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AN APPLICATION OF ALMOST INCREASING SEQUENCES

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Abstract. In this paper using any almost increasing sequence a result of Mishra and Srivastava [5] on $|C,1|_k$ summability factors has been generalized for $|C,\alpha;\delta|_k$ summability factors under weaker conditions.

1. Introduction. Let $\sum a_n$ be a given infinite series with the sequence of its partial sums (s_n). We denote by t_n^α n-th Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), i.e.,

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be $|C,\alpha|_k$ summable, $k \geqslant 1$ and $\alpha > -1$, if (see [3])

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n^\alpha|^k < \infty, \tag{3}$$

and it is said to be summable $|C,\alpha;\delta|_k$, $k \geqslant 1$, $\alpha > -1$ and $\delta \geqslant 0$, if (see [4])

$$\sum_{n=1}^{\infty} n^{\delta k-1}|t_n^\alpha|^k < \infty. \tag{4}$$

It is known that for $k \geqslant 1$ and $0 < \alpha \leqslant 1$ (see [7])

$$\sum_{n=1}^{m} \frac{1}{n}|t_n^\alpha|^k = O\{\sum_{n=1}^{m} \frac{|s_n|^k}{n^{(\alpha-1)k+1}}\}. \tag{5}$$

Mishra and Srivastava [5] have proved the following theorem for $|C,1|_k$ summability factors of infinite series.

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THEOREM A. *Let* (X_n) *be a positive non-decreasing sequence and* (β_n), (λ_n) *sequences such that*

$$|\Delta\lambda_n| \leqslant \beta_n \tag{6}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{7}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty \tag{8}$$

$$|\lambda_n|X_n = O(1) \quad as \quad n \to \infty. \tag{9}$$

If

$$\sum_{n=1}^{m} \frac{1}{n}|s_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n\lambda_n$ *is* $|C,1|_k$ *summable,* $k \geqslant 1$.

2. The aim of this paper is to generalize Theorem A under weaker conditions for $|C,\alpha;\delta|_k$ summability. For this we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leqslant b_n \leqslant Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. So we are weakening the hypotheses of the theorem replacing the increasing sequence by an almost increasing sequence.

Now, we shall prove the following theorem.

THEOREM. *Let* (X_n) *be an almost increasing sequence and the sequences* (β_n) *and* (λ_n) *such that conditions (6)-(9) of Theorem A are satisfied. If the sequence* (u_n^α), *defined by (see [6])*

$$u_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1 \\ \max_{1\leqslant v\leqslant n} |t_v^\alpha|, & 0<\alpha<1 \end{cases} \tag{11}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k-1}(u_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{12}$$

then the series $\sum a_n\lambda_n$ *is* $|C,\alpha;\delta|_k$ *summable,* $k \geqslant 1$ *and* $0 \leqslant \delta < \alpha \leqslant 1$.

We need the following lemmas.

LEMMA 1 ([2]). *If* $0 < \alpha \leqslant 1$ *and* $1 \leqslant v \leqslant n$, *then*

$$|\sum_{p=0}^{v} A_{n-p}^{\alpha-1}a_p| \leqslant \max_{1\leqslant m\leqslant v} |\sum_{p=0}^{m} A_{m-p}^{\alpha-1}a_p|. \tag{13}$$

LEMMA 2. *Under the conditions on* (X_n), (β_n) *and* (λ_n) *as in the statement of the theorem, the following conditions hold, when (8) is satisfied:*

$$n\beta_n X_n = O(1) \quad as \quad n \to \infty. \tag{14}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty \tag{15}$$

Proof. Let $Ac_n \leqslant X_n \leqslant Bc_n$, where (c_n) is an increasing sequence. In this case

$$\begin{aligned} nX_n\beta_n &\leqslant nBc_n|\sum_{v=n}^{\infty}\Delta\beta_v| \leqslant nBc_n\sum_{v=n}^{\infty}|\Delta\beta_v| \\ &\leqslant B\sum_{v=n}^{\infty} vc_v|\Delta\beta_v| \leqslant \frac{B}{A}\sum_{v=n}^{\infty} v|\Delta\beta_v|X_v < \infty. \end{aligned}$$

Hence $n\beta_n X_n = O(1) \quad as \quad n \to \infty$. Again

$$\begin{aligned} \sum_{n=1}^{\infty} X_n\beta_n &\leqslant B\sum_{n=1}^{\infty} c_n\beta_n = B\sum_{n=1}^{\infty} c_n|\sum_{v=n}^{\infty}\Delta\beta_v| \\ &\leqslant B\sum_{n=1}^{\infty} c_n\sum_{v=n}^{\infty}|\Delta\beta_v| = B\sum_{v=1}^{\infty}|\Delta\beta_v|\sum_{n=1}^{v} c_n \\ &\leqslant B\sum_{v=1}^{\infty} vc_v|\Delta\beta_v| \leqslant \frac{B}{A}\sum_{v=1}^{\infty} vX_v|\Delta\beta_v| < \infty. \end{aligned}$$

Hence $\sum_{n=1}^{\infty} X_n\beta_n < \infty$.

3. Proof of the Theorem. Let (T_n^α) be the n-th (C,α) mean, with $0 < \alpha \leqslant 1$, of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v\lambda_v. \tag{16}$$

Applying Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v} A_{n-p}^{\alpha-1} pa_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v,$$

so that making use of Lemma 1, we have

$$\begin{aligned} |T_n^\alpha| &\leqslant \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v||\sum_{p=1}^{v} A_{n-p}^{\alpha-1} pa_p| + \frac{|\lambda_n|}{A_n^\alpha}|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v| \\ &\leqslant \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha u_v^\alpha|\Delta\lambda_v| + |\lambda_n|u_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say.} \end{aligned}$$

Since

$$|T_{n,1}^{\alpha}+T_{n,2}^{\alpha}|^{k}\leqslant 2^{k}(|T_{n,1}^{\alpha}|^{k}+|T_{n,2}^{\alpha}|^{k}),$$

to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1}|T_{n,r}^{\alpha}|^{k}<\infty \quad \text{for} \quad r=1,2, \quad by \quad (3).$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1}|T_{n,1}^{\alpha}|^{k} &\leqslant \sum_{n=2}^{m+1} n^{\delta k-1}(A_{n}^{\alpha})^{-k}\{\sum_{v=1}^{n-1} A_{v}^{\alpha}u_{v}^{\alpha}\beta_{v}\}^{k}\\
&\leqslant \sum_{n=2}^{m+1} n^{\delta k-1}(A_{n}^{\alpha})^{-k}\{\sum_{v=1}^{n-1}(A_{v}^{\alpha})^{k}(u_{v}^{\alpha})^{k}\beta_{v}\}\times\{\sum_{v=1}^{n-1}\beta_{v}\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k-\alpha k-1}\{\sum_{v=1}^{n-1} v^{\alpha k}(u_{v}^{\alpha})^{k}\beta_{v}\}\\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(u_{v}^{\alpha})^{k}\beta_{v}\sum_{n=v+1}^{m+1}\frac{1}{n^{1+\alpha k-\delta k}}\\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(u_{v}^{\alpha})^{k}\beta_{v}\int_{v}^{\infty}\frac{dx}{x^{1+\alpha k-\delta k}}\\
&= O(1)\sum_{v=1}^{m} v^{\delta k}(u_{v}^{\alpha})^{k}\beta_{v}=O(1)\sum_{v=1}^{m} v\beta_{v}v^{\delta k-1}(u_{v}^{\alpha})^{k}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_{v})\sum_{r=1}^{v} r^{\delta k-1}(u_{r}^{\alpha})^{k}+O(1)m\beta_{m}\sum_{v=1}^{m} v^{\delta k-1}(u_{v}^{\alpha})^{k}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_{v})|X_{v}+O(1)m\beta_{m}X_{m}\\
&= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_{v}|X_{v}+O(1)\sum_{v=1}^{m-1}|\beta_{v+1}|X_{v}+O(1)m\beta_{m}X_{m}\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 2.

Finally, since $|\lambda_n| = O(1)$, by (9), we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1}|T_{n,2}^{\alpha}|^k &= \sum_{n=1}^{m} |\lambda_n|^{k-1}|\lambda_n| n^{\delta k-1}(u_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m} |\lambda_n| n^{\delta k-1}(u_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} v^{\delta k-1}(u_v^{\alpha})^k + O(1)|\lambda_m| \sum_{n=1}^{m} n^{\delta k-1}(u_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 2. Therefore, we get that

$$\sum_{n=1}^{m} \frac{1}{n}|T_{n,r}^{\alpha}|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2.$$

This completes the proof of the Theorem.

REMARK. It should be noted that if we take $\alpha = 1$ and $\delta = 0$ in this theorem, then we get Theorem A under weaker conditions. In fact, in this case the condition (12) reduces to the condition (10), by (5) and (11).

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