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A GENERAL EXISTENCE THEOREM FOR THE SINGULAR EQUATION $(\varphi_p(y'))' + f(t,y) = 0$

HAISHEN LÜ AND DONAL O'REGAN

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Abstract. Some theorems concerning the existence of positive solution for the singular equation $(\varphi_p(y'))' + f(t,y) = 0$, $y(0) = y(1) = 0$, are established. The results are obtained using the lower-upper solution approach.

1. Introduction

In this article we present existence results for the boundary value problem

$$\begin{cases} (\varphi_p(y'))' + f(t,y) = 0, \ 0 < t < 1, \\ y(0) = y(1) = c, \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$ and $c \geqslant 0$. Equations of the above form occur in the study of the $p-$Laplace equation [2], non-Newtonial fluid theory [3], and the turbulent flow of a gas in a porous medium [4].

Problem (1.1) has been studied by many authors (see [4–11] and references therein) usually under non-singular conditions. Recently, [5, 11], existence results were given for the singular boundary value problem

$$\begin{cases} (\varphi_p(y'))' + q(t)f(t,y) = 0, \ 0 < t < 1, \\ y(0) = A, \ \ y(1) = B, \end{cases} \tag{1.2}$$

where $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is continuous, and

$$q \in C(0,1), \ \int_0^1 q(s)ds < +\infty. \tag{1.3}$$

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In this paper, we give new general existence results for the problem (1.1). The function $f(t,y)$ can be singular at both end points $t=0$, and $t=1$. In addition, we only require q to satisfy

$$\begin{cases} q \in C(0,1), \quad \text{with} \\ \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} q(r)\,dr\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s q(r)\,dr\right) ds < +\infty, \end{cases} \tag{1.4}$$

where φ_p^{-1} is the inverse function of φ_p.

It is obvious that condition (1.3) is a special case of condition (1.4). On the other hand, if $p=2$, condition (1.4) is equivalent to

$$q \in C(0,1), \quad \text{and} \quad \int_0^1 t(1-t)\,q(t)\,dt < +\infty$$

2. Main Results

Consider the two-point boundary value problem

$$\begin{cases} (\varphi_p(y'))' + f(t,y) = 0, \ 0<t<1, \\ y(0) = y(1) = c, \end{cases} \tag{2.1}$$

where $f : D \to \mathbf{R}$ is continuous function and $D \subset (0,1)\times\mathbf{R}$. By a solution y of (2.1), we mean a function $y \in C([0,1],\mathbf{R}) \cap C^1((0,1),\mathbf{R})$ with $\varphi_p(y') \in C^1((0,1),\mathbf{R})$ and such that $(t,y(t)) \in D$ for all $t \in (0,1)$, $(\varphi_p(y'))' + f(t,y) = 0$ for all $t \in (0,1)$, and $y(0)=y(1)=c$.

DEFINITION 1. Let $\alpha \in C([0,1],\mathbf{R}) \cap C^1((0,1),\mathbf{R})$, and $\varphi_p(\alpha') \in C^1((0,1),\mathbf{R})$. α is called a *lower solution* for problem (2.1) if $(t,\alpha(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{cases} (\varphi_p(\alpha'(t)))' + f(t,\alpha(t)) \geqslant 0, \ 0<t<1, \\ \alpha(0) \leqslant c, \ \alpha(1) \leqslant c. \end{cases}$$

Let $\beta \in C([0,1],\mathbf{R}) \cap C^1((0,1),\mathbf{R})$, and $\varphi_p(\beta') \in C^1((0,1),\mathbf{R})$. β is called an *upper solution* for problem (2.1) if $(t,\beta(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{cases} (\varphi_p(\beta'(t)))' + f(t,\beta(t)) \leqslant 0, \ 0<t<1, \\ \beta(0) \geqslant c, \ \beta(1) \geqslant c. \end{cases}$$

Also, if α, $\beta \in C([0,1],\mathbf{R})$ are such that $\alpha(t) \leqslant \beta(t)$, for all $t \in [0,1]$, we define the set $D_\alpha^\beta := \{(t,x) \in (0,1)\times\mathbf{R} : \alpha(t) \leqslant x \leqslant \beta(t)\}$.

LEMMA 2.1. *Let $f : (0,1)\times\mathbf{R} \to \mathbf{R}$ be continuous, $q \in C(0,1)$ and assume the following conditions hold:*

$$|f(t,y)| \leqslant M\,q(t), \ (t,y) \in (0,1)\times\mathbf{R}, \tag{2.2}$$

and

$$\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} q(r)\,dr\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s q(r)\,dr\right) ds < +\infty; \tag{2.3}$$

here $M > 0$ is a constant. Then the BVP (2.1) has at least one solution for any $c \in \mathbf{R}$.

In order to prove the existence of solutions to (2.1), we consider the boundary value problem

$$\begin{cases} (\varphi_p(y'))' + \eta_n(t) f(t,y) = 0, \ 0 < t < 1, \\ y(0) = y(1) = c, \end{cases} \tag{2.1^n}$$

where $n \geqslant 4$ is a natural number, $\eta_n(t)$ is a continuous function such that $0 \leqslant \eta_n(t) \leqslant 1$ on $[0,1]$ and

$$\eta_n(t) = \begin{cases} 0, & 0 < t < \frac{1}{2n}, \\ 1, & \frac{1}{n} \leqslant t \leqslant 1 - \frac{1}{n}, \\ 0, & t > 1 - \frac{1}{2n}. \end{cases}$$

We need the following.

LEMMA 2.2. *There exists a unique solution $V \in C([0,1],\mathbf{R}) \cap C^1((0,1),\mathbf{R})$ to the problem*

$$\begin{cases} (\varphi_p(V'))' + Mq(t) = 0, \ 0 < t < 1, \\ V(0) = V(1) = c. \end{cases} \tag{2.4}$$

Proof. We prove existence (the proof of uniqueness is elementary). Set for $t \in (0,1)$

$$x(t) := \int_0^t \varphi_p^{-1}\left(\int_s^t q(r)\,dr\right) ds - \int_t^1 \varphi_p^{-1}\left(\int_t^s q(r)\,dr\right) ds, \ \ 0 < t < 1.$$

Clearly, $x(t)$ is continuous and nondecreasing in $(0,1)$ and $x(0+) < 0 < x(1-)$. Thus, $x(t)$ has zeros in $(0,1)$. Let ξ be a zero of $x(t)$ in $(0,1)$. Then

$$\int_0^\xi \varphi_p^{-1}\left(\int_s^\xi q(r)\,dr\right) ds = \int_\xi^1 \varphi_p^{-1}\left(\int_\xi^s q(r)\,dr\right) ds. \tag{2.5}$$

Put

$$V(t) := \begin{cases} c + \int_0^t \varphi_p^{-1}\left(\int_s^\xi Mq(r)\,dr\right) ds, \ 0 \leqslant t \leqslant \xi, \\ c + \int_t^1 \varphi_p^{-1}\left(\int_\xi^s M\,q(r)\,dr\right) ds, \ \xi \leqslant t \leqslant 1. \end{cases} \tag{2.6}$$

Then, V is a well defined function on $[0,1]$. Moreover,

$$V'(t) := \begin{cases} \varphi_p^{-1}\left(\int_s^\xi M\,q(r)dr\right) ds, \ 0 < t \leqslant \xi, \\ -\varphi_p^{-1}\left(\int_\xi^s M\,q(r)\,dr\right) ds, \ \xi \leqslant t < 1. \end{cases}$$

Therefore, $(\varphi_p(V'))' + Mq(t) = 0, \ 0 < t < 1$ and $V(0) = V(1) = c$. □

An argument similar to that in Lemma 2.2 yields our next result.

LEMMA 2.3. *There exists a unique solution* $v \in C([0,1],\mathbf{R}) \cap C^1((0,1),\mathbf{R})$ *to the problem*

$$\begin{cases} (\varphi_p(v'))' - Mq(t) = 0, \ 0 < t < 1, \\ v(0) = v(1) = c. \end{cases} \tag{2.7}$$

LEMMA 2.4. *Let* $u_n(t)$ *be a solution to* $(2.1)_n$. *Then*

$$v(t) \leqslant u_n(t) \leqslant V(t), \ 0 \leqslant t \leqslant 1.$$

Proof. We shall prove only $u_n(t) \leqslant V(t)$ on $[0,1]$ since the argument is essentially the same for the case $u_n(t) \geqslant v(t)$ on $[0,1]$.

To see this, suppose

$$u_n(t) \nleqslant V(t) \quad \text{for } \ t \in (0,1).$$

Since $u_n(0) = V(0), u_n(1) = V(1)$, then there exists $t_0 \in (0,1)$ with $u_n(t_0) > V(t_0)$ and hence there would exist an interval (a,b) such that $u_n(t) > V(t)$ in (a,b) and $u_n(a) - V(a) = u_n(b) - V(b) = 0$. Then $u_n(t) - V(t)$ has a positive maximum at a point $B \in (a,b)$. Note $u_n'(B) = V'(B)$. Let $m = u_n(B) - V(B)$. Integrate both sides of the equality (2.4) and $(2.1)_n$ from s to B, $a < s < B$, to get

$$V'(s) = \varphi_p^{-1}\left(\varphi_p(V'(B)) + M\int_s^B q(r)\,dr\right),$$

and

$$u_n'(s) = \varphi_p^{-1}\left(\varphi_p(u_n'(B)) + \int_s^B \eta_n(r) f(r,y(r))\,dr\right).$$

Now integrate both side of the above equality from a to B, to obtain

$$V(B) - V(a) = \int_a^B \varphi_p^{-1}\left(\varphi_p(V'(B)) + M\int_s^B q(r)\,dr\right) ds,$$

and

$$u_n(B) - u_n(a) = \int_a^B \varphi_p^{-1}\left(\varphi_p(u_n'(B)) + \int_s^B \eta_n(r) f(r,y(r))\,dr\right) ds.$$

This leads to $0 < m = u_n(B) - V(B) \leqslant 0$, a contradiction. □

LEMMA 2.5. *For each fixed* $n \geqslant 4$, $(2.1)_n$ *has a solution* $u_n \in C^1([0,1],\mathbf{R})$.

Proof. This is a consequence of Theorem 2.1 in [5] since

$$|\eta_n(t) f(t,y)| \leqslant M\eta_n(t)\, q(t),$$

with

$$\eta_n(t) f(t,y) \quad \text{continuous for} \quad (t,y) \in [0,1] \times \mathbf{R}$$

and $\int_0^1 \eta_n(t)\, q(t)\, dt < +\infty$. □

LEMMA 2.6. *$\{u_n\}_{n=4}^{\infty}$ is equicontinuous on* $[0,1]$.

Proof. Let $\varepsilon > 0$ be given. From the continuity of v, V on $[0,1]$, it follows that there exists a constant $\delta_1 \in \left(0, \frac{1}{4}\right)$ such that

$$c - \varepsilon < v(t) \leqslant V(t) < c + \varepsilon \quad \text{for} \quad t \in [0, 2\delta_1],$$

and

$$c - \varepsilon < v(t) \leqslant V(t) < c + \varepsilon \quad \text{for} \quad t \in [1 - 2\delta_1, 1].$$

Fix $n \in \{4, 5, \ldots\}$. Now

$$|u_n(t_1) - u_n(t_2)| < 2\varepsilon, \text{ for } t_1, t_2 \in [0, 2\delta_1] \text{ or } t_1, t_2 \in [1 - 2\delta_1, 1].$$

Next consider $t_1, t_2 \in [\delta_1, 1 - \delta_1]$. It is clear that there is an n^* such that $\frac{1}{n^*} \leqslant \delta_1$. Then, for all $n > n^*$, we have

$$\begin{aligned} 0 &= (\varphi_p(u_n'(t)))' + \eta_n(t) f(t, u_n(t)) \\ &= (\varphi_p(u_n'(t)))' + f(t, u_n(t)), \quad \text{for all } t \in [a, b]; \end{aligned}$$

here $a = \delta_1$, $b = 1 - \delta_1$. Thus u_n is a solution of (2.1) for all $t \in [a, b]$ and $n \geqslant n^*$. In addition

$$\sup\{|f(t, x)| : t \in [a, b], x \in \mathbf{R}\} \leqslant \sup\{M q(t) : t \in [a, b]\} < +\infty.$$

Also notice

$$u_n(t) = u_n(a) + \int_a^t \varphi_p^{-1}\left(\tau_n + \int_r^b f(s, u_n(s))\, ds\right) dr, \ a \leqslant t \leqslant b,$$

where τ_n is a solution of the equation

$$\int_a^b \varphi_p^{-1}\left(\tau_n + \int_r^b f(s, u_n(s))\, ds\right) dr = u_n(b) - u_n(a).$$

By the mean value theorem, there exists a $\xi_n \in [a, b]$ with

$$\varphi_p^{-1}\left(\tau_n + \int_{\xi_n}^b f(s, u_n(s))\, ds\right) dr = \frac{u_n(b) - u_n(a)}{b - a}.$$

That is

$$\tau_n = -\int_{\xi_n}^b f(s, u_n(s))\, ds + \varphi_p\left(\frac{u_n(b) - u_n(a)}{b - a}\right).$$

From Lemma 2.4 we have $v(t) \leqslant u_n(t) \leqslant V(t)$. Therefore there is a constant $C > 0$ such that

$$|\tau_n| \leqslant M \int_a^b q(t) + \varphi_p\left(\frac{|u_n(b) - u_n(a)|}{b - a}\right) \leqslant C.$$

Then for $t_1, t_2 \in [\delta_1, 1-\delta_1]$ we have

$$\begin{aligned} |u_n(t_1) - u_n(t_2)| &= \left| \int_{t_2}^{t_1} \varphi_p^{-1} \left(\tau_n + \int_r^b f\,(s, u_n\,(s))\;ds \right) dr \right| \\ &\leqslant L\,|t_1 - t_2|\,; \end{aligned}$$

here $L = \varphi_p^{-1}\left(C + M\int_a^b q\,(s)\;ds\right)$.

Put $\delta_2 = \frac{\varepsilon}{L}$. Then if $t_1, t_2 \in [\delta_1, 1-\delta_1]$, and $|t_1 - t_2| < \delta_2$ we have

$$|u_n(t_1) - u_n(t_2)| \leqslant L\,|t_1 - t_2| < L\delta_2 = \varepsilon.$$

Finally set $\delta = \min\{\delta_1, \delta_2\}$. Then if $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$ we have

$$|u_n(t_1) - u_n(t_2)| < 3\,\varepsilon. \tag{2.8}$$

This completes the proof. □

Lemma 2.4 and Lemma 2.6, and the Ascoli–Arzela theorem, guarantees that there exists a uniformly convergent subsequence of $\{u_n\}_{n=4}^{\infty}$, denoted again by $\{u_n\}_{n=4}^{\infty}$, which converges uniformly to u on $[0,1]$. It is clear that $u\,(0) = u\,(1) = c$.

Proof of Lemma 2.1. Now let $\Gamma = [a,b] \subset (0,1)$ be a compact interval. It is clear that there is an $n^* = n^*\,(\Gamma)$ such that $\Gamma \subset \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$, for all $n > n^*$, and as a result

$$\begin{aligned} 0 &= (\varphi_p(u_n'\,(t)))' + \eta_n\,(t) f\,(t, u_n\,(t)) \\ &= (\varphi_p(u_n'\,(t)))' + f\,(t, u_n\,(t)), \quad \text{for all } t \in [a,b]. \end{aligned}$$

Thus u_n is a solution of (2.1) for all $t \in [a,b]$ and $n \geqslant n^*$. In addition

$$\sup\{|f\,(t,x)| : t \in [a,b], x \in \mathbf{R}\} \leqslant \sup\{Mq\,(t) : t \in [a,b]\} < +\infty.$$

Also notice

$$u_n\,(t) = u_n\,(a) + \int_a^t \varphi_p^{-1}\left(\tau_n + \int_r^b f\,(s, u_n\,(s))\;ds\right) dr, \; a \leqslant t \leqslant b,$$

where τ_n is a solution of the equation

$$\int_a^b \varphi_p^{-1}\left(\tau_n + \int_r^b f\,(s, u_n\,(s))\;ds\right) dr = u_n\,(b) - u_n\,(a).$$

By the mean value theorem, there exists a $\xi_n \in [a,b]$ with

$$\varphi_p^{-1}\left(\tau_n + \int_{\xi_n}^b f\,(s, u_n\,(s))\;ds\right) dr = \frac{u_n\,(b) - u_n\,(a)}{b-a}.$$

That is

$$\tau_n = -\int_{\xi_n}^b f\,(s, u_n\,(s))\;ds + \varphi_p\left(\frac{u_n\,(b) - u_n\,(a)}{b-a}\right).$$

From Lemma 2.4 we have $v(t) \leqslant u_n(t) \leqslant V(t)$. Therefore there is a constant $Q > 0$ such that

$$|\tau_n| \leqslant M \int_a^b q(t) + \varphi_p\left(\frac{|u_n(b) - u_n(a)|}{b-a}\right) \leqslant Q.$$

Thus there exists a convergent subsequence of $\{\tau_n\}_{n=n^*}^{\infty}$, denoted again by $\{\tau_n\}_{n=n^*}^{\infty}$, which converges to τ. In addition we know that there exists a convergent subsequence of $\{u_n\}_{n=n^*}^{\infty}$, denoted again by $\{u_n\}_{n=n^*}^{\infty}$, which converges uniformly to u on $[a,b]$.

Therefore,

$$u(t) = u(a) + \int_a^t \varphi_p^{-1}\left(\tau + \int_r^b f(s, u(s))\, ds\right) dr,\ a \leqslant t \leqslant b,$$

where τ is a solution of the equation

$$\int_a^b \varphi_p^{-1}\left(\tau + \int_r^b f(s, u(s))\, ds\right) dr = u(b) - u(a).$$

The above implies that u is a solution of (2.1) on the interval $\Gamma = [a,b]$. Since Γ is arbitrary, we find that

$$u \in C^1((0,1), \mathbf{R}), \text{ and } \left(\varphi_p(u')\right)' + f(t, u(t)) = 0 \text{ for } t \in (0,1).$$

Finally since $u \in C^1((0,1), \mathbf{R})$ and $u(0) = u(1) = c$, we have that u is a solution of (2.1). □

THEOREM 2.1. *Suppose*

(H1). $f : (0,1) \times \mathbf{R} \to \mathbf{R}$ *is continuous holds. In addition, assume* $\alpha \in C[0,1]$, $\beta \in C[0,1]$ *are lower and upper solution of problem* (2.1), *and suppose the following conditions are satisfied:*

(H2). $\alpha(t) \leqslant \beta(t)$, *for all* $0 \leqslant t \leqslant 1$,

(H3). $\alpha(0) \leqslant c \leqslant \beta(0)$, $\alpha(1) \leqslant c \leqslant \beta(1)$,

and

(H4). there exists a continuous function $q \in C(0,1)$, *with*

$$|f(t,y)| \leqslant q(t),\ \forall (t,y) \in D_{\alpha\beta},$$

and

$$\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} q(r)\, dr\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s q(r)\, dr\right) ds < +\infty.$$

Then the BVP (2.1) *has at least one solution* $y \in C[0,1]$, $\varphi_p(y') \in C^1(0,1)$, *and* $\alpha(t) \leqslant y(t) \leqslant \beta(t)$, $0 \leqslant t \leqslant 1$.

Proof. Let

$$g(t,y) = \begin{cases} f(t, \beta(t)) + q(t)\ \frac{(\beta(t) - y)}{(1 + y - \beta(t))}, & y > \beta(t), \\ f(t,y), & \alpha(t) \leqslant y \leqslant \beta(t), \\ f(t, \alpha(t)) + q(t)\ \frac{(\alpha(t) - y)}{(1 - y + \alpha(t))}, & y < \alpha(t). \end{cases}$$

Then $g:(0,1)\times\mathbf{R}\to\mathbf{R}$ is continuous and $|g(t,y)|\leqslant 2q(t)$, for $(t,y)\in(0,1)\times\mathbf{R}$. Lemma 2.1 implies that there exists at least one $y\in C^1[0,1]$, $\varphi_p(y')\in C^1(0,1)$, which satisfies

$$\begin{cases} (\varphi_p(y'))'+g(t,y)=0, & 0<t<1, \\ y(0)=c, \ \ y(1)=c. \end{cases} \tag{2.9}$$

If we can prove that y satisfies $\alpha(t)\leqslant y(t)\leqslant\beta(t)$ for $t\in(0,1)$ then y is a solution of problem (2.1).

We now prove that $\alpha(t)\leqslant y(t)\leqslant\beta(t)$ for $t\in[0,1]$. Notice

$$\alpha(0)\leqslant y(0)\leqslant\beta(0) \ \text{ and } \ \alpha(1)\leqslant y(1)\leqslant\beta(1).$$

Suppose $\alpha(t)\nleqslant y(t)$, for $t\in(0,1)$. Then exists $t_0\in(0,1)$, with

$$\min_{t\in[0,1]}(y(t)-\alpha(t))=y(t_0)-\alpha(t_0)<0, \quad \text{and} \ \ y'(t_0)=\alpha'(t_0). \tag{2.10}$$

On the other hand, $g(t_0,y(t_0))-f(t_0,\alpha(t_0))>0$, and by continuity there exists $\varepsilon_0>0$ with

$$g(t,y(t))-f(t,\alpha(t))>0 \ \text{ for all } \ t\in[t_0,t_0+\varepsilon_0). \tag{2.11}$$

Fix any $t\in[t_0,t_0+\varepsilon_0)$. From (2.9), (2.10), and (2.11) we have

$$\begin{aligned}\varphi_p(y'(t)) &= \varphi_p(y'(t_0))-\int_{t_0}^{t}g(s,y(s))\,ds<\varphi_p(\alpha'(t_0))-\int_{t_0}^{t}f(s,\alpha(s))\,ds \\ &\leqslant \varphi_p(\alpha'(t_0))-\int_{t_0}^{t}(\varphi_p(\alpha'(s)))'\,ds=\varphi_p(\alpha'(t)).\end{aligned}$$

Thus, $y'(t)-\alpha'(t)<0$ for $t\in[t_0,t_0+\varepsilon_0)$. This is a contradiction. Therefore, $\alpha(t)\leqslant y(t)$. Similiarly, $y(t)\leqslant\beta(t)$. □

EXAMPLE 1. Consider the following boundary value problem:

$$\begin{cases} (\varphi_p(y'))'+\lambda f(t,y)=0, & 0<t<1, \\ y(0)=y(1)=0, \end{cases} \tag{2.12}$$

where $\lambda\geqslant 0$.

Suppose the following conditions are satisfied:

(H5). $f:(0,1)\times\mathbf{R}^+\to\mathbf{R}^+$ is continuous,

(H6). there exists a continuous function $q\in C(0,1)$ with

$$\int_0^{\frac{1}{2}}\varphi_p^{-1}\left(\int_s^{\frac{1}{2}}q(r)\,dr\right)ds+\int_{\frac{1}{2}}^{1}\varphi_p^{-1}\left(\int_{\frac{1}{2}}^{s}q(r)\,dr\right)ds<+\infty,$$

and for each given $\eta>0$ there is a positive constant M_η satisfying

$$f(t,y)\leqslant M_\eta q(t), \ \forall(t,y)\in(0,1)\times[0,\eta].$$

Then, there exists a positive constant λ^* such that the BVP (2.12) has at least a positive solution for $0<\lambda<\lambda^*$.

Proof. It follows from $(H6)$ that for each given $\eta > 0$,

$$\int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_s^{\frac{1}{2}} M_\eta q(r)\, dr\right) ds + \int_{\frac{1}{2}}^1 \varphi_p^{-1}\left(\int_{\frac{1}{2}}^s M_\eta q(r)\, dr\right) ds < +\infty.$$

The result follows from Theorem 2.1 if we can find a lower solution α and an upper solution β of (2.12) satisfying $(H2)$ and $(H3)$.

To see this we first consider

$$\begin{cases} (\varphi_p(y'))' + q(t) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases}$$

It is clear that its solution β can be written as

$$\beta(t) := \begin{cases} \int_0^t \varphi_p^{-1}\left(\int_s^{\xi^*} q(r)\, dr\right) ds, & 0 \leqslant t \leqslant \xi^*, \\ \int_t^1 \varphi_p^{-1}\left(\int_{\xi^*}^s q(r)\, dr\right) ds, & \xi^* \leqslant t \leqslant 1, \end{cases}$$

where ξ^* is a zero of the function

$$x(t) := \int_0^t \varphi_p^{-1}\left(\int_s^t q(r)\, dr\right) ds - \int_t^1 \varphi_p^{-1}\left(\int_t^s q(r)\, dr\right) ds, \quad 0 < t < 1.$$

Let $\beta_0 = \max_{t\in[0,1]} \beta(t) > 0$. Then, since

$$\begin{aligned} (\varphi_p(\beta'))' + \lambda f(t, \beta(t)) &= -q(t) + \lambda f(t, \beta(t)) \\ &\leqslant -q(t) + \frac{1}{M_{\beta_0}} f(t, \beta(t)) \\ &\leqslant -q(t) + \frac{1}{M_{\beta_0}} M_{\beta_0} q(t) = 0, \quad \text{on} \quad (0,1) \end{aligned}$$

for $0 < \lambda \leqslant \frac{1}{M_{\beta_0}}$, so it follows that β is an upper solution of (2.12) if $0 < \lambda \leqslant \frac{1}{M_{\beta_0}}$.

On the other hand, $\alpha \equiv 0$ is obviously a lower solution of (2.12) and satisfies $\alpha(t) \leqslant \beta(t)$ for all $t \in [0,1]$. From Theorem 2.1, we deduce that (2.12) has a positive solution y if $0 < \lambda \leqslant \frac{1}{M_{\beta_0}}$. □

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Haisshen Lü
Department of Mathematics
Lanzhou University
Lanzhou, Gansu
730000, P. R. China

Donal O'Regan
Department of Mathematics
National University of Ireland
Galway, Ireland