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ON A RESULT OF LEINDLER

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Abstract. In this paper we give an alternative proof of a result of Leindler on Hardy type inequalities. This method can also be used to obtain alternative and simpler proofs of other results.

1. Introduction

In [2] Hardy and Littlewood proved the following:

THEOREM A [2]. *If* $a_n \geqslant 0, n = 1, 2, \ldots,$ *then we have the following:*

$$\sum_{n=1}^{\infty} n^{-c} \left(\sum_{k=1}^{n} a_k \right)^p \leqslant K \sum_{n=1}^{\infty} n^{-c(1-p)} a_n^p \left(\sum_{k=n}^{\infty} k_{-c} \right)^p \quad (c > 1, p \geqslant 1); \tag{1}$$

$$\sum_{n=1}^{\infty} n^{-c} \left(\sum_{k=n}^{\infty} a_k \right)^p \leqslant K \sum_{n=1}^{\infty} n^{-c(1-p)} a_n^p \left(\sum_{k=1}^{n} n^{-c} \right)^p \quad (c > 1, p \geqslant 1); \tag{2}$$

$$\sum_{n=1}^{\infty} n^{-c(1-p)} a_n^p \left(\sum_{k=n}^{\infty} n^{-c} \right)^p \leqslant K \sum_{n=1}^{\infty} n^{-c} \left(\sum_{k=1}^{n} a_k \right)^p \quad (c > 1, 0 < p \leqslant 1); \tag{3}$$

$$\sum_{n=1}^{\infty} n^{-c(1-p)} a_n^p \left(\sum_{k=1}^{n} n^{-c} \right)^p \leqslant K \sum_{n=1}^{\infty} n^{-c} \left(\sum_{k=n}^{\infty} a_k \right)^p \quad (c > 1, 0 < p \leqslant 1). \tag{4}$$

Theorem A was generalized by Leindler [4] where he obtained analogues of (1), (2), (3), and (4). In a subsequent paper, Leindler [5] improved the inequalities and showed that, in (1) and (2), the constant $K = p^p$ was best possible.

Before we state the theorems of Leindler, we introduce the following notations: for given sequences $\{a_n\}$ and $\{\lambda_n\}$, we denote A_{mn} and Λ_{mn} by

$$A_{mn} = \sum_{i=m}^{n} a_i \quad \text{and} \quad \Lambda_{mn} = \sum_{i=m}^{n} \lambda_i \ (1 \leqslant m \leqslant n < \infty).$$

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THEOREM B [4]. *Let $a_n \geqslant 0$ and $\lambda_n \geqslant 0 (n = 1, 2, \dots)$ be given. Let $\nu_1 < \nu_2 < \dots < \nu_n < \dots$ denotes indices for which $\lambda_{\nu_n} > 0, n = 1, 2, \dots$. Let N be the number of positive terms of the sequence $\{\lambda_n\}$, provided this number is finite, in the contrary case set $N = \infty$. Set $\nu_0 = 0$, and if $N < \infty$ then $\nu_{N+1} = \infty$. Then we have the following inequalities*

$$\sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leqslant p^p \sum_{n=1}^{N} \lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}, \quad p \geqslant 1; \tag{5}$$

$$\sum_{n-1}^{\infty} \lambda_n A_{n,\infty}^p \leqslant p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}, \quad p \geqslant 1; \tag{6}$$

$$\sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n} \leqslant 8 \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p, \quad 0 < p \leqslant 1; \tag{7}$$

$$\sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}}^p \leqslant 9 \sum_{n-1}^{\infty} \lambda_n A_{n,\infty}^p, \quad 0 < p \leqslant 1. \tag{8}$$

The constants p^p is best possible for (5) and (6).

In [5] Leindler showed that the inequalities (7) and (8) can be improved. Precisely, he proved:

THEOREM C [5]. *Under the assumptions of Theorem B the opposite inequalities of (5) and (6) hold for $0 < p \leqslant 1$, and the constant p^p is also best possible in this case.*

Theorem B and Theorem C imply the following:

COROLLARY. *If $\lambda_n > 0$ and $a_n \geqslant 0$, then the following hold:*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{k=1}^{n} a_k \right)^p \leqslant p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=n}^{\infty} \lambda_k \right)^p, \quad p \geqslant 1; \tag{9}$$

$$\sum_{n-1}^{\infty} \lambda_n \left(\sum_{k=n}^{\infty} a_k \right)^p \leqslant p^p \sum_{n-1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{k=1}^{n} \lambda_k \right)^p, \quad p \geqslant 1. \tag{10}$$

If $0 < p < 1$ the sign of the inequalities are reversed. All constants are best possible.

The main goal of this paper is to provide an alternative proof of Theorem C. In fact, we shall prove the following theorem which is the result of Leindler [4] and [5].

THEOREM 1. *Under the conditions of Theorem B, inequalities (5) and (6) hold when $p \geqslant 1$. If $0 < p \leqslant 1$ the sign of the inequalities in (5) and (6) are reversed.*

2. Proof of the Theorem

We need the following lemmas for the proof.

LEMMA 1. (see [1] Lemma 1, and [3] Lemma 5.1). *If* $p > 1$ *and* $z_n \geqslant 0$, $n = 1,2,\ldots,$ *then*

$$\left(\sum_{k=1}^{n} z_k\right)^p \leqslant p \sum_{k=1}^{n} z_k \left(\sum_{j=1}^{k} z_j\right)^{p-1}. \tag{11}$$

If $0 < p \leqslant 1$, the sign of the inequality is reversed.

LEMMA 2 [7]. *If* $p > 1$ *and* $z_n \geqslant 0$, $n = 1,2,\ldots,$ *then for every natural number* $m > n$,

$$\left(\sum_{k=n}^{m} z_k\right)^p \leqslant p \sum_{k=n}^{m} z_k \left(\sum_{j=k}^{m} z_j\right)^{p-1}. \tag{12}$$

If $0 < p \leqslant 1$, the sign of the inequality is reversed.

Proof of Theorem 1. Let $0 < p \leqslant 1$. First we shall prove

$$\sum_{n=1}^{\infty} \lambda_n A_{1n}^p \geqslant p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty} A_{\nu_{n-1}+1,\nu_n}^p. \tag{13}$$

For $p = 1$, both sides of (13) are equal as can be seen from changing the order of summation. If $0 < p < 1$, we follow Leindler [5] and set the following notations:

$$\alpha_n := A_{\nu_{n-1}+1,\nu_n}, \quad \beta_0 = 0, \quad \beta_n := \sum_{k=1}^{n} \alpha_k, \quad \delta_n := \lambda_{\nu_n},$$

and $R_n := \Lambda_{\nu_n,\infty}$ for every n in $1 \leqslant n \leqslant N$. If $N < \infty$, then let $R_{N+1} := \delta_{N+1} = 0$.

For any positive integer $m, m \leqslant N$, we have

$$\sum_{k=1}^{\nu_m} \lambda_k \left(\sum_{i=1}^{k} a_i\right)^p = \sum_{n=1}^{m} \lambda_{\nu_n} \left(\sum_{k=1}^{n} \alpha_i\right)^p = \sum_{n=1}^{m} \delta_n (\beta_n)^p. \tag{14}$$

Using Lemma 1 for $0 < p < 1$, we have

$$\begin{aligned} \sum_{n=1}^{m} \delta_n (\beta_n)^p &= \sum_{n=1}^{m} \delta_n \left(\sum_{k=1}^{n} \alpha_k\right)^p \geqslant p \sum_{n=1}^{m} \delta_n \sum_{k=1}^{n} \alpha_k \left(\sum_{j=1}^{k} \alpha_j\right)^{p-1} \\ &= p \sum_{n=1}^{m} \delta_n \sum_{k=1}^{n} \alpha_k (\beta_k)^{p-1} \\ &= p \sum_{k=1}^{m} \alpha_k (\beta_k)^{p-1} \sum_{n=k}^{m} \delta_n. \end{aligned}$$

Multiplying and dividing by $\delta_k^{\frac{1}{p}-1}$ and using Hölder inequality for $0 < p < 1$, we get for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\sum_{n=1}^{m} \delta_n(\beta_n)^p \geqslant p \left(\sum_{k=1}^{m} \alpha_k^p \delta_k^{1-p} \left(\sum_{n=k}^{m} \delta_n \right)^p \right)^{1/p} \left(\sum_{k=1}^{m} \delta_k(\beta_k)^p \right)^{1/q}. \qquad (15)$$

We divide both sides by the second factor on the right side of (15) to get

$$\left(\sum_{n=1}^{m} \delta_n(\beta_n)^p \right)^{1/p} \geqslant p \left(\sum_{k=1}^{m} \alpha_k^p \delta_k^{1-p} \left(\sum_{n=k}^{m} \delta_n \right)^p \right) 1/p. \qquad (16)$$

Substituting the expressions for δ_n, β_n and α_n, we see that the opposite of inequality (5) holds.

Note that if the second factor on the right of (15) is zero, then the result obviously holds.

To prove inequality (5) one can follow the same reasoning and use (11) and Hölder inequality for $p > 1$.

To prove the opposite of inequality (6), i.e.,

$$\sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \geqslant p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p$$

we define (as in Leindler [5])

$$\alpha_n^* := A_{\nu_n,\nu_{n+1}-1}, \quad \gamma_n := \sum_{k=n}^{N} \alpha_k^* \quad \text{and} \quad \Lambda_{1,\nu_N} = \sum_{k=1}^{n} \delta_k.$$

Using the case $0 < p < 1$ of Lemma 2 and following the method of proof given above, we can prove the opposite of inequality (6). Inequality (6) is obtained in the same manner when we use (12) and Hölder inequality for $p > 1$, appropriately.

REMARK. The advantage of our method of proof is that it unifies the method of proof for $p > 1$ and $0 < p < 1$, an important result of Leindler, and can be used to prove similar inequalities.

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