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PERTURBATION BOUNDS FOR CERTAIN OPERATOR FUNCTIONS

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Abstract. Some inequalities for norms and spectral radius involving operator monotone functions have been obtained. Lieb's concavity for trace of certain function has also been considered in general form.

1. Introduction

In what follows, $\mathbf{H}$ denotes a finite dimensional Hilbert space. $\mathscr{L}(\mathbf{H})$ is the space of bounded linear operators on $\mathbf{H}$, while $\mathscr{L}_+(\mathbf{H})$ is the cone of positive semidefinite operators on $\mathbf{H}$ and $\mathscr{P}(\mathbf{H})$ is the cone of positive operators on $\mathbf{H}$. Let I, J be intervals in $\mathbf{R}$. $\mathscr{S}_I(\mathbf{H})$ will denote the set of all selfadjoint operators in $\mathscr{L}(\mathbf{H})$ whose spectrum in contained in I. For an operator $A \in \mathscr{L}(\mathbf{H})$, $\operatorname{spec}(A)$ denotes the spectrum of A. The trace of an $A \in \mathscr{L}(\mathbf{H})$ is denoted by $\operatorname{tr}(A)$. A real valued function f defined on an interval I is called operator monotone on I if for $A, B \in \mathscr{S}_I(\mathbf{H})$ with $A \geqslant B$, we have $f(A) \geqslant f(B)$, where $f(A)$ is defined by familiar functional calculus. A real valued function f defined on an interval I is called operator concave on I if

$$f(\lambda A + (1-\lambda)B) \geqslant \lambda f(A) + (1-\lambda)f(B)$$

for all $A, B \in \mathscr{S}_I(\mathbf{H})$ and $0 \leqslant \lambda \leqslant 1$. A function f on I is called submultiplicative (respectively supermultiplicative) if for all $x, y \in I$

$$f(xy) \leqslant f(x)f(y) \text{ (respectively } f(xy) \geqslant f(x)f(y))$$

whenever $xy \in I$. In Section 2, we shall obtain some inequalities for spectral radius involving operator monotone functions. A general class of functions which contains operator monotone functions is also considered in this section. Some inequalities for norms for functions in this class have been proved. In Section 3, we consider Lieb's concavity theorem in general form.

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2. Inequalities for Spectral Radius and Norms

The spectral radius for an $A \in \mathscr{L}(\mathbf{H})$, is denoted by $\mathrm{spr}(A)$ and is defined by

$$\mathrm{spr}(A) = \sup\{|\lambda| : \lambda \in \mathrm{spec}(A)\}.$$

If A is normal, i.e., $AA^* = A^*A$ then $\mathrm{spr}(A) = \|A\|$, where $\|\cdot\|$ is the operator norm on $\mathscr{L}(\mathbf{H})$. A norm $|||\cdot|||$ on $\mathscr{L}(\mathbf{H})$ is called symmetric, or unitarily invariant, if $|||A||| = |||UAV|||$ for all $A \in \mathscr{L}(\mathbf{H})$ and all unitary operators $U, V \in \mathscr{L}(\mathbf{H})$. The operator (spectral) norm $\|\cdot\|$ on $\mathscr{L}(\mathbf{H})$ is such a norm. In [5,8], it is proved that for $A, B \in \mathscr{P}(\mathbf{H})$,

$$\mathrm{spr}(A^sB^s) \leqslant (\mathrm{spr}(AB))^s \tag{1}$$

and

$$\|A^sB^s\| \leqslant \|AB\|^s \tag{2}$$

for all $0 \leqslant s \leqslant 1$. The function $f(x) = x^s$, $0 \leqslant s \leqslant 1$ is operator monotone on $(0,\infty)$. Thus one might conjecture that for all $A, B \in \mathscr{P}(\mathbf{H})$ and for all positive operator monotone functions f on $(0,\infty)$

$$\mathrm{spr}(f(A)f(B)) \leqslant f(\mathrm{spr}(AB)) \tag{3}$$

and

$$\|f(A)f(B)\| \leqslant f(\|AB\|). \tag{4}$$

However, this fails. Observe that if either of the inequalities (3) or (4) is to hold, then the function f must be supermultiplicative. Since an operator monotone function need not be supermultiplicative, the inequalities (3) or (4) need not be true for an operator monotone function in general. In this section, we shall prove suitable generalizations of inequalities (1) and (2).

THEOREM 2.1. *Let f be a positive operator monotone function on $I = (0,\infty)$. Let $A, B \in \mathscr{P}(\mathbf{H})$ and $\mathrm{spr}(AB) \leqslant 1$. Then*

$$\mathrm{spr}(f(A)(f(B^{-1}))^{-1} \leqslant 1.$$

Proof.

$$\begin{aligned}
\mathrm{spr}(AB) \leqslant 1 &\implies \mathrm{spr}(B^{1/2}AB^{1/2}) \leqslant 1 \\
&\implies B^{1/2}AB^{1/2} \leqslant I \\
&\implies A \leqslant B^{-1} \\
&\implies f(A) \leqslant f(B^{-1}) \\
&\implies (f(B^{-1}))^{-1/2}f(A)(f(B^{-1}))^{-1/2} \leqslant I \\
&\implies \mathrm{spr}\{(f(B^{-1}))^{-1/2}f(A)(f(B^{-1}))^{-1/2}\} \leqslant 1 \\
&\implies \mathrm{spr}(f(A)(f(B^{-1}))^{-1}) \leqslant 1.
\end{aligned}$$

THEOREM 2.2. *Let f be a positive operator monotone function on $I = (0,\infty)$ and let f be submultiplicative. Let $A,B \in \mathscr{P}(\mathbf{H})$. Then*

$$\operatorname{spr}(f(A)(f(B^{-1}))^{-1}) \leqslant f(\operatorname{spr}(AB)).$$

Proof. Let $\operatorname{spr}(AB) = \alpha$. Then $\operatorname{spr}(\frac{A}{\alpha}B) = 1$. Therefore, by Theorem 2.1,

$$\operatorname{spr}\Big(f\left(\frac{A}{\alpha}\right)(f(B^{-1}))^{-1}\Big) \leqslant 1.$$

Now

$$f(A) = f\left(\frac{A}{\alpha}\alpha\right) \leqslant f\left(\frac{A}{\alpha}\right)f(\alpha),$$

since f is submultiplicative. Consequently,

$$0 \leqslant (f(B^{-1}))^{-1/2}\frac{f(A)}{f(\alpha)}(f(B^{-1}))^{-1/2} \leqslant (f(B^{-1}))^{-1/2}f\left(\frac{A}{\alpha}\right)(f(B^{-1}))^{-1/2},$$

and hence

$$\operatorname{spr}\{(f(B^{-1}))^{-1/2}\frac{f(A)}{f(\alpha)}(f(B^{-1}))^{-1/2}\} \leqslant \operatorname{spr}\{(f(B^{-1}))^{-1/2}f\left(\frac{A}{\alpha}\right)(f(B^{-1}))^{-1/2}\}.$$

Thus

$$\begin{aligned}
\operatorname{spr}(f(A)(f(B^{-1}))^{-1}) &= \operatorname{spr}(f(\alpha)(f(B^{-1}))^{-1/2}\frac{f(A)}{f(\alpha)}(f(B^{-1}))^{-1/2}) \\
&\leqslant \operatorname{spr}(f(\alpha)(f(B^{-1}))^{-1/2}f\left(\frac{A}{\alpha}\right)(f(B^{-1}))^{-1/2}) \\
&= \operatorname{spr}(f(\alpha)f\left(\frac{A}{\alpha}\right)(f(B^{-1}))^{-1}) \\
&\leqslant f(\alpha) = f(\operatorname{spr}(AB)).
\end{aligned}$$

Next we consider a more general class of functions and prove some norm inequalities for functions in this class. Let I be an interval in $\mathbf{R}$. Consider the class of functions:

$$\mathscr{L}_I = \{f : I \to [0,\infty) \text{ such that } A,B \in \mathscr{S}_I(\mathbf{H}),\ A^2 \geqslant B^2 \text{ implies } (f(A))^2 \geqslant (f(B))^2\}.$$

The following lemma shows that the class $\mathscr{L}_I$ is a positive convex cone.

LEMMA 2.3. *Let $f,g \in \mathscr{L}_I$. Then (i) $\alpha f \in \mathscr{L}_I$ for all $\alpha \geqslant 0$. (ii) $f+g \in \mathscr{L}_I$.*

Proof. (i) Let $A,B \in \mathscr{S}_I(\mathbf{H})$ be such that $A^2 \geqslant B^2$. Then $(f(A))^2 \geqslant (f(B))^2$, which implies, $(\alpha f(A))^2 \geqslant (\alpha f(B))^2$. Thus $\alpha f \in \mathscr{L}_I$.

(ii) Let $A,B \in \mathscr{S}_I(\mathbf{H})$ and $A^2 \geqslant B^2$. Then $(f(A))^2 \geqslant (f(B))^2$ and $(g(A))^2 \geqslant (g(B))^2$. Therefore

$$\begin{aligned}
(f(A)+g(A))^2 &= (f(A))^2 + (g(A))^2 + 2f(A)g(A) \\
&\geqslant (f(B))^2 + (g(B))^2 + 2f(B)g(B) \\
&= (f(B)+g(B))^2
\end{aligned}$$

since $f(A)g(A) \geqslant f(B)g(B)$ [see 3]. So $f+g \in \mathscr{L}_I$.

PROPOSITION 2.4. *Let $I=(0,\infty)$. Then every positive operator monotone function on I is in $\mathscr{L}_I$.*

Proof. Let f be a positive operator monotone function on I. Then by [11, Theorem 1], f admits the integral representation

$$f(x)=a+bx+\int_0^\infty \frac{x}{x+t}d\mu(t)$$

where $a,b\geqslant 0$ and μ is a finite positive measure on $(0,\infty)$. From Lemma 2.3 and the above integral representation it follows that we need only to prove that for each $t>0$, the function $f_t(x)=\frac{x}{x+t}$ is in $\mathscr{L}_I$. Let $A,B\in\mathscr{P}(\mathbf{H})$ with $A^2\geqslant B^2$. Then

$$\begin{aligned}(f_t(A))^2\geqslant (f_t(B))^2 &\Leftrightarrow [A(A+tI)^{-1}]^2\geqslant [B(B+tI)^{-1}]^2\\ &\Leftrightarrow (I+tA^{-1})^{-2}\geqslant (I+tB^{-1})^{-2}\\ &\Leftrightarrow (I+tA^{-1})^{2}\leqslant (I+tB^{-1})^{2}\\ &\Leftrightarrow I+t^2A^{-2}+2tA^{-1}\leqslant I+t^2B^{-2}+2tB^{-1}.\end{aligned}$$

The last inequality is true. Thus $f_t\in\mathscr{L}_I$.

THEOREM 2.5. *Let $A,B\in\mathscr{P}(\mathbf{H})$, $I=(0,\infty)$ and $\|AB\|\leqslant 1$. Then for all positive $f\in\mathscr{L}_I$,*

$$\|f(A)(f(B^{-1}))^{-1}\|\leqslant 1.$$

Proof.

$$\begin{aligned}\|AB\|\leqslant 1 &\implies \|BA^2B\|\leqslant 1\\ &\implies BA^2B\leqslant I\\ &\implies A^2\leqslant B^{-2}\\ &\implies (f(A))^2\leqslant (f(B^{-1}))^2\\ &\implies (f(B^{-1}))^{-1}(f(A))^2(f(B^{-1}))^{-1}\leqslant I\\ &\implies \|(f(B^{-1}))^{-1}(f(A))^2(f(B^{-1}))^{-1}\|\leqslant 1\\ &\implies \|f(A)(f(B^{-1}))^{-1}\|\leqslant 1.\end{aligned}$$

THEOREM 2.6. *Let $A,B\in\mathscr{P}(\mathbf{H})$ and $I=(0,\infty)$. Then for all positive submultiplicative $f\in\mathscr{L}_I$,*

$$\|f(A)(f(B^{-1}))^{-1}\|\leqslant f(\|AB\|).$$

Proof. Let $\|AB\|=\alpha$. Then $\|\frac{A}{\alpha}B\|=1$. Therefore, by Theorem 2.5,

$$\left\|f\left(\frac{A}{\alpha}\right)(f(B^{-1}))^{-1}\right\|\leqslant 1.$$

Now

$$f(A)=f\left(\frac{A}{\alpha}\alpha\right)\leqslant f\left(\frac{A}{\alpha}\right)f(\alpha),$$

since f is submultiplicative. Consequently,

$$(f(B^{-1}))^{-1}\left(\frac{f(A)}{f(\alpha)}\right)^2(f(B^{-1}))^{-1} \leqslant (f(B^{-1}))^{-1}\left(f\left(\frac{A}{\alpha}\right)\right)^2(f(B^{-1}))^{-1}.$$

Thus

$$\begin{aligned}\|f(A)(f(B^{-1}))^{-1}\|^2 &= \left\|(f(\alpha))^2(f(B^{-1}))^{-1}\left(\frac{f(A)}{f(\alpha)}\right)^2(f(B^{-1}))^{-1}\right\| \\ &\leqslant (f(\alpha))^2\left\|(f(B^{-1}))^{-1}\left(f\left(\frac{A}{\alpha}\right)\right)^2(f(B^{-1}))^{-1}\right\| \\ &= (f(\alpha))^2\left\|f\left(\frac{A}{\alpha}\right)(f(B^{-1}))^{-1}\right\|^2 \\ &\leqslant (f(\alpha))^2 = (f(\|AB\|))^2.\end{aligned}$$

Hence

$$\|f(A)(f(B^{-1}))^{-1}\| \leqslant f(\|AB\|).$$

We prove our next result for all unitarily invariant norms. A basic property of unitarily invariant norms is that they are symmetric gauge functions of the singular values of the operator. For a positive semidefinite operator T its singular values are the same as its eigen values. Let $T \in \mathscr{L}_+(\mathbf{H})$ and let its eigen values be enumerated as

$$\lambda_1(T) \geqslant \lambda_2(T) \geqslant \cdots \geqslant \lambda_n(T).$$

Then the Ky Fan $k-norms$ of T are defined as

$$\|T\|_k = \sum_{j=1}^{k} \lambda_j(T),$$

$k = 1,2,\ldots,n$.

LEMMA 2.7. *Let* $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$ *and* $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n \geqslant 0$ *be such that* $\prod_{j=1}^{k} x_j \leqslant \prod_{j=1}^{k} y_j$, $k = 1,2,\ldots,n$. *Then* $\sum_{j=1}^{k} x_j \leqslant \sum_{j=1}^{k} y_j$, $k = 1,2,\ldots,n$.

LEMMA 2.8. *Let* $A,B \in \mathscr{L}_+(\mathbf{H})$. *Then* $|||A||| \leqslant |||B|||$ *for all unitarily invariant norms* $|||\cdot|||$ *if and only if* $\|A\|_k \leqslant \|B\|_k$, $k = 1,2,\ldots,n$.

For a proof of the above lemmas, the reader is referred to [4,9].

THEOREM 2.9. *Let* $I = (0,\infty)$ *and* $f \in \mathscr{L}_I$ *be submultiplicative and positive. Then*

$$|||(f(B^{-1}))^{-1}f(A)(f(B^{-1}))^{-1}||| \leqslant |||(f((BAB)^{1/2}))^2|||$$

for all $A,B \in \mathscr{P}(\mathbf{H})$.

Proof. By Theorem 2.6, we have

$$\begin{aligned}\|(f(B^{-1}))^{-1}f(A)(f(B^{-1}))^{-1}\| &= \|[(f(A))^{1/2}(f(B^{-1}))^{-1}]^*[(f(A))^{1/2}(f(B^{-1})^{-1}]\| \\ &= \|[(f(A))^{1/2}(f(B^{-1}))^{-1}]\|^2 \\ &\leqslant \|f(A^{1/2})(f(B^{-1}))^{-1}\|^2 \\ &\leqslant (f(\|A^{1/2}B\|))^2 \\ &= (f(\|BAB\|^{1/2}))^2,\end{aligned}$$

that is,

$$\lambda_1((f(B^{-1}))^{-1}f(A)(f(B^{-1}))^{-1} \leqslant (f((\lambda_1(BAB))^{1/2}))^2.$$

Replacing A and B by their antisymmetric tensor powers, we obtain for $k = 1, 2, \ldots, n$,

$$\prod_{j=1}^{k} \lambda_j((f(B^{-1}))^{-1}f(A)(f(B^{-1}))^{-1}) \leqslant \prod_{j=1}^{k} (f((\lambda_j(BAB))^{1/2}))^2.$$

By Lemma 2.7, this leads to

$$\sum_{j=1}^{k} \lambda_j((f(B^{-1}))^{-1}f(A)(f(B^{-1}))^{-1}) \leqslant \sum_{j=1}^{k} (f((\lambda_j(BAB))^{1/2}))^2,$$

$k = 1, 2, \ldots, n$. Thus

$$\|(f(B^{-1}))^{-1}f(A)(f(B^{-1}))^{-1}\|_k \leqslant \|(f((BAB)^{1/2}))^2\|_k$$

$k = 1, 2, \ldots, n$. Hence by Lemma 2.8,

$$|||(f(B^{-1}))^{-1}f(A)(f(B^{-1}))^{-1}||| \leqslant |||(f((BAB)^{1/2}))^2|||.$$

This completes the proof of the theorem.

The following corollary which is a special case of the above theorem when $f(x) = x^r$, $0 \leqslant r \leqslant 1$ is proved in [2].

COROLLARY 2.10. *Let* $A, B \in \mathscr{P}(\mathbf{H})$. *Then*

$$|||B^r A^r B^r||| \leqslant |||(BAB)^r|||,$$

for all $0 \leqslant r \leqslant 1$.

REMARK 2.11. Let $I = (0, \infty)$. Observe that a function $f \in \mathscr{L}_I$ if and only if $x \to (f(\sqrt{x}))^2$ is operator monotone on I. Consequently, if $f \in \mathscr{L}_I$, then $x \to f(\sqrt{x})$ is operator monotone on I. Since the functions $x \to \frac{x}{1+x}$ and $x \to \log(1+x)$ are in $\mathscr{L}_I$. Therefore, the functions $x \to (\frac{\sqrt{x}}{1+\sqrt{x}})^2$ and $x \to (\log(1+\sqrt{x}))^2$ are operator monotone on I and hence by [3, Theorem 2.3], the function $x \to \frac{\sqrt{x}}{1+\sqrt{x}} \log(1+\sqrt{x})$ is operator monotone on I. In this way one can obtain more examples of operator monotone functions and hence inequalities for spectral radius and norm for suitable functions.

REMARK 2.12. Let $I = (0, \infty)$. Let f be a positive supermultiplicitave operator monotone function on I. Then $x \to (f(x^{-1}))^{-1}$ is a submultiplicative operator monotone function on I. Since for a function h which depends upon the eigen values of an operator, we have $h(AB) = h(BA)$ for $A, B \in \mathscr{P}(\mathbf{H})$. Consequently, we obtain

$$\mathrm{spr}(f(A)(f(B^{-1}))^{-1}) \leqslant (f((\mathrm{spr}(AB))^{-1}))^{-1}$$

and

$$\|f(A)(f(B^{-1}))^{-1}\| \leqslant (f(\|AB\|^{-1}))^{-1},$$

for $A, B \in \mathscr{P}(\mathbf{H})$.

3. Lieb's Concavity

The geometric mean # introduced by Pusz and Woronowicz [13] (see also [10]), for $A,B \in \mathscr{P}(\mathbf{H})$ is defined as

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Thus if A commutes with B, then

$$A\#B = (AB)^{1/2}.$$

If $A,B \in \mathscr{L}_+(\mathbf{H})$, then the geometric mean is defined by continuity. The following results hold:

(i) $(\alpha A)\#(\alpha B) = \alpha(A\#B)$ for all $\alpha \geqslant 0$.

(ii) $(A+C)\#(B+D) \geqslant (A\#B)+(C\#D)$.

(iii) $A \geqslant C$, $B \geqslant D \implies A\#B \geqslant C\#D$.

A map $\phi : \mathscr{S}_I(\mathbf{H}) \times \mathscr{S}_J(\mathbf{H}) \to \mathbf{R}$ is called jointly concave if

$$\phi(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2) \geqslant \lambda\phi(A_1,B_1) + (1-\lambda)\phi(A_2,B_2)$$

for all $A_1,A_2 \in \mathscr{S}_I(\mathbf{H})$; $B_1,B_2 \in \mathscr{S}_J(\mathbf{H})$ and $0 \leqslant \lambda \leqslant 1$. The space $\mathscr{L}(\mathbf{H})$ is a Hilbert space, where the inner product is defined as

$$\langle A,B\rangle = \mathrm{tr}(A^*B),$$

$A,B \in \mathscr{L}(\mathbf{H})$.

For our main theorem in this section, we need the following lemmas.

LEMMA 3.1. *Let $A,B \in \mathscr{P}(\mathbf{H})$. Let $\mathscr{A}$ and $\mathscr{B}$ be the left and the right multiplication operators on the space $\mathscr{L}(\mathbf{H})$ induced by A and B respectively, i.e.,*

$$\mathscr{A}(X) = AX \text{ and } \mathscr{B}(X) = XB.$$

Then $\mathscr{A}$ and $\mathscr{B}$ are positive operators on $\mathscr{L}(\mathbf{H})$.

LEMMA 3.2. *Let R_1,R_2,S_1,S_2,T_1,T_2 be in $\mathscr{P}(\mathbf{H})$. Suppose R_1 commutes with R_2, S_1 commutes with S_2 and T_1 commutes with T_2, and $R_1 \geqslant \lambda S_1 + (1-\lambda)T_1$, $R_2 \geqslant \lambda S_2 + (1-\lambda)T_2$, $0 \leqslant \lambda \leqslant 1$. Then for all $0 \leqslant s,t$; $s+t \leqslant 1$*

$$R_1^sR_2^t \geqslant \lambda S_1^sS_2^t + (1-\lambda)T_1^sT_2^t.$$

Proof. Let E be the set of all (s,t); $s,t \geqslant 0$ for which inequality

$$R_1^sR_2^t \geqslant \lambda S_1^sS_2^t + (1-\lambda)T_1^sT_2^t$$

is true. Clearly $(0,0)$, $(0,1)$ and $(1,0)$ are in E, and E is closed. We shall prove that E is convex. Let (s_1,t_1), $(s_2,t_2) \in E$. Then

$$R_1^{s_1}R_2^{t_1} \geqslant \lambda S_1^{s_1}S_2^{t_1} + (1-\lambda)T_1^{s_1}T_2^{t_1}$$

and

$$R_1^{s_2}R_2^{t_2} \geqslant \lambda S_1^{s_2}S_2^{t_2} + (1-\lambda)T_1^{s_2}T_2^{t_2}.$$

Using (iii), (ii) and (i) respectively, we get

$$\begin{aligned} R_1^{(s_1+s_2)/2}R_2^{(t_1+t_2)/2} &= (R_1^{s_1}R_2^{t_1})\#(R_1^{s_2}R_2^{t_2}) \\ &\geqslant (\lambda S_1^{s_1}S_2^{t_1} + (1-\lambda)T_1^{s_1}T_2^{t_1})\#(\lambda S_1^{s_2}S_2^{t_2} + (1-\lambda)T_1^{s_2}T_2^{t_2}) \\ &\geqslant \lambda\{(S_1^{s_1}S_2^{t_1})\#(S_1^{s_2}S_2^{t_2})\} + (1-\lambda)\{(T_1^{s_1}T_2^{t_1})\#(T_1^{s_2}T_2^{t_2})\} \\ &= \lambda S_1^{(s_1+s_2)/2}S_2^{(t_1+t_2)/2} + (1-\lambda)T_1^{(s_1+s_2)/2}T_2^{(t_1+t_2)/2}. \end{aligned}$$

Therefore $((s_1+s_2)/2,(t_1+t_2)/2) \in E$. So E is convex. This proves the lemma.

THEOREM 3.3. *Let f and g be positive operator concave functions on I and J respectively. Let $X \in \mathscr{L}(\mathbf{H})$ and $s,t \geqslant 0$ be such that $s+t \leqslant 1$. Then the map*

$$\phi(A,B) = \operatorname{tr}\{X^*(f(A))^s X(g(B))^t\}$$

is jointly concave on $\mathscr{S}_I(\mathbf{H}) \times \mathscr{S}_J(\mathbf{H})$.

Proof. Let $A_1,A_2 \in \mathscr{S}_I(\mathbf{H})$; $B_1,B_2 \in \mathscr{S}_J(\mathbf{H})$ and $0 \leqslant \lambda \leqslant 1$. Let $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}$ be the left multiplication operators induced by $f(A_1)$, $f(A_2)$ and $f(\lambda A_1+(1-\lambda)A_2)$ respectively; and $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}$ be the right multiplication operators induced by $g(B_1)$, $g(B_2)$ and $g(\lambda B_1+(1-\lambda)B_2)$ respectively. By Lemma 3.1, $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}$, $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}$ are positive operators. Moreover, $\mathscr{A}_1$ commutes with $\mathscr{B}_1$, $\mathscr{A}_2$ commutes with $\mathscr{B}_2$ and $\mathscr{A}$ commutes with $\mathscr{B}$. Also, we have

$$\mathscr{A} \geqslant \lambda\mathscr{A}_1 + (1-\lambda)\mathscr{A}_2$$

and

$$\mathscr{B} \geqslant \lambda\mathscr{B}_1 + (1-\lambda)\mathscr{B}_2,$$

since f and g are operator concave on I and J respectively. Hence by Lemma 3.2,

$$\mathscr{A}^s\mathscr{B}^t \geqslant \lambda\mathscr{A}_1^s\mathscr{B}_1^t + (1-\lambda)\mathscr{A}_2^s\mathscr{B}_2^t$$

for $0 \leqslant s,t$; $s+t \leqslant 1$. Thus for every $X \in \mathscr{L}(\mathbf{H})$,

$$\begin{aligned} &\langle X, (f(\lambda A_1+(1-\lambda)A_2))^s X(g(\lambda B_1+(1-\lambda)B_2))^t\rangle \\ &\qquad \geqslant \langle X, \lambda(f(A_1))^s X(g(B_1))^t + (1-\lambda)(f(A_2))^s X(g(B_2))^t\rangle \end{aligned}$$

or that

$$\begin{aligned} &\operatorname{tr}\{X^*(f(\lambda A_1+(1-\lambda)A_2))^s X(g(\lambda B_1+(1-\lambda)B_2))^t\} \\ &\qquad \geqslant \lambda\operatorname{tr}\{X^*(f(A_1))^s X(g(B_1))^t\} + (1-\lambda)\operatorname{tr}\{X^*(f(A_2))^s X(g(B_2))^t\}. \end{aligned}$$

This completes the proof.

COROLLARY 3.4. *Let f and g be positive operator concave functions on $I = (0,\infty)$, and $J = (0,\infty)$ respectively. Let $X \in \mathscr{L}(\mathbf{H})$ and $s,t \geqslant 0$ be such that $s+t \leqslant 1$. Then the map*

$$\psi(A,B) = \operatorname{tr}\{X^* f(A^s) X g(B^t)\}$$

is jointly concave on $\mathscr{S}_I(\mathbf{H}) \times \mathscr{S}_J(\mathbf{H})$.

Proof. The proof of the corollary follows from Theorem 3.3, since the functions $f_1(x) = (f(x^s))^{1/s}$ and $g_1(x) = (g(x^t))^{1/t}$ are also operator concave on $(0,\infty)$.

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