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ON MAXIMUM MODULUS OF POLYNOMIALS AND RELATED ENTIRE FUNCTIONS WITH RESTRICTED ZEROS

N. K. GOVIL AND M. A. QAZI

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Abstract. Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n, and $M(p,r) = \max\limits_{|z|=r} |p(z)|$. As a generalization of a well known result of Rivlin [5] and some other results ([1], [4]) in this direction, Jain [2] besides proving some other results, also proved that if the polynomial $p(z) = a_0 + \sum_{\nu=m}^{n} a_\nu z^\nu$ has no zeros in $|z| < k$, $k > 0$, then for $0 \leqslant r < R \leqslant k$, $M(p,r) \geqslant \left(\dfrac{r^m + k^m}{R^m + k^m}\right)^{n/m} M(p,R)$. In this paper we present very simple proofs of this and other results of Jain [2].

Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n, and $M(p,r) = \max_{|z|=r} |p(z)|$. Then the following result of Rivlin [5] concerning the size of $M(p,r)$ is well known.

THEOREM A. [5] *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, $p(z) \neq 0$ in $|z| < 1$ then*

$$M(p,r) \geqslant \left(\frac{1+r}{2}\right)^n M(p,1), \qquad r \leqslant 1. \tag{1}$$

The result is best possible with equality only for the polynomial $p(z) = \left(\dfrac{\alpha + \beta z}{2}\right)^n$, $|\alpha| = |\beta|$.

Govil [1] generalized the above result of Rivlin [5] by proving

THEOREM B. [1, Theorem 1] *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for $0 \leqslant r \leqslant R \leqslant 1$,*

$$M(p,r) \geqslant \left(\frac{1+r}{1+R}\right)^n M(p,R). \tag{2}$$

The result is best possible with equality only for the polynomial $p(z) = \left(\dfrac{\alpha + \beta z}{2}\right)^n$, $|\alpha| = |\beta|$.

The above result of Govil [1] was generalized by Qazi [4] (see also [3, Theorem 1.7.8, p. 445]) who proved

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THEOREM C. [4] *If* $p(z) = a_0 + \sum_{\nu=m}^{n} a_\nu z^\nu$ *has no zeros in* $|z| < 1$*, then for* $0 \leqslant s \leqslant S \leqslant 1$*,*

$$M(p,s) \geqslant \left(\frac{1+s^m}{1+S^m}\right)^{n/m} M(p,S); \tag{3}$$

more precisely,

$$M(p,s) \geqslant \exp\left(-n\int_s^S \frac{t^m + \frac{m}{n}\frac{|a_m|}{|a_0|}t^{m-1}}{t^{m+1} + \frac{m}{n}\frac{|a_m|}{|a_0|}(t^m+t)+1}\,dt\right) M(p,S). \tag{4}$$

An entire function f is said to be of exponential type τ if it is either of order less than 1 or it is of order 1, type at most τ. The Phragmén-Lindelöf indicator function of a function f of exponential type is defined to be

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r} \qquad (0 \leqslant \theta < 2\pi).$$

The boundedness of an entire function of exponential type on one line carries with it boundedness on every parallel line. Assuming that f is bounded on the real axis, we set

$$\mu(f;y) := \sup_{-\infty<x<\infty} |f(x+iy)|.$$

If $p(z)$ is a polynomial of degree n, then $f(z) := p(e^{iz})$ is an entire function of exponential type $\tau = n$, bounded on the real axis. Besides $h_{f'}(\pi/2) \leqslant -1$, $h_{g'}(\pi/2) \leqslant -1$ where $g(z) := e^{i\tau z}\overline{f(\overline{z})}$. If $p(z) \neq 0$ for $|z| < 1$ then $f(z) \neq 0$ for $\mathbb{I}m\, z > 0$ and $h_f(\pi/2) = 0$. Thus as an extension of Theorem C to functions of exponential type, Qazi [4] proved

THEOREM D. [4] *Let* f *be an entire function of exponential type* τ *bounded on the real axis. Further, let* $h_f(\pi/2) = 0$*,* $h_{f'}(\pi/2) \leqslant -\lambda < 0$*,* $h_{g'}(\pi/2) \leqslant -\lambda < 0$*, where* $g(z) := e^{i\tau z}\overline{f(\overline{z})}$*. If* $f(z) \neq 0$ *for* $\operatorname{Im} z > 0$*, then*

$$\mu(f;y^*) \geqslant \left(\frac{1+e^{-\lambda y^*}}{1+e^{-\lambda\eta^*}}\right)^{\tau/\lambda} \mu(f;\eta^*) \ \text{for}\ \ 0 \leqslant \eta^* < y^* < \infty. \tag{5}$$

Recently, Jain [2] has proved the following respective generalizations of the above Theorems C and D of Qazi [4]:

THEOREM C$'$. [2, Theorem 1, p. 269] *If* $p(z) = a_0 + \sum_{\nu=m}^{n} a_\nu z^\nu$ *has no zeros in* $|z| < k$*,* $k > 0$*, then for* $0 \leqslant r < R \leqslant k$*,*

$$M(p,r) \geqslant \left(\frac{r^m + k^m}{R^m + k^m}\right)^{n/m} M(p,R); \tag{3'}$$

more precisely,

$$M(p,r) \geqslant \exp\left(-n\int_r^R \frac{t^m + \frac{m}{n}\frac{|a_m|}{|a_0|}k^{m+1}t^{m-1}}{t^{m+1} + k^{m+1} + \frac{m}{n}\frac{|a_m|}{|a_0|}(k^{m+1}t^m + k^{2m}t)+1}\,dt\right) M(p,R) \tag{4'}$$

There is equality in (3$'$) for $p(z) = (z^m + k^m)^{n/m}$*, where* n *is a multiple of* m*.*

THEOREM D$'$. [2, Theorem 2, p. 270] *Let $f(z)$ be an entire function of exponential type τ bounded on the real axis. Further, let $h_f(\pi/2) = 0$, $h_{f'}(\pi/2) \leqslant -\lambda < 0$, $h_{g'}(\pi/2) \leqslant -\lambda < 0$, where $g(z) = e^{i\tau z}\overline{f(\bar{z})}$. If $f(z) \neq 0$ for $\operatorname{Im} z > \gamma$ for some real number γ, then for $\gamma \leqslant \eta \leqslant y < \infty$*

$$\mu(f;y) \geqslant \left(\frac{e^{-\lambda\gamma}+e^{-\lambda y}}{e^{-\lambda\gamma}+e^{-\lambda\eta}}\right)^{\tau/\lambda} \mu(f;\eta). \tag{5'}$$

For $k = 1$, Theorem C$'$ reduces to Theorem C. To obtain Theorem D from Theorem D$'$ take $\gamma = 0$.

In order to prove Theorems C$'$ and D$'$, Jain in [2] develops machinary analogous to the proofs of Theorems C and D in [4]. In this paper we observe that although Theorems C$'$ and D$'$ are generalizations of Theorems C and D respectively, because Theorem C$'$ for $k = 1$ and Theorem D$'$ for $\gamma = 0$ reduce to Theorems C and D respectively, Theorems C$'$ and D$'$ can in fact be obtained very easily from Theorems C and D, just by using suitable transformations, and this can be done as follows.

If $p(z) \neq 0$ for $|z| < k$, $k > 0$ then the polynomial $p(kz) \neq 0$ for $|z| < 1$. Further, if $0 \leqslant r < R \leqslant k$, then $0 \leqslant \frac{r}{k} < \frac{R}{k} \leqslant 1$ and applying (3) to the polynomial $p(kz)$ with $s = r/k$ and $S = R/k$, we get

$$M(p,r) \geqslant \left(\frac{k^m+r^m}{k^m+R^m}\right)^{n/m} M(p,R),$$

which is (3$'$).

Similarly, if we apply (4) to $p(kz)$ with $s = r/k$ and $S = R/k$, we will get

$$\begin{aligned} M(p,r) &\geqslant \exp\left(-n\int_{r/k}^{R/k} \frac{t^m + \frac{m}{n}\frac{|a_m|}{|a_0|}k^m t^{m-1}}{t^{m+1} + \frac{m}{n}\frac{|k^m a_m|}{|a_0|}(t^m+t)+1}\,dt\right) M(p,R) \\ &= \exp\left(-n\int_{r}^{R} \frac{t^m + \frac{m}{n}\frac{|a_m|}{|a_0|}k^{m+1} t^{m-1}}{t^{m+1} + k^{m+1} + \frac{m}{n}\frac{|a_m|}{|a_0|}(k^{m+1}t^m+k^{2m}t)}\,dt\right) M(p,R) \end{aligned}$$

which is (4$'$).

If $f(z)$ is an entire function of exponential type τ with $f(z) \neq 0$ for $\operatorname{Im} z > \gamma$ then the function $F(z) = f(z+i\gamma)$ is an entire function of exponential type τ with $F(z) \neq 0$ for $\operatorname{Im} z > 0$. Also, if $G(z) = e^{i\tau z}\overline{F(\bar{z})}$ then as is easy to verify, $h_f(\pi/2) = 0$, $h_{f'}(\pi/2) \leqslant -\lambda < 0$, $h_{g'}(\pi/2) \leqslant -\lambda < 0$ implies $h_F(\pi/2) = 0$, $h_{F'}(\pi/2) \leqslant -\lambda < 0$, $h_{G'}(\pi/2) \leqslant -\lambda < 0$ respectively. Note that $\gamma \leqslant \eta \leqslant y < \infty$ implies $0 \leqslant \eta - \gamma \leqslant y - \gamma < \infty$, hence if we apply Theorem D to $F(z) = f(z+i\gamma)$ with $\eta^* = \eta - \gamma$ and $y^* = y - \gamma$, we get

$$\mu(F;y-\gamma) \geqslant \left(\frac{1+e^{-\lambda(y-\gamma)}}{1+e^{-\lambda(\eta-\gamma)}}\right)^{\tau/\lambda} \mu(F;\eta-\gamma).$$

Since $\mu(F;y-\gamma)=\mu(f;y)$ and $\mu(F;\eta-\gamma)=\mu(f;\eta)$, the above inequality is clearly equivalent to

$$\mu(f;y)\geqslant\left(\frac{e^{-\lambda\gamma}+e^{-\lambda y}}{e^{-\lambda\gamma}+e^{-\lambda\eta}}\right)^{\tau/\lambda}\mu(f;\eta),$$

and Theorem D$'$ is thus established.

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N. K. Govil
Department of Mathematics
Auburn University
Auburn, Alabama 36849 – 5310
U.S.A.
e-mail: govilnk@auburn.edu

M. A. Qazi
Department of Mathematics
Tuskegee University
Tuskegee, Alabama 36088
U.S.A.
e-mail: maqazi@tusk.edu