{\rho_\Phi} : x \geqslant 0\}$. Then ρ_Φ has the property $(UPFE)$ with respect to D and K.*

Proof. Fix $x, y \in X_\rho$ such that $x \geqslant y$, $w, z \in D$ and $t \in \Omega$. Set $x_1 = x(t)$, $x_2 = w(t)$, $x_3 = y(t)$, $x_4 = z(t)$ and define $\varphi_t(c) = \Phi(t, c)$ for $c \in \mathbb{R}_+$. Since $x_1 \geqslant x_3$, we can apply Lemma 3.7 to function φ_t obtaining

$$\Phi(t, |(x-w)(t)|) + \Phi(t, |(y-z)(t)|) \leqslant \Phi(t, |(x - w \wedge z)(t)|) + \Phi(t, |(y - w \vee z)(t)|). \quad (6)$$

Integrating (6) over t we get the inequality (2), and the proof is complete.

EXAMPLE 3.9. Let $\Omega = [0, 2]$ and define $\Phi_1(t) = t^2$, $\Phi_2(t) = t$ for $t \in \mathbb{R}_+$. Since Φ_1 and Φ_2 are convex functions, Theorems 3.4 and 3.8 show that the modulars ρ_{Φ_1} and ρ_{Φ_2} satisfy the properties of four elements with respect to the set D of all simple functions defined on Ω.

Now, consider the modular $\rho = \max\{\rho_{\Phi_1}, \rho_{\Phi_2}\}$. Choose any $\alpha \in (3, +\infty)$, $\beta \in (0, 1)$ and put $x = w = \alpha I_{[0,1)}$, $y = \beta I_{[0,1)}$, $z = \alpha I_{[1,2]}$, where I_A denotes the characteristic function of A. Then $x \geqslant y$ and $w, z \in D$, but we have

$$\begin{aligned}\rho(x-w) + \rho(y-z) &= \max\{\alpha^2 + \beta^2, \alpha + \beta\} = \alpha^2 + \beta^2 \\ &< \alpha^2 + \beta = \rho(x - w \vee z) + \rho(y - w \wedge z).\end{aligned}$$

Taking $x = z = \alpha I_{[0,1)}$, $y = \beta I_{[0,1)}$ and $w = \alpha I_{[0,1)} + \beta I_{[1,2]}$, we get

$$\rho(x-w) + \rho(y-z) = \beta + (\alpha - \beta)^2 > \beta^2 + (\alpha - \beta)^2 = \rho(x - w \wedge z) + \rho(y - w \vee z).$$

From this it follows that ρ does not satisfy the properties $(LPFE)$ and $(UPFE)$ with respect to D.

It is known that $(LPFE)$ implies the isotonicity of the projection operator (see [5], [6] for more details). We will now show that the property $(UPFE)$ is related to the antiisotonicity of the antiprojection operator.

Let D be a non-empty subset of a modular space X_ρ and choose $x \in X_\rho$. We will denote by $P_D^a(x)$ the set of all elements $y \in D$ such that

$$\rho(x - y) = \sup_{d \in D} \rho(x - d).$$

The set $P_D^a(x)$ is called the *antiprojection of x onto D*.

THEOREM 3.10. *Suppose that X_ρ is a modular space ordered by a pointed convex cone K, $D \subset X_\rho$ a non-empty latticially closed subset and $x, y \in X_\rho$ with $x \geqslant y$. If both $P_D^a(x)$, $P_D^a(y)$ are non-empty and if ρ satisfies the property (UPFE) with respect to D and K, then there exist $w \in P_D^a(x)$ and $v \in P_D^a(y)$ such that $v \geqslant w$.*

Proof. Choose any $w \in P_D^a(x)$ and $z \in P_D^a(y)$. Then $w \vee z, w \wedge z \in D$. Since the property (UPFE) is satisfied, we have

$$\rho(x-w) + \rho(y-z) \leqslant \rho(x - w \wedge z) + \rho(y - w \vee z) \leqslant \rho(x-w) + \rho(y - w \vee z).$$

Consequently, $\rho(y-z) \leqslant \rho(y-w \vee z)$. Taking $v = w \vee z$, we get $v \in P_D^a(y)$ and $v \geqslant w$, which is our assertion.

COROLLARY 3.11. *Let X_{ρ_Φ} be an Orlicz-Musielak space as in Theorem 3.8 and $D \subset X_{\rho_\Phi}$ a latticially closed subset. Then for any $x, y \in X_{\rho_\Phi}$ such that $x \geqslant y$ and $P_D^a(x)$, $P_D^a(y)$ are both singletons, we have $P_D^a(y) \geqslant P_D^a(x)$.*

4. The reversed properties of four elements in Orlicz-Musielak spaces

In this section we wish to investigate when the inequalities (1) and (2) hold in the reversed direction (with some other constants). First we review some of the standard facts on Orlicz-Musielak modulars. For more details we refer the reader to [15].

Let $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a φ-function with a parameter.

DEFINITION 4.1. We say that Φ has the property (Δ_2) if there exist $K \geqslant 2$ and a non-negative function $h \in L^1(\Omega)$ such that

$$\Phi(t, 2x) \leqslant K\Phi(t, x) + h(t) \quad \text{for all } t \in \Omega \text{ and } x \in \mathbb{R}_+. \tag{7}$$

This is an important condition which implies many interesting properties of the modular ρ_Φ. The following remark summarizes a few of them.

REMARK 4.2. If Φ satisfies (Δ_2) and X is defined as in Definition 2.2, then

(a) $X_{\rho_\Phi} = \{f \in X : \rho_\Phi(f) < \infty\}$,

(b) $X_{\rho_\Phi} = \{f \in X : \rho_\Phi(\alpha f) < \infty \text{ for every } \alpha > 0\}$,

(c) ρ_Φ is a continuous modular, i.e. $\lim_{\alpha \to 1} \rho_\Phi(\alpha x) = \rho_\Phi(x)$ for all $x \in X_{\rho_\Phi}$.

If, moreover, the measure μ is atomless and σ-finite, then the properties (a) and (b) are equivalent to (Δ_2).

In case of Φ independent of t and $\mu(\Omega) = \infty$, (Δ_2) is equivalent to the condition

$$\Phi(2x) \leqslant K\Phi(x) \text{ for all } x \geqslant 0.$$

If Φ is independent of t and $\mu(\Omega) < \infty$, then (7) means exactly that Φ has the property (Δ_2) at ∞, i.e.

$$\Phi(2x) \leqslant K\Phi(x) \text{ for all } x \geqslant x_0 \text{ (with some } x_0 > 0).$$

We will show that modulars satisfying the condition (Δ_2) have some other properties, related to the inequalities studied in Section 3. Our proof is based on several intermediate results.

LEMMA 4.3. *Suppose that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function satisfying $\varphi(0) = 0$ and a, b, c are non-negative numbers. If there exist $K \geqslant 2$ and $M \geqslant 0$ such that $\varphi(2x) \leqslant K\varphi(x) + M$ for any $x \in \mathbb{R}_+$, then*

$$\varphi(a+b) \leqslant \frac{K}{2}(\varphi(a) + \varphi(b)) + M \tag{8}$$

and

$$\varphi(a+b+c) \leqslant A(\varphi(a)+\varphi(b)+\varphi(c))+N, \tag{9}$$

where $A=\frac{K^2}{4}$, $N=(\frac{K}{2}+1)M$.

Proof. By assumption, $\frac{K}{2}\leqslant A$. We have

$$\begin{aligned}\varphi(a+b) &\leqslant \frac{1}{2}\varphi(2a)+\frac{1}{2}\varphi(2b) \leqslant \frac{1}{2}(K\varphi(a)+M)+\frac{1}{2}(K\varphi(b)+M)\\ &= \frac{K}{2}(\varphi(a)+\varphi(b))+M.\end{aligned}$$

Hence

$$\begin{aligned}\varphi(a+b+c) &\leqslant \frac{K}{2}(\varphi(a+b)+\varphi(c))+M\\ &\leqslant \frac{K}{2}(\frac{K}{2}(\varphi(a)+\varphi(b))+M+\varphi(c))+M\\ &= \frac{K^2}{4}(\varphi(a)+\varphi(b))+\frac{K}{2}\varphi(c)+(\frac{K}{2}+1)M\\ &\leqslant A(\varphi(a)+\varphi(b)+\varphi(c))+N,\end{aligned}$$

and the lemma follows.

LEMMA 4.4. *Let* φ *be as in Lemma 4.3. Then for all* $x_1,x_2,x_3,x_4\in\mathbb{R}$ *with* $x_1\geqslant x_3$, *we have*

$$\begin{aligned}\varphi(|x_1-x_2|)+\varphi(|x_3-x_4|) &\leqslant A[\varphi(|x_1-x_2\vee x_4|)\\ &\quad+\varphi(|x_3-x_2\wedge x_4|)+B\varphi((x_1\wedge x_4-x_3\vee x_2)\vee 0)]+N, \end{aligned}\tag{10}$$

where $A=\frac{K^2}{4}$, $B=1+\frac{4}{K^2}$, $N=(\frac{K}{2}+1)M$.

Proof. If $x_2\geqslant x_4$, then the third term on the right hand side of (10) is equal to zero and (10) is satisfied. So we may assume that $x_4\geqslant x_2$. We need to consider six cases.

i) $x_4\geqslant x_2\geqslant x_1\geqslant x_3$. Set $a=x_1-x_3$, $b=x_2-x_1$, $c=x_4-x_2$. Observe that $a,b,c\geqslant 0$. By (4) and (9), we have

$$\begin{aligned}&\varphi(x_2-x_1)+\varphi(x_4-x_3)=\varphi(b)+\varphi(a+b+c)\\ &\quad\leqslant \varphi(b)+A(\varphi(a)+\varphi(b)+\varphi(c))+N\leqslant A(\varphi(a)+2\varphi(b)+\varphi(c))+N\\ &\quad\quad\leqslant A(\varphi(a+b)+\varphi(b+c))+N=A(\varphi(x_2-x_3)+\varphi(x_4-x_1))+N,\end{aligned}$$

which is our claim.

ii) $x_1\geqslant x_3\geqslant x_4\geqslant x_2$. Taking $a=x_4-x_2$, $b=x_3-x_4$, $c=x_1-x_3$ and reasoning as in the previous case, we get (10).

iii) $x_4\geqslant x_1\geqslant x_2\geqslant x_3$. Define $a=x_2-x_3$, $b=x_1-x_2$, $c=x_4-x_1$. From (9) it follows that

$$\begin{aligned}&\varphi(x_1-x_2)+\varphi(x_4-x_3)=\varphi(b)+\varphi(a+b+c)\\ &\quad\leqslant \varphi(b)+A(\varphi(a)+\varphi(b)+\varphi(c))+N=A(\varphi(a)+\varphi(c)+B\varphi(b))+N\\ &\quad\quad=A(\varphi(x_2-x_3)+\varphi(x_4-x_1)+B\varphi(x_1-x_2))+M,\end{aligned}$$

and (10) is proved.

iv) $x_1 \geqslant x_4 \geqslant x_3 \geqslant x_2$. We argue as in part (iii), choosing $a = x_3 - x_2$, $b = x_4 - x_3$, $c = x_1 - x_4$.

v) $x_4 \geqslant x_1 \geqslant x_3 \geqslant x_2$. Put $a = x_3 - x_2$, $b = x_1 - x_3$, $c = x_4 - x_1$. Then, according to (8), we have

$$\begin{aligned}\varphi(x_1 - x_2) + \varphi(x_4 - x_3) &= \varphi(a+b) + \varphi(b+c)\\ &\leqslant \frac{K}{2}(\varphi(a) + \varphi(b)) + M + \frac{K}{2}(\varphi(b) + \varphi(c)) + M\\ &\leqslant A(\varphi(a) + \varphi(c) + \frac{4}{K}\varphi(b)) + N\\ &\leqslant A(\varphi(a) + \varphi(c) + B\varphi(b)) + N\\ &= A(\varphi(x_3 - x_2) + \varphi(x_4 - x_1) + B\varphi(x_1 - x_3)) + N,\end{aligned}$$

as required.

vi) $x_1 \geqslant x_4 \geqslant x_2 \geqslant x_3$. The proof is completely similar as that of part (v), with $a = x_1 - x_4$, $b = x_4 - x_2$, $c = x_2 - x_3$.

We are thus led to the following analogue of Theorem 3.4.

THEOREM 4.5. *Suppose that X_{ρ_Φ} is an Orlicz-Musielak space generated by a function $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies the conditions (φ 3) and (Δ_2). Then, for all $x, y, w, z \in X_{\rho_\Phi}$ such that $x \geqslant y$, we have*

$$\begin{aligned}\rho_\Phi(x - w) + \rho_\Phi(y - z) \leqslant A[\rho_\Phi(x - w \vee z) + \rho_\Phi(y - w \wedge z)\\ + B\rho_\Phi((x \wedge z - y \vee w) \vee 0)] + C, \quad (11)\end{aligned}$$

where $A = \frac{K^2}{4}$, $B = 1 + \frac{4}{K^2}$, $C = (\frac{K}{2} + 1)\int_\Omega h(t)\, d\mu(t)$.

Proof. Choose any $t \in \Omega$ and put $x_1 = x(t)$, $x_2 = w(t)$, $x_3 = y(t)$, $x_4 = z(t)$, $\varphi_t(c) = \Phi(t, c)$ for $c \in \mathbb{R}_+$. From (7) we see that $\varphi_t(2c) \leqslant K\varphi_t(c) + h(t)$. Lemma 4.4 now implies

$$\begin{aligned}\Phi(t, |(x - w)(t)|) + \Phi(t, |(y - z)(t)|) \leqslant A(\Phi(t, |(x - w \vee z)(t)|)\\ + \Phi(t, |(y - w \wedge z)(t)|) + B\Phi(t, ((x \wedge z - y \vee w) \vee 0)(t)) + (\frac{K}{2} + 1)h(t). \quad (12)\end{aligned}$$

Intergrating (12) over t, we complete the proof.

We can also rephrase Corollary 3.11 as follows.

THEOREM 4.6. *Let X_{ρ_Φ} satisfy the hypotheses of Theorem 4.5. Then, for all $x, y, w, z \in X_{\rho_\Phi}$ such that $x \geqslant y$, we have*

$$\begin{aligned}\rho_\Phi(x - w) + \rho_\Phi(y - z) \geqslant A[\rho_\Phi(x - w \wedge z) + \rho_\Phi(y - w \vee z)\\ - B\rho_\Phi((x \wedge w - y \vee z) \vee 0)] - C, \quad (13)\end{aligned}$$

where $A = \frac{4}{K^2}$, $B = 1 + \frac{K^2}{4}$, $C = \frac{2(K+2)}{K^2}\int_\Omega h(t)\, d\mu(t)$.

The proof goes by arguing as in the previous case and using the corresponding

LEMMA 4.7. *Assume that* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a convex function such that* $\varphi(0) = 0$ *and* $\varphi(2x) \leqslant K\varphi(x) + M$ *for some* $K \geqslant 2$, $M \geqslant 0$ *and for all* $x \geqslant 0$. *If* $x_1, x_2, x_3, x_4 \in \mathbb{R}$ *and* $x_1 \geqslant x_3$, *then*

$$\begin{aligned}\varphi(|x_1 - x_2|) + \varphi(|x_3 - x_4|) \geqslant A[&\varphi(|x_1 - x_2 \wedge x_4|) \\ &+ \varphi(|x_3 - x_2 \vee x_4|) - B\varphi((x_1 \wedge x_2 - x_3 \vee x_4) \vee 0)] - N, \quad (14)\end{aligned}$$

with $A = \frac{4}{K^2}$, $B = 1 + \frac{K^2}{4}$ *and* $N = \frac{2M(K+2)}{K^2}$.

Proof of Lemma 4.7. We check at once that (14) is satisfied when $x_4 \geqslant x_2$. So it suffices to consider the same six cases as in Lemma 3.7. Define $\widetilde{K} = \frac{K^2}{4}$ and $\widetilde{N} = (\frac{K}{2} + 1)M$.

i) $x_2 \geqslant x_4 \geqslant x_1 \geqslant x_3$. Put $a = x_1 - x_3$, $b = x_4 - x_1$, $c = x_2 - x_4$. As in the proof of Lemma 4.4, we get

$$\begin{aligned}\varphi(x_4 - x_1) + \varphi(x_2 - x_3) &= \varphi(b) + \varphi(a + b + c) \\ &\leqslant \widetilde{K}(\varphi(a + b) + \varphi(b + c)) + \widetilde{N} \\ &= \widetilde{K}(\varphi(x_4 - x_3) + \varphi(x_2 - x_1)) + \widetilde{N},\end{aligned}$$

which, divided by $\widetilde{K}$, gives the desired conclusion.

iii) $x_2 \geqslant x_1 \geqslant x_4 \geqslant x_3$. Set $a = x_4 - x_3$, $b = x_1 - x_4$, $c = x_2 - x_1$. Then

$$\begin{aligned}\varphi(x_1 - x_4) + \varphi(x_2 - x_3) &= \varphi(b) + \varphi(a + b + c) \\ &\leqslant \widetilde{K}(\varphi(a) + \varphi(b) + (1 + \frac{1}{\widetilde{K}})\varphi(b)) + \widetilde{N} \\ &= \widetilde{K}(\varphi(x_4 - x_3) + \varphi(x_2 - x_1) + (1 + \frac{1}{\widetilde{K}})\varphi(x_1 - x_4)) + \widetilde{N},\end{aligned}$$

and the formula (14) follows easily.

v) $x_2 \geqslant x_1 \geqslant x_3 \geqslant x_4$. Considering $a = x_3 - x_4$, $b = x_1 - x_3$ and $c = x_2 - x_1$, we obtain

$$\begin{aligned}\varphi(x_1 - x_4) + \varphi(x_2 - x_3) &= \varphi(a + b) + \varphi(b + c) \\ &\leqslant \widetilde{K}(\varphi(a) + \varphi(c) + \frac{4}{K}\varphi(b)) + \widetilde{N} \\ &\leqslant \widetilde{K}(\varphi(x_3 - x_4) + \varphi(x_2 - x_1) + (1 + \frac{1}{\widetilde{K}})\varphi(x_2 - x_4)) + \widetilde{N},\end{aligned}$$

and, in consequence, the inequality (14).

The proof of the remaining cases is the same as above so we omit the details.

REMARK 4.8. In the case of L_p-spaces Theorems 4.5 and 4.6 have been proved in [13].

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(Received June 5, 2000)

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