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UNIFIED APPROACH TO REFINEMENTS OF JENSEN'S INEQUALITIES

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Abstract. In this note we introduce a notion of M-dominated k-sample mean, where M is fixed multiset, which unifies several refinements of the Jensen's inequality for any mid-convex function. The approach treats on equal basis sample means over samples with of without repetitions.

1. Introduction

Let $f : C \to \mathbf{R}$ be a real-valued function defined on a convex set C in a real linear space X, $x_i \in C$ $(i = 1, \ldots, n)$. Let $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ be a fixed multiset having $m_j =: \nu_j(M)$ elements equal to j, for $1 \leqslant j \leqslant n$. Then we define M-dominated k-sample mean of f by

$$f_{k,n}^M = f_{k,n}^M(x_1, \ldots, x_n) = \frac{1}{N_k(M)} \sum_{\substack{I \subset M \\ |I|=k}} f\left(\frac{x_I}{k}\right) \tag{1}$$

where for every submultiset $I = \{1^{p_1}, 2^{p_2}, \ldots, n^{p_n}\}$ of M (i. e. $p_j \leqslant m_j$) x_I denotes $\sum_{i \in I} x_i = \sum_{j=1}^{n} p_j x_j$, $|I|$ denotes $\sum_{i=1}^{n} p_i$ and $N_k(M)$ denotes the k-th rank number of M $(= \#\{I \subset M \,|\, |I| = k\})$.

If f is mid-convex, then the well-known Jensen's inequality reads as follows

$$f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \leqslant \frac{1}{n}\sum_{j-1}^{n} f(x_j) \tag{2}$$

together with a simple, but important, consequence

$$f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \leqslant \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{x_1 + \cdots + \widehat{x_j} + \cdots + x_n}{n-1}\right) \tag{3}$$

(where $\widehat{x_j}$ means that x_j is omitted) (c. f. Key Lemma in [3]).

The following Proposition makes possible unification of results on Jensen's inequalities based on samples with/without repetitions.

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PROPOSITION. *With notations as above, we have*

$$N_{k+1}(M)f^M_{k+1,n} = \sum_{I\subset M,|I|=k+1} f\left(\frac{1}{k+1}x_I\right) \leqslant \frac{1}{k+1}\sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right) \tag{4}$$

where $c_I = \sum\limits_{\substack{1\leqslant j\leqslant n\\ \nu_j(I)<m_j}} (\nu_j(I)+1)$.

Proof. Applying the Jensen inequality in the form (3) to the terms in the left hand side in (4) yields:

$$\sum_{I\subset M,|I|=k+1} f\left(\frac{1}{k+1}x_I\right) \leqslant \frac{1}{k+1}\sum_{I\subset M,|I|=k+1}\sum_{i\in I} f\left(\frac{1}{k}x_{I\setminus\{i\}}\right).$$

Then, the right hand side can be rewritten as

$$= \frac{1}{k+1}\sum_{I\subset M,|I|=k} c_I f\left(\frac{1}{k}x_I\right)$$

where $c_I = \#\{I'\subset M : |I'| = k+1 \text{ and } I' = I\cup\{j\} \text{ for some } 1\leqslant j\leqslant n\}$. If $I = \{\underbrace{1\leqslant 1\leqslant 1\cdots\leqslant 1}_{\nu_1(I)} \leqslant \underbrace{2\leqslant 2\leqslant\cdots\leqslant 2}_{\nu_2(I)} \leqslant\cdots\leqslant \underbrace{n\leqslant\cdots\leqslant n}_{\nu_n(I)}\}$, then j can be "added" iff $\nu_j(I)+1\leqslant \nu_j(M) = m_j$ and this can be done in $\nu_j(I)+1$ ways. Hence

$$c_I = \sum_{\substack{1\leqslant j\leqslant n\\ \nu_j(I)<m_j}} (\nu_j(I)+1), \tag{5}$$

so the proof follows.

Recall now the functions $f_{k,n}$ and $\overline{f}_{k,n}$ associated to a mid-convex function f, as defined in [3]:

$$f_{k,n} = f_{k,n}(x_1,\dots,x_n) = \binom{n}{k}^{-1}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \quad (1\leqslant k\leqslant n) \tag{6}$$

$$\overline{f}_{k,n} = \overline{f}_{k,n}(x_1,\dots,x_n) = \binom{n+k-1}{k}^{-1}\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \quad (k\geqslant 1) \tag{7}$$

COROLLARY. *If f is mid-convex, then the following refinements of the Jensen's inequality (c. f. [3,2,1])*

a) $f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) = f_{n,n}\leqslant\cdots\leqslant f_{k+1,n}\leqslant f_{k,n}\leqslant\cdots\leqslant f_{1,n} = \frac{1}{n}\sum_{i=1}^n f(x_i)$

b) $f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) = \overline{f}_{\infty,n}\leqslant\cdots\leqslant \overline{f}_{k+1,n}\leqslant \overline{f}_{k,n}\leqslant\cdots\leqslant \overline{f}_{1,n} = \frac{1}{n}\sum_{i=1}^n f(x_i)$ *hold true.*

Proof. Both cases follow from the Proposition above as follows.

For a) we take M to be the following multiset (actually a set) $M = \{1, 2, \ldots, n\}$. Here $\nu_j(M) = 1$, $1 \leqslant j \leqslant n$, so for $I \subset M$, $|I| = k$ from (5) we obtain $c_I = \sum_{\substack{1 \leqslant j \leqslant n \\ \nu_j(I) < 1}} (\nu_j(I) + 1) = n - k$. Now, from (4)

$$\binom{n}{k+1} f_{k+1,n} = N_{k+1}(M) f^M_{k+1,n} \leqslant \frac{1}{k+1} \sum_{I \subset M, |I| = k} (n-k) f\left(\frac{1}{k} x_I\right) = \frac{n-k}{k+1} \binom{n}{k} f_{k,n}$$

i. e. $f_{k+1,n} \leqslant f_{k,n}$.

For b) we take M to be the multiset $M = \{1^\infty, 2^\infty, \ldots, n^\infty\}$ here $\nu_j(M) = \infty$, $1 \leqslant j \leqslant n$, so for $I \subset M, |I| = k$ from (5) we get

$$c_I = \sum_{\substack{1 \leqslant j \leqslant n \\ \nu_j(I) < \infty}} (\nu_j(I) + 1) = n + \sum_{j=1}^{n} \nu_j(I) = n + k.$$

Now, from (4):

$$\begin{aligned} \binom{n+k}{k+1} \overline{f}_{k+1,n} = N_{k+1}(M) f^M_{k+1,n} &\leqslant \frac{1}{k+1} \sum_{I \subset M, |I| = k} (n+k) f\left(\frac{1}{k} x_I\right) \\ &= \frac{n+k}{k+1} \binom{n+k-1}{k} \overline{f}_{k,n} \end{aligned}$$

i. e. $\overline{f}_{k+1,n} \leqslant \overline{f}_{k,n}$.

For a wider list of interesting specializations of the refinements stated in the Corollary above, which involve mixtures of the geometric and arithmetic means based on samples with (or without) repetitions, for readers convenience, we refer to the paper [3].

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