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WEIGHTED $L^p - L^q$ INEQUALITIES FOR OSCILLATORY INTEGRAL OPERATORS

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1. Introduction

Let $\Delta = \{(x, x) : x \in \mathbf{R}\}$. Suppose that $K \in C^\infty(\mathbf{R}^2 \backslash \Delta)$ satisfying

$$\left| \frac{\partial^{j+l} K}{\partial x^j \partial y^l}(x, y) \right| \leqslant A_{j,l} |x - y|^{-(j+l)} \tag{1}$$

for $j, l \geqslant 0$ and $(x, y) \in \mathbf{R}^2 \backslash \Delta$.

For $a, b > 0$, define the non-convolutional oscillatory integral operator $T_{a,b}$:

$$(T_{a,b} f)(x) = \int_R e^{i|x|^a |y|^b} K(x, y) f(y) dy \tag{2}$$

initially for $f \in \mathscr{S}(\mathbf{R})$, the space of Schwartz functions on $\mathbf{R}$.

Such operators often arise in harmonic analysis (see e.g. [7–12]). It should be noted that when $a = b = 1$ and $K \equiv 1$ the operator in (2) is essentially the Fourier transform.

The main problem under investigation concerns the boundedness properties of the operators $\{T_{a,b}\}$. It has been observed that for $a, b > 0$ the $L^p(\mathbf{R}) \to L^p(\mathbf{R})$ boundedness cannot hold in general unless $p = \frac{a+b}{a}$ (see below). On the other hand, it has been established in [5–7] and [1] that $T_{a,b}$ is indeed bounded from $L^{\frac{a+b}{a}}(\mathbf{R})$ to $L^{\frac{a+b}{a}}(\mathbf{R})$ whenever $a, b \geqslant 1$.

In this paper we examine what happens when the restriction $a, b \geqslant 1$ is lifted. The fact that $T_{a,b}$ may fail to be bounded from $L^{\frac{a+b}{a}}(\mathbf{R})$ to $L^{\frac{a+b}{a}}(\mathbf{R})$ can be seen easily by taking $a = b = 1/2$ and $K \equiv 1$. In this case $\frac{a+b}{a} = 2$, but $T_{1/2,1/2}$ is not a bounded operator from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ (see Section 9). On the other hand, it will be shown that a more comprehensive theory exists if one considers the broader class of weighted L^p spaces with power weights, which we shall now describe.

Let $\sigma, \gamma \in \mathbf{R}$. Recall that the space $L^p(\mathbf{R}, |x|^\sigma)$ represents the collection of Lebesgue measurable functions f satisfying

$$\|f\|_{p,|x|^\sigma} = \left(\int_{\mathbf{R}} |f(x)|^p |x|^\sigma dx \right)^{1/p} < \infty.$$

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By a simple argument one obtains that in general $T_{a,b}$ cannot be a bounded operator from $L^p(\mathbf{R}, |x|^\gamma)$ to $L^q(\mathbf{R}, |x|^\sigma)$ unless

$$\frac{1}{b} - \frac{1+\gamma}{bp} - \frac{1+\sigma}{aq} = 0 \tag{3}$$

(see Section 9). Clearly in the special case $\sigma = \gamma = 0$ and $p = q$, condition (3) is reduced to $p = \frac{a+b}{a}$. However, as pointed out earlier, condition (3) does not always imply the $L^p(\mathbf{R}, |x|^\gamma) \to L^q(\mathbf{R}, |x|^\sigma)$ boundedness of $T_{a,b}$ (e.g. $a = b = 1/2$ and $\sigma = \gamma = 0$). The following theorem (Theorem 2.1), which is the main result of this paper, shows when the the $L^p(\mathbf{R}, |x|^\gamma) \to L^q(\mathbf{R}, |x|^\sigma)$ boundedness of $T_{a,b}$ holds under condition (3).

2. Main result and some of its implications

THEOREM 2.1. *Let $a, b > 0$. Let K and $T_{a,b}$ be given as in (1) and (2). Let $\sigma, \gamma \in \mathbf{R}$ and $p, q \in (1, \infty)$ such that $p \leqslant q$ and $\frac{1}{b} - \frac{1+\gamma}{bp} - \frac{1+\sigma}{aq} = 0$. If $-1 < \sigma < a - 1$ and $\gamma > (p-1)(1-b)$, then there exists $A = A(a, b, \sigma, \gamma, p, q) > 0$ such that*

$$\|T_{a,b}f\|_{q,|x|^\sigma} \leqslant A\|f\|_{p,|x|^\gamma} \tag{4}$$

for all $f \in L^p(\mathbf{R}, |x|^\gamma)$.

REMARKS.

(1) When $p = q$, condition (3) and the above theorem are reduced to $p = (1+\gamma) + \frac{b}{a}(1+\sigma)$ and $T_{a,b}$ mapping $L^p(\mathbf{R}, |x|^\gamma)$ to $L^p(\mathbf{R}, |x|^\sigma)$ boundedly when $-1 < \sigma < a - 1$ and $\gamma > \frac{(1-b)(1+\sigma)}{a}$.

(2) If in addition to $p = q$, we let $\sigma = \gamma$ (therefore $p = (1+\sigma)(1+\frac{b}{a})$), then it follows that $T_{a,b}$ is bounded from $L^p(\mathbf{R}, |x|^\sigma)$ to $L^p(\mathbf{R}, |x|^\sigma)$ whenever $a + b > 2$ and $\frac{1-b}{a+b-1} < \sigma < a - 1$.

Furthermore, if σ is taken to be 0, then the $L^{\frac{a+b}{a}}(\mathbf{R}) \to L^{\frac{a+b}{a}}(\mathbf{R})$ boundedness is recovered for $a, b > 1$.

(3) Finally, it should be pointed out that if $a = b$ (e.g. $a = b = \frac{1}{2}$ as mentioned earlier) and $\gamma = -\sigma \in (1-a, 1)$, then $T_{a,a}$ is a bounded operator from the Hilbert space $L^2(\mathbf{R}, |x|^\gamma)$ to its dual space $L^2(\mathbf{R}, |x|^{-\gamma})$.[1]

A few words about the organization of the paper are in order. We shall first establish (4) in the case $p = q$ (Proposition 3.1). This will be accomplished in Sections 3–7. The full statement of Theorem 2.1 will then be obtained by applying Stein's theorem on interpolation of analytic family of operators. See Section 8. The final section contains a few additional remarks including the derivation of condition (3).

[1]The author is indebted to Chris Lennard for a helpful discussion.

3. Dyadic decomposition and almost orthogonality

PROPOSITION 3.1. *Let $a,b>0$. Let K and $T_{a,b}$ be given as in (1) and (2). Let σ, $\gamma \in \mathbf{R}$ such that $-1<\sigma<a-1$ and $\gamma > \frac{(1-b)(1+\sigma)}{a}$. Then for $p = (1+\gamma)+\frac{b}{a}(1+\sigma)$ $T_{a,b}$ is a bounded operator from $L^p(\mathbf{R},|x|^\gamma)$ to $L^p(\mathbf{R},|x|^\sigma)$.*

An initial step in our proof of Proposition 3.1 involves a dyadic decomposition of the operator $T_{a,b}$. To implement this idea, we shall begin by selecting a real-valued function $\psi \in C_0^\infty(\mathbf{R})$ such that $\mathrm{supp}(\psi) \subseteq (1/2,4)$ and

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) \equiv 1$$

for all $x>0$. For each $k \in \mathbf{N}$ define ψ_k by

$$\psi_k(x) = \psi(2^{-k}x).$$

We also define ψ_0 by

$$\psi_0(x) = \sum_{k=-\infty}^{0} \psi(2^{-k}x).$$

Let

$$\Gamma = \{(j,0) : 0 \leqslant j \leqslant 3\} \cup \{(0,k) : 0 \leqslant k \leqslant 3\},$$
$$H(x,y) = \sum_{(j,k)\in\Gamma} \psi_j(x)\psi_k(y),$$

and

$$h(x) = \sum_{j=4}^{\infty} \psi_j(x).$$

For given $a,b,$ and K, we shall define the operators $\{T_{jk} : j,k \geqslant 1\}$, R_1, R_2, and R_3 by

$$(T_{jk}f)(x) = \psi_j(x)\int_0^\infty e^{ix^a y^b} K(x,y)\psi_k(y)f(y)dy, \tag{5}$$

$$(R_1 f)(x) = \int_0^\infty e^{ix^a y^b} K(x,y)H(x,y)f(y)dy, \tag{6}$$

$$(R_2 f)(x) = \psi_0(x)\int_0^\infty e^{ix^a y^b} K(x,y)h(y)f(y)dy, \tag{7}$$

$$(R_3 f)(x) = h(x)\int_0^\infty e^{ix^a y^b} K(x,y)\psi_0(y)f(y)dy. \tag{8}$$

Therefore one obtains the following for $x>0$:

$$\int_0^\infty e^{ix^a y^b} K(x,y)f(y)dy = (R_1 f)(x) + (R_2 f)(x) + (R_3 f)(x) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(T_{jk}f)(x). \tag{9}$$

We shall begin with the estimates for the operators T_{jk} when $|j-k| \leqslant 3$. We have the following:

PROPOSITION 3.2. *Let* $j,k \in \mathbf{N}$ *and* $|j-k| \leqslant 3$. *Then there exists a constant* $A > 0$ *independent of* j *and* k *such that*

$$\|T_{jk}f\|_2 \leqslant A2^{(1-\frac{a+b}{2})j}\|f\|_2 \tag{10}$$

for all $f \in L^2(\mathbf{R})$.

The main tool we shall use to prove Proposition 3.2 is the Cotlar-Stein lemma on almost orthogonality ([2][4]).

LEMMA 3.3. (Cotlar-Stein) *Let* $\{S_m : m \in \mathbf{Z}\}$ *be a collection of bounded operators on* $L^2(\mathbf{R})$. *Let* $\{\omega(\ell) : \ell \in \mathbf{Z}\}$ *be a sequence of nonnegative real numbers such that*

$$A = \sum_{\ell \in \mathbf{Z}} \sqrt{\omega(\ell)} < \infty,$$

$$\|S_m^* S_n\| \leqslant \omega(m-n),$$

and

$$\|S_m S_n^*\| \leqslant \omega(m-n)$$

for all $m,n \in \mathbf{Z}$. *If a linear operator* S *on* $L^2(\mathbf{R})$ *can be written as*

$$Sf = \sum_{m \in \mathbf{Z}} S_m f,$$

then S *is a bounded operator on* $L^2(\mathbf{R})$ *and satisfies* $\|S\| \leqslant A$.

Proposition 3.2 will be obtained as a special case of the following.

PROPOSITION 3.4. *Let* K *satisfy (1). For* $\lambda > 1$, $\delta > 0$, *and* $\beta \in [\frac{1}{8}, 8]$ *define the operator* $S = S_{\lambda,\delta,\beta}$ *by*

$$Sf(x) = \psi(x)\int_0^\infty e^{i\lambda x^a y^b} K(\delta x, \delta y)\psi(\beta y)f(y)dy. \tag{11}$$

Then there exists $A > 0$ *independent of* $\lambda, \delta,$ *and* β *such that*

$$\|Sf\|_2 \leqslant A\lambda^{-\frac{1}{2}}\|f\|_2 \tag{12}$$

for all $f \in L^2(\mathbf{R})$.

Proof of Proposition 3.4. First we select a real-valued function $\phi \in C_0^\infty(\mathbf{R})$ such that $\mathrm{supp}(\phi) \subseteq (-1,1)$ and

$$\sum_{m \in \mathbf{Z}} \phi(t-m) = 1 \tag{13}$$

for all $t \in \mathbf{R}$. For $m \in \mathbf{Z}$ we define S_m by

$$S_m f(x) = \psi(x)\int_0^\infty e^{i\lambda x^a y^b} K(\delta x, \delta y)\phi(\lambda^{\frac{1}{2}}(x-y)-m)\psi(\beta y)f(y)dy. \tag{14}$$

By (11), (13), and (14) we see that

$$Sf = \sum_{m\in\mathbf{Z}} S_m f. \tag{15}$$

Let

$$\Omega_m(x,y) = e^{i\lambda x^a y^b} K(\delta x, \delta y)\phi(\lambda^{\frac{1}{2}}(x-y)-m)\psi(x)\psi(\beta y).$$

Then by (1) and Schur's lemma we obtain

$$\|S_m\| \leqslant (A_{0,0}\|\phi\|_1)\lambda^{-\frac{1}{2}} \tag{16}$$

for all $m \in \mathbf{Z}$. For $m, n \in \mathbf{Z}$ we have

$$S_m^* S_n f(x) = \int_{\mathbf{R}} L_{mn}(x,y) f(y)dy \tag{17}$$

where

$$\begin{aligned} L_{mn}(x,y) &= \chi_{[0,\infty)}(y)\int_0^\infty \overline{\Omega_m(z,x)}\Omega_n(z,y)dz \\ &= \psi(\beta x)\psi(\beta y)\int_0^\infty e^{i\lambda z^a(y^b-x^b)}\overline{K(\delta z,\delta x)}K(\delta z,\delta y)\times \\ &\qquad \times \phi(\lambda^{\frac{1}{2}}(z-x)-m)\phi(\lambda^{\frac{1}{2}}(z-y)-n)[\psi(z)]^2 dz. \end{aligned}$$

Let

$$\begin{aligned} g_{mn}(x,y,z) = {} & \overline{K(\delta z,\delta x)}K(\delta z,\delta y)\times \\ & \times \phi(\lambda^{\frac{1}{2}}(z-x)-m)\phi(\lambda^{\frac{1}{2}}(z-y)-n)[\psi(z)]^2. \end{aligned}$$

By employing integration by parts we get

$$L_{mn}(x,y) = \frac{-\psi(\beta x)\psi(\beta y)}{i\lambda a(y^b-x^b)}\int_0^\infty e^{i\lambda z^a(y^b-x^b)}\frac{\partial}{\partial z}\Big[\frac{g_{mn}(x,y,z)}{z^{a-1}}\Big]dz$$

$$= \frac{\psi(\beta x)\psi(\beta y)}{[i\lambda a(y^b-x^b)]^4}\int_0^\infty e^{i\lambda z^a(y^b-x^b)}\Big[c_0 z^{-4a} g_{mn}(x,y,z) + c_1 z^{1-4a}\frac{\partial g_{mn}}{\partial z}(x,y,z)$$

$$+c_2 z^{2-4a}\frac{\partial^2 g_{mn}}{\partial z^2}(x,y,z) + c_3 z^{3-4a}\frac{\partial^3 g_{mn}}{\partial z^3}(x,y,z) + c_4 z^{4-4a}\frac{\partial^4 g_{mn}}{\partial z^4}(x,y,z)\Big]dz$$

where $c_0, c_1, c_2, c_3,$ and c_4 are constants that depend on a only.

Assume that $|n|, |m| \geqslant 2$ and $|n-m| \geqslant 4$. Then whenever $g_{mn}(x,y,z) \neq 0$ we have $\frac{1}{2} < z < 4$ and

$$|\lambda^{\frac{1}{2}}(z-x)-m| < 1$$

and

$$|\lambda^{\frac{1}{2}}(z-y)-n| < 1.$$

Thus

$$|z-x| > \lambda^{-\frac{1}{2}}(|m|-1) \geqslant \lambda^{-\frac{1}{2}},$$

$$|z-y|>\lambda^{-\frac{1}{2}}(|n|-1)\geqslant\lambda^{-\frac{1}{2}},$$

and

$$|x-y|>\lambda^{-\frac{1}{2}}[|n-m|-2]\geqslant\frac{|n-m|\lambda^{-\frac{1}{2}}}{2}. \tag{18}$$

Therefore by (1) we have

$$|z^{j-4a}\frac{\partial^j g_{mn}}{\partial z^j}(x,y,z)|\leqslant A_j\lambda^{j/2}$$

for $0\leqslant j\leqslant 4$. By the preceding inequalities we obtain

$$|L_{mn}(x,y)|\leqslant\frac{A\lambda^2|\psi(\beta x)\psi(\beta y)|}{(\lambda|y^b-x^b|)^4}\int_{|z-(x+m\lambda^{-\frac{1}{2}})|<\lambda^{-\frac{1}{2}}}dz=\frac{2A|\psi(\beta x)\psi(\beta y)|}{\lambda^{\frac{5}{2}}|y^b-x^b|^4}.$$

Since $\operatorname{supp}(\psi)\subseteq(\frac{1}{2},4)$ and $\frac{1}{8}\leqslant\beta\leqslant 8$, we have

$$\frac{1}{16}<x,y<32$$

whenever $L_{mn}(x,y)\neq 0$. Thus

$$|y^b-x^b|\geqslant c_0|x-y|$$

with $c_0=b2^{4(1-b)}$. By (18) we see that

$$L_{mn}(x,y)=0$$

when $|x-y|\leqslant\frac{1}{2}|n-m|\lambda^{-\frac{1}{2}}$, and

$$|L_{mn}(x,y)|\leqslant\frac{C}{\lambda^{\frac{5}{2}}|x-y|^4}$$

for all $x,y\in\mathbf{R}$. Thus

$$\sup_{x\in\mathbf{R}}\int_{\mathbf{R}}|L_{mn}(x,y)|dy+\sup_{y\in\mathbf{R}}\int_{\mathbf{R}}|L_{mn}(x,y)|dx\leqslant C\lambda^{-1}|n-m|^{-3}.$$

It follows from Schur's lemma that

$$\|S_m^*S_n\|\leqslant\frac{C}{\lambda(1+|n-m|)^3}$$

whenever $|m|,|n|\geqslant 2$ and $|m-n|\geqslant 4$.

For $|m-n|<4$ by (16) we have

$$\|S_m^*S_n\|\leqslant\|S_m^*\|\|S_n\|\leqslant(A_{0,0}\|\phi\|_1)^2\lambda^{-1}.$$

Therefore,

$$\|S_m^*S_n\|\leqslant\frac{C}{\lambda(1+|n-m|)^3} \tag{19}$$

whenever $|m|, |n| \geqslant 2$. Similarly,

$$\|S_m S_n^*\| \leqslant \frac{C}{\lambda(1+|n-m|)^3} \tag{20}$$

for all $|m|, |n| \geqslant 2$.

Let $\omega(\ell) = \frac{C}{\lambda(1+|\ell|)^3}$. Then by Lemma 3.3

$$\|\sum_{|m|\geqslant 2} S_m\| \leqslant \sqrt{\frac{C}{\lambda}} \sum_{\ell \in Z} \frac{1}{(1+|\ell|)^{\frac{3}{2}}} = \tilde{A}\lambda^{-\frac{1}{2}}. \tag{21}$$

By (15), (16), and (21)

$$\|S\| \leqslant \|S_0\| + \|S_{-1}\| + \|S_1\| + \|\sum_{|m|\geqslant 2} S_m\| \leqslant A\lambda^{-\frac{1}{2}},$$

which completes the proof of Proposition 3.4.

Now we shall prove Proposition 3.2.

Proof of Proposition 3.2. Assume that $|j-k| \leqslant 3$ and $f \in L^2(\mathbf{R})$. By definition

$$(T_{jk}f)(2^j x) = 2^j \psi(x) \int_0^\infty e^{i2^{j(a+b)}x^a y^b} K(2^j x, 2^j y)\psi(2^{j-k}y) f(2^j y)dy.$$

Since $|j-k| \leqslant 3$, we have $2^{j-k} \in [\frac{1}{8}, 8]$. By applying Proposition 3.4 with $\lambda = 2^{j(a+b)}$, $\delta = 2^j$, and $\beta = 2^{j-k}$, we obtain

$$\left(\int_{\mathbf{R}} |T_{jk}f(2^j x)|^2 dx\right)^{\frac{1}{2}} \leqslant A 2^j 2^{-j(\frac{a+b}{2})} \left(\int_{\mathbf{R}} |f(2^j y)|^2 dy\right)^{\frac{1}{2}},$$

which implies that

$$\|T_{jk}f\|_2 \leqslant A 2^{(1-\frac{a+b}{2})j} \|f\|_2$$

for all $f \in L^2(\mathbf{R})$. Proposition 3.2 is proved.

4. The method of T^*T

We shall now turn our attention to the operators T_{jk} with $|j-k| \geqslant 4$. The method that we use is to consider $T_{jk}^* T_{jk}$, which reduces matters to related oscillatory integrals.

PROPOSITION 4.1. *Let $j, k \in \mathbf{N}$ with $|j-k| \geqslant 4$. Suppose that K satisfies (1). Let T_{jk} be given as in (5). Then there exists a constant $A > 0$ independent of j and k such that*

$$\|T_{jk}\| \leqslant A 2^{[(1-a)j+(1-b)k]/2} \sqrt{j+k}. \tag{22}$$

Proof of Proposition 4.1. First let us assume that $j \geqslant k+4$. By definition we have

$$T_{jk}^* T_{jk} f(x) = \int_{\mathbf{R}} W_{jk}(x,y) f(y) dy \tag{23}$$

where

$$W_{jk}(x,y) = \psi_k(x)\psi_k(y)\int_0^\infty e^{iz^a(y^b-x^b)}\overline{K(z,x)}K(z,y)[\psi_j(z)]^2dz.$$

When $\psi_k(x)\psi_k(y)[\psi_j(z)]^2 \neq 0$ we have

$$2^{k-1} < x, y < 2^{k+2} \qquad \text{and} \qquad 2^{j-1} < z < 2^{j+2}.$$

By $j \geqslant k+4$ we have

$$\left|\frac{\partial}{\partial z}[\overline{K(z,x)}K(z,y)]\right| \leqslant C[|z-x|^{-1} + |z-y|^{-1}] \leqslant 4C2^{-j}.$$

Thus by integration by parts

$$\begin{aligned}|W_{jk}(x,y)| &= \frac{|\psi_k(x)\psi_k(y)|}{a|x^b-y^b|}\left|\int_0^\infty e^{iz^a(y^b-x^b)}\frac{d}{dz}\{z^{1-a}\overline{K(z,x)}K(z,y)[\psi_j(z)]^2\}dz\right| \\ &\leqslant \frac{C|\psi_k(x)\psi_k(y)|}{a|x^b-y^b|}\int_{2^{j-1}}^{2^{j+2}}(z^{-a}+2^{-j}z^{1-a})dz \\ &\leqslant C2^{(1-a)j}|\psi_k(x)\psi_k(y)||x^b-y^b|^{-1}. \end{aligned} \tag{24}$$

We also have

$$\begin{aligned}|W_{jk}(x,y)| &\leqslant |\psi_k(x)\psi_k(y)|\int_{2^{j-1}}^{2^{j+2}}|\overline{K(z,x)}K(z,y)|[\psi(2^{-j}z)]^2dz \\ &\leqslant C2^j|\psi_k(x)\psi_k(y)|. \end{aligned} \tag{25}$$

Let

$$W_{jk}^1(x,y) = \begin{cases} W_{jk}(x,y) & \text{if } x \geqslant y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$W_{jk}^2(x,y) = \begin{cases} W_{jk}(x,y) & \text{if } y > x \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$W_{jk}(x,y) = W_{jk}^1(x,y) + W_{jk}^2(x,y). \tag{26}$$

Then by (24) and (25)

$$\begin{aligned}\sup_{y\in\mathbf{R}}\int_{\mathbf{R}}|W_{jk}^1(x,y)|dx &= \sup_{v\in\mathbf{R}}\int_{\mathbf{R}}|W_{jk}^1(2^ku,2^kv)|2^kdu \\ &\leqslant 2^k\sup_{v>0}|\psi(v)|[\int_v^{(v^b+2^{-aj-bk})^{\frac{1}{b}}}2^j|\psi(u)|du + 2^{-bk+(1-a)j}\times \\ &\times\int_{(v^b+2^{-aj-bk})^{\frac{1}{b}}}^\infty(u^b-v^b)^{-1}|\psi(u)|du]\end{aligned}$$

$$\leqslant C2^k \sup_{v>0} |\psi(v)| \left[2^j [(v^b + 2^{-aj-bk})^{\frac{1}{b}} - v] + 2^{-bk+(1-a)j} \int_{2^{-aj-bk}}^{4^b} \frac{dt}{t} \right]$$
$$\leqslant C2^{[(1-a)j+(1-b)k]}(1 + aj + bk) \leqslant C2^{[(1-a)j+(1-b)k]}(j+k).$$

We also have

$$\sup_{x\in\mathbf{R}} \int_{\mathbf{R}} |W^1_{jk}(x,y)|dy = \sup_{u\in\mathbf{R}} \int_{\mathbf{R}} |W^1_{jk}(2^k u, 2^k v)| 2^k dv$$
$$\leqslant 2^k \sup_{u>0} |\psi(u)| \left[\int_{u^b - 2^{-aj-bk} \leqslant v^b \leqslant u^b} 2^j |\psi(v)| dv + 2^{-bk+(1-a)j} \times \right.$$
$$\left. \times \int_{v^b \leqslant u^b - 2^{-aj-bk}} (u^b - v^b)^{-1} |\psi(v)| dv \right]$$
$$\leqslant C2^{[(1-a)j+(1-b)k]}(j+k).$$

Similar estimates can be obtained for $W^2_{jk}(x,y)$; therefore, by (26) we have

$$\sup_{x\in\mathbf{R}} \int_{\mathbf{R}} |W_{jk}(x,y)|dy \leqslant C2^{[(1-a)j+(1-b)k]}(j+k) \tag{27}$$

and

$$\sup_{y\in\mathbf{R}} \int_{\mathbf{R}} |W_{jk}(x,y)|dx \leqslant C2^{[(1-a)j+(1-b)k]}(j+k). \tag{28}$$

It then follows from (23), (27), (28), and Schur's lemma that

$$\|T_{jk}\| = \|T^*_{jk} T_{jk}\|^{\frac{1}{2}} \leqslant A2^{\frac{[(1-a)j+(1-b)k]}{2}} \sqrt{j+k}$$

when $j \geqslant k+4$.

If $k \geqslant j+4$ for $x, y \in \operatorname{supp}(\psi_k)$ and $z \in \operatorname{supp}(\psi_j)$, we have

$$\left| \frac{\partial}{\partial z} [\overline{K(z,x)} K(z,y)] \right| \leqslant C \left(\frac{1}{|z-x|} + \frac{1}{|z-y|} \right) \leqslant C2^{-k} \leqslant C2^{-j}.$$

Therefore all the arguments used to treat the case $j \geqslant k+4$ can be applied here to obtain

$$\|T_{jk}\| \leqslant A2^{\frac{[(1-a)j+(1-b)k]}{2}} \sqrt{j+k}$$

when $k \geqslant j+4$. This finishes the proof of Proposition 4.1.

5. Interpolation and weighted estimates

We shall now use the inequalities obtained in the previous sections and the method of interpolation to establish the desired estimates for $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} T_{jk}$.

PROPOSITION 5.1. *Let* $a,b>0$. *Let* $-1<\sigma<a-1$, $\gamma>\frac{(1+\sigma)(1-b)}{a}$, *and* $p=(1+\gamma)+\frac{b}{a}(1+\sigma)$. *Let* T_{jk} *be given as in (5). If* $p\leqslant 2$, *then there exists* $A=A(a,b,\sigma,\gamma)>0$ *such that*

$$\|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}T_{jk}f\|_{p,|x|^{\sigma}}\leqslant A\|f\|_{p,|x|^{\gamma}} \tag{29}$$

for all $f\in\mathscr{S}(\mathbf{R})$.

Proof of Proposition 5.1. By assumption we have

$$p>1+\frac{(1+\sigma)(1-b)}{a}+\frac{b}{a}(1+\sigma)=1+\frac{(1+\sigma)}{a}>1. \tag{30}$$

Thus $p\in(1,2]$. We have

$$\|T_{jk}f\|_1\leqslant C2^j\|f\|_1. \tag{31}$$

Let $p_0=q_0=1$, $p_1=q_1=2$, $t=\frac{2(p-1)}{p}\in(0,1]$. Then by Proposition 3.2 and Riesz's convexity theorem, we obtain that

$$\|T_{jk}f\|_p\leqslant C2^{(1-t)j}2^{t(1-\frac{a+b}{2})j}\|f\|_p \tag{32}$$

when $|j-k|\leqslant 3$. Thus

$$\begin{aligned}\|\sum_{|j-k|\leqslant 3}T_{jk}f\|_{p,|x|^{\sigma}}&\leqslant C\sum_{|j-k|\leqslant 3}2^{\frac{j\sigma}{p}}\left(\int_{\mathbf{R}}|T_{jk}f(x)|^p dx\right)^{\frac{1}{p}}\\&\leqslant C\sum_{|j-k|\leqslant 3}2^{\frac{j\sigma}{p}}2^{(1-t)j}2^{t(1-\frac{a+b}{2})j}\left(\int_{\mathbf{R}}|f(y)\chi_{[2^{k-1},2^{k+2}]}(y)|^p dy\right)^{\frac{1}{p}}\\&\leqslant C\left(\sum_{j=1}^{\infty}2^{j[\frac{\sigma-\gamma}{p}+1-\frac{(a+b)(p-1)}{p}]}\right)\|f\|_{p,\gamma}.\end{aligned}$$

Since

$$\frac{\sigma-\gamma}{p}+1-\frac{(a+b)(p-1)}{p}<0,$$

we obtain that

$$\|\sum_{|j-k|\leqslant 3}T_{jk}f\|_{p,|x|^{\sigma}}\leqslant A\|f\|_{p,\gamma}. \tag{33}$$

Similarly, by (22), (31), and interpolation we have

$$\|T_{jk}f\|_p\leqslant C2^{(1-t)j}2^{\frac{t[(1-a)j+(1-b)k]}{2}}(j+k)^{t/2}\|f\|_p \tag{34}$$

when $|j-k| \geqslant 4$. Hence

$$\begin{aligned}
\| \sum_{|j-k|\geqslant 4} T_{jk}f \|_{p,|x|^\sigma} &\leqslant C \sum_{|j-k|\geqslant 4} 2^{\frac{j\sigma}{p}} \|T_{jk}f\|_p \\
&\leqslant C \sum_{|j-k|\geqslant 4} 2^{\frac{j\sigma}{p}} 2^{(1-t)j} 2^{\frac{t[(1-a)j+(1-b)k]}{2}} (j+k)^{t/2} \left(\int_{2^{k-1}}^{2^{k+1}} |f(y)|^p dy \right)^{\frac{1}{p}} \\
&\leqslant C \left(\sum_{|j-k|\geqslant 4} 2^{j[\frac{\sigma+1}{p} - \frac{a(p-1)}{p}]} 2^{k[-\frac{\gamma}{p} + \frac{(1-b)(p-1)}{p}]} (j+k)^{\frac{p-1}{p}} \right) \|f\|_{p,|x|^\gamma}.
\end{aligned}$$

Since

$$\epsilon_1 = \frac{a(p-1)}{p} - \frac{\sigma+1}{p} = \frac{a}{p}\left[\gamma - \frac{(1+\sigma)(1-b)}{a}\right] > 0,$$
$$\epsilon_2 = \frac{\gamma}{p} - \frac{(1-b)(p-1)}{p} = \frac{b}{p}\left[\gamma - \frac{(1+\sigma)(1-b)}{a}\right] > 0,$$

and

$$(j+k)^{\frac{p-1}{p}} \leqslant C 2^{-\frac{(\epsilon_1 j + \epsilon_2 k)}{2}},$$

we obtain that

$$\| \sum_{|j-k|\geqslant 4} T_{jk}f \|_{p,|x|^\sigma} \leqslant C \left(\sum_{j,k\geqslant 1} 2^{-\frac{\epsilon_1 j}{2}} 2^{-\frac{\epsilon_2 k}{2}} \right) \|f\|_{p,|x|^\gamma} \leqslant A\|f\|_{p,|x|^\gamma}. \tag{35}$$

By (33) and (35) we see that

$$\| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} T_{jk}f \|_{p,|x|^\sigma} \leqslant A\|f\|_{p,|x|^\gamma},$$

which proves Proposition 5.1.

6. The estimates for R_1, R_2, and R_3

PROPOSITION 6.1. *Let $a,b > 0$. Let $\sigma > -1$, $\gamma > -\frac{(1+\sigma)b}{a}$, $p = (1+\gamma) + \frac{b}{a}(1+\sigma)$, and R_1 be given as in (6). Then there exists $A > 0$ such that*

$$\|R_1 f\|_{p,|x|^\sigma} \leqslant A\|f\|_{p,|x|^\gamma} \tag{36}$$

for all $f \in \mathscr{S}(\mathbf{R})$.

Proof of Proposition 6.1. By $-\frac{\gamma}{p-1} + 1 = \frac{b(1+\sigma)}{a(p-1)} > 0$,

$$|R_1 f(x)| \leqslant C\chi_{[0,32)}(x)\|f\|_{p,|x|^\gamma}.$$

By $\sigma > -1$, one obtains (36).

PROPOSITION 6.2. *Let* $a, b > 0$. *Let* $-1 < \sigma \leqslant a-1$, $\gamma > -\frac{(1+\sigma)b}{a}$, *and* $p = (1+\gamma) + \frac{b}{a}(1+\sigma)$. *Define* S_{ab} *by*

$$(S_{ab}f)(x) = \chi_{[0,\infty)}(x) \int_0^\infty e^{ix^a y^b} f(y)dy$$

for $f \in \mathscr{S}(\mathbf{R})$. *Then there exists* $A = A(a, b, \sigma, \gamma) > 0$ *such that*

$$\|S_{ab}f\|_{p,|x|^\sigma} \leqslant A\|f\|_{p,|x|^\gamma}$$

for all $f \in \mathscr{S}(\mathbf{R})$.

Proof of Proposition 6.2. For $f \in \mathscr{S}(\mathbf{R})$ let

$$g(u) = \frac{1}{b}\chi_{[0,\infty)}(u)u^{\frac{1}{b}-1}f(u^{\frac{1}{b}}).$$

Then

$$\overline{S_{a,b}f(x)} = \hat{g}(x^a)\chi_{[0,\infty)}(x).$$

By $0 \leqslant 1 - \frac{\sigma+1}{a} < 1$ and Pitt's Inequality ([3]),

$$\begin{aligned}\|S_{ab}f\|^p_{p,|x|^\sigma} &= \frac{1}{a}\int_0^\infty |\hat{g}(\xi)|^p |\xi|^{-[1-\frac{(1+\sigma)}{a}]}d\xi \\ &\leqslant C\int_0^\infty |g(u)|^p |u|^{1-\frac{(1+\sigma)}{a}+(p-2)}du \leqslant C\|f\|^p_{p,|x|^\gamma},\end{aligned}$$

where we used the fact that

$$b\left[p-1-\frac{(1+\sigma)}{a}\right] + p(1-b) + b - 1 = p - 1 - \frac{b}{a}(1+\sigma) = \gamma.$$

This proves Proposition 6.2.

PROPOSITION 6.3. *Let* $a, b > 0$. *Let* $-1 < \sigma \leqslant a-1$, $\gamma > -\frac{(1+\sigma)b}{a}$, $p = (1+\gamma) + \frac{b}{a}(1+\sigma)$, *and* R_2 *be given as in (7). Then there exists* $A > 0$ *such that*

$$\|R_2f\|_{p,|x|^\sigma} \leqslant A\|f\|_{p,|x|^\gamma} \tag{37}$$

for all $f \in \mathscr{S}(\mathbf{R})$.

Proof of Proposition 6.3. First we observe that

$$\operatorname{supp}(\psi_0) \subseteq (0,4) \qquad \text{and} \qquad \operatorname{supp}(h) \subseteq (8,\infty).$$

Let $m = [\frac{b(1+\sigma)}{ap}]$. Then for $x \in \operatorname{supp}(\psi_0)$ and $y \in \operatorname{supp}(h)$ there exists a $t = t(x,y) \in (0,x)$ such that

$$K(x,y) = \sum_{\ell=0}^{m} \frac{1}{\ell!}\frac{\partial^\ell K}{\partial x^\ell}(0,y)x^\ell + \frac{1}{(m+1)!}\frac{\partial^{m+1}K}{\partial^{m+1}}(t,y)x^{\ell+1}. \tag{38}$$

By (1) for $\ell = 0, 1, ..., m$ we have

$$\left|\frac{\partial^\ell K}{\partial x^\ell}(0,y)\right| \leqslant \frac{C}{|y|^\ell} \leqslant C \tag{39}$$

and

$$\left|\frac{\partial^{m+1} K}{\partial x^{m+1}}(t,y)\right| \leqslant \frac{C}{|t-y|^{m+1}} \leqslant C|y|^{-(m+1)}. \tag{40}$$

For $\ell = 0, 1, ..., m,$ define $R_2^{(\ell)}$ by

$$R_2^{(\ell)} f(x) = \frac{x^\ell}{\ell!}\psi_0(x)\int_0^\infty e^{ix^a y^b}\frac{\partial^\ell K}{\partial x^\ell}(0,y)h(y)f(y)dy.$$

Then by $\sigma > -1$ and (38)

$$\|R_2 f - \sum_{\ell=0}^m R_2^{(\ell)} f\|_{p,|x|^\sigma}^p \leqslant C\int_0^4\left|\int_8^\infty |f(y)||y|^{-(m+1)}dy\right|^p |x|^\sigma dx$$

$$\leqslant C\left(\int_8^\infty |f(y)|^p y^\gamma dy\right)\left(\int_8^\infty y^{-[(m+1)p+\gamma]/(p-1)}dy\right)^{p-1} \leqslant C\|f\|_{p,|x|^\gamma}^p, \tag{41}$$

where we also used

$$\frac{(m+1)p+\gamma}{p-1} > \frac{\frac{b}{a}(1+\sigma)+\gamma}{p-1} = 1.$$

By Proposition 6.2 and (39) we have

$$\|R_2^{(\ell)} f\|_{p,|x|^\sigma} \leqslant C\|f\|_{p,|x|^\gamma} \tag{42}$$

for $\ell = 0, 1, ..., m$. Therefore by (41) and (42) we obtain

$$\|R_2 f\|_{p,|x|^\sigma} \leqslant C\|f\|_{p,|x|^\gamma}$$

for all $f \in \mathscr{S}(\mathbf{R})$, which proves Proposition 6.3.

PROPOSITION 6.4. *Let* $a, b > 0$. *Let* $-1 < \sigma \leqslant a-1$, $\gamma > -\frac{(1+\sigma)b}{a}$, $p = (1+\gamma) + \frac{b}{a}(1+\sigma)$, *and* R_3 *be given as in (8). Then there exists* $A > 0$ *such that*

$$\|R_3 f\|_{p,|x|^\sigma} \leqslant A\|f\|_{p,|x|^\gamma} \tag{43}$$

for all $f \in \mathscr{S}(\mathbf{R})$.

Proof of Proposition 6.4. Let $n = [\frac{1+\sigma}{p}]$. For $x \in \text{supp}(h) \subseteq (8,\infty)$ and $y \in \text{supp}(\psi_0) \subseteq (0,4)$ there exists $\tau = \tau(x,y) \in (0,y)$ such that

$$K(x,y) = \sum_{\ell=0}^n \frac{1}{\ell!}\frac{\partial^\ell K}{\partial y^\ell}(x,0)y^\ell + \frac{1}{(n+1)!}\frac{\partial^{n+1}K}{\partial y^{n+1}}(x,\tau)y^{n+1}. \tag{44}$$

By (1) we have the following for $\ell = 0, 1, ..., n$:

$$\left|\frac{\partial^\ell K}{\partial y^\ell}(x,0)\right| \leqslant \frac{C}{|x|^\ell} \leqslant C \tag{45}$$

and

$$\left|\frac{\partial^{n+1}K}{\partial y^{n+1}}(x,\tau)\right| \leqslant \frac{C}{|x-\tau|^{n+1}} \leqslant C|x|^{-(n+1)}. \tag{46}$$

For $\ell = 0,1,\dots,n$, define $R_3^{(\ell)}$ by

$$R_3^{(\ell)}f(x) = \frac{1}{\ell!}\frac{\partial^\ell K}{\partial y^\ell}(x,0)h(x)\int_0^\infty e^{ix^a y^b} y^\ell \psi_0(y)f(y)dy. \tag{47}$$

By (8), (44), (46), and (47) and $-(n+1)p+\sigma < -1$, we have

$$\|R_3 f - \sum_{\ell=0}^{n} R_3^{(\ell)} f\|_{p,|x|^\sigma} \leqslant C\|f\|_{p,|x|^\gamma}. \tag{48}$$

By Proposition 6.2 and (45) we get

$$\|\sum_{\ell=0}^{n} R_3^{(\ell)} f\|_{p,|x|^\sigma} \leqslant C\sum_{\ell=0}^{n}\left(\int_0^\infty |y^\ell \psi_0(y)f(y)|^p |y|^\gamma dy\right)^{\frac{1}{p}} \leqslant C\|f\|_{p,|x|^\gamma}. \tag{49}$$

Therefore by (48) and (49) we obtain that

$$\|R_3 f\|_{p,|x|^\sigma} \leqslant C\|f\|_{p,|x|^\gamma}$$

for all $f \in \mathscr{S}(\mathbf{R})$, which proves Proposition 6.4.

7. The $L^p(\mathbf{R},|x|^\gamma) \to L^p(\mathbf{R},|x|^\sigma)$ boundedness

We are finally ready to present the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $a,b>0$. Let $-1<\sigma<a-1$, $\gamma > \frac{(1+\sigma)(1-b)}{a}$, and $p = (1+\gamma)+\frac{b}{a}(1+\sigma)$. Let K and $T_{a,b}$ be given as in (1) and (2). First we shall establish the $L^p(\mathbf{R},|x|^\gamma) \to L^p(\mathbf{R},|x|^\sigma)$ boundedness when $p \leqslant 2$.

Assume that $p \leqslant 2$. For $j = 1,2,3,4$ define T_j by

$$T_1 f(x) = \chi_{(0,\infty)}(x)\int_0^\infty e^{ix^a y^b} K(x,y)f(y)dy,$$

$$T_2 f(x) = \chi_{(0,\infty)}(x)\int_0^\infty e^{ix^a y^b} K(x,-y)f(-y)dy,$$

$$T_3 f(x) = \chi_{(0,\infty)}(x)\int_0^\infty e^{ix^a y^b} K(-x,y)f(y)dy,$$

and

$$T_4 f(x) = \chi_{(0,\infty)}(x)\int_0^\infty e^{ix^a y^b} K(-x,-y)f(-y)dy.$$

Then we have

$$T_{a,b}f(x) = T_1 f(x) + T_2 f(x) + T_3 f(-x) + T_4 f(-x). \tag{50}$$

Since $K(x,y)$ satisfies (1) so does $K(-x,-y)$. When $x,y>0$ we have $|x+y|>|x-y|$; therefore, $K(-x,y)$ and $K(x,-y)$ also satisfy (1) in $(0,\infty)\times(0,\infty)$. By (9), (50), and Propositions 5.1, 6.1, 6.3, and 6.4, we obtain

$$\|T_{a,b}f\|_{p,|x|^\sigma} \leqslant A\|f\|_{p,|x|^\gamma}$$

for all $f\in\mathscr{S}(\mathbf{R})$.

We now turn to the remaining case: $p>2$. Let $q=\frac{p}{p-1}<2$. Let $a_1=b>0$, $b_1=a>0$, $\sigma_1=-q\gamma/p$, and $\gamma_1=-q\sigma/p$. Then we have

$$(1+\gamma_1)+\frac{b_1}{a_1}(1+\sigma_1)=q\left[\left(1-\frac{1}{p}\right)\left(1+\frac{a}{b}\right)-\frac{1}{p}\left(\sigma+\frac{a\gamma}{b}\right)\right]=q$$

and

$$\sigma_1+1=1-\frac{q\gamma}{p}=\frac{b(1+\sigma)}{a(p-1)}>0.$$

On the other hand,

$$\sigma_1-(a_1-1)=-\frac{b}{p-1}\left[\gamma-\frac{(1+\sigma)(1-b)}{a}\right]<0.$$

In addition,

$$\gamma_1-\frac{(1+\sigma_1)(1-b_1)}{a_1}=\frac{[(a-1)-\sigma]}{a(p-1)}>0.$$

Thus we have $-1<\sigma_1<a_1-1$ and $\gamma_1>\frac{(1+\sigma_1)(1-b_1)}{a_1}$. Let

$$R_{a,b}f(x)=\int_{\mathbf{R}}e^{i|x|^b|y|^a}K(y,x)f(y)dy. \tag{51}$$

Since $K(y,x)$ satisfies (1) and $q<2$, we obtain that

$$\|R_{a,b}f\|_{q,|x|^{\sigma_1}} \leqslant A\|f\|_{q,|x|^{\gamma_1}}. \tag{52}$$

Let $\Gamma_q=\{g\in\mathscr{S}(\mathbf{R}):\|g\|_q=1\}$. For a given g we let $\tilde{g}(y)=|y|^{\sigma/p}g(y)$. Then for $f\in\mathscr{S}(\mathbf{R})$

$$\begin{aligned}\|T_{a,b}f\|_{p,|x|^\sigma}&=\sup_{g\in\Gamma_q}\left|\int_{\mathbf{R}}T_{a,b}f(y)|y|^{\sigma/p}g(y)dy\right|=\sup_{g\in\Gamma_q}\left|\int_{\mathbf{R}}f(x)R_{a,b}\tilde{g}(x)dx\right|\\&\leqslant A\|f\|_{p,|x|^\gamma}\sup_{g\in\Gamma_q}\left(\int_{\mathbf{R}}[|y|^{\frac{\sigma}{p}}|g(y)|]^q|y|^{-\frac{q\sigma}{p}}dy\right)^{1/q}=A\|f\|_{p,|x|^\gamma}.\end{aligned}$$

This finishes the proof of Proposition 3.1.

8. Analytic interpolation and proof of main result

Finally we are ready to present the proof of our main result Theorem 2.1. This can be achieved by combining Proposition 3.1 with Stein's theorem on interpolation of analytic families of operators. First let us recall Stein's result.

Let $u_0, u_1 \in \mathbf{R}$ so that $u_0 < u_1$. Let

$$D = \{s \in \mathbf{C} \;\mid\; u_0 < Re(s) < u_1\}.$$

Let $k(\cdot,\cdot,\cdot)$ be a function on $\mathbf{R}\times\mathbf{R}\times\overline{D}$, and for each $s \in \overline{D}$ define $k_s : \mathbf{R}\times\mathbf{R} \to \mathbf{C}$ by

$$k_s(x,y) = k(x,y,s).$$

Suppose that there are $L, A > 0$ such that

(i) $\operatorname{supp}(k_s) \subseteq [-L,L]^2$ for all $s \in \overline{D}$;

(ii) $|k(x,y,s)| \leqslant A$ for all $(x,y,s) \in \mathbf{R}\times\mathbf{R}\times\overline{D}$;

(iii) for each $(x,y) \in \mathbf{R}\times\mathbf{R}$ the function

$$s \to k(x,y,s)$$

is continuous in $\overline{D}$ and analytic in D.

For each $s \in \overline{D}$, define the operator U^s by

$$(U^s f)(x) = \int_{\mathbf{R}} k_s(x,y) f(y)\,dy. \tag{53}$$

LEMMA 8.1 ([12]). *Let* $1 \leqslant p_j, q_j \leqslant \infty$ *and* $\{U^s\}$ *be given as above. Suppose that there are* $M_0, M_1 > 0$ *such that*

$$\|U^s f\|_{q_j} \leqslant M_j \|f\|_{p_j} \tag{54}$$

whenever $Re(s) = u_j$ *for* $j = 0,1$. *Then for every* $t \in [0,1]$

$$\|U^{u_0(1-t)+u_1 t} f\|_{q_t} \leqslant M_0^{1-t} M_1^t \|f\|_{p_t} \tag{55}$$

where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Proof of Theorem 2.1. Since the case $p = q$ is covered by Proposition 3.1, we may assume that $1 < p < q < \infty$. Let $N \in \mathbf{N}$ be fixed, $I_N = [\frac{1}{N}, N]$, and $D = \{s \in \mathbf{C} : 0 < Re(s) < 1\}$. For each $s \in \overline{D}$ define $k_s(x,y)$ by

$$k_s(x,y) = e^{i|x|^a|y|^b} K(x,y)\chi_{I_N}(|x|)\chi_{I_N}(|y|)|x|^{s\sigma/(q+1-\frac{q}{p})}|y|^{-s\gamma/[p(1+\frac{1}{q}-\frac{1}{p})]}.$$

Let $p_0 = 1$, $q_0 = \infty$, $p_1 = q_1 = q + 1 - q/p$, and $\{U^s\}$ be defined as in (53).

When $Re(s) = 0$, we have $s = i\theta$ for some $\theta \in \mathbf{R}$ and

$$|(U^s f)(x)| = \left| \int_{\mathbf{R}} e^{i|x|^a|y|^b} K(x,y)\chi_{I_N}(x)\chi_{I_N}(y) \times \right.$$
$$\left. \times\, |x|^{i\theta\sigma/(q+1-\frac{q}{p})}|y|^{-i\theta\gamma/[p(1+\frac{1}{q}-\frac{1}{p})]} f(y)\,dy \right| \leqslant A\|f\|_1. \tag{56}$$

When $Re(s) = 1$, we write $s = 1 + i\theta$ for some $\theta \in \mathbf{R}$. Then

$$|U^s f(x)| = \chi_{I_N}(|x|)|x|^{\sigma/q_1}|T_{a,b}(f_\theta)(x)| \tag{57}$$

where

$$f_\theta(y) = \chi_{I_N}(|y|)|y|^{-(1+i\theta)\gamma q/(pp_1)}f(y).$$

Let $\sigma_1 = \sigma$ and $\gamma_1 = q\gamma/p$. Then we have

$$\gamma_1 - \frac{(1+\sigma_1)(1-b)}{a} = \frac{q}{bp}[\gamma - (p-1)(1-b)] > 0,$$

and

$$1 + \gamma_1 + \frac{b}{a}(1+\sigma_1) - p_1 = \frac{q\gamma}{p} + \frac{b(1+\sigma)}{a} - q + \frac{q}{p} = 0.$$

Thus we have $-1 < \sigma_1 < a-1$, $\gamma_1 > \frac{(1+\sigma_1)(1-b)}{a}$, and $p_1 = 1 + \gamma_1 + \frac{b}{a}(1+\sigma_1)$. By Proposition 3.1 the operator $T_{a,b}$ is bounded from $L^{p_1}(\mathbf{R}, |x|^{\gamma_1})$ to $L^{p_1}(\mathbf{R}, |x|^{\sigma_1})$. Thus by (57) we have

$$\begin{aligned}\|U^s f\|_{q_1} &\leqslant \left(\int_{\mathbf{R}} |(T_{a,b}f_\theta)(x)|^{p_1}|x|^{\sigma_1}dx\right)^{1/p_1}\\ &\leqslant A\left(\int_{\mathbf{R}} |f_\theta(y)|^{p_1}|y|^{\gamma_1}dy\right)^{1/p_1} \leqslant A\|f\|_{p_1}\end{aligned} \tag{58}$$

whenever $Re(s) = 1$.

Let $t = 1 + \frac{1}{q} - \frac{1}{p} \in (0,1)$. Then

$$\frac{1}{p_t} = \frac{1-t}{1} + \frac{t}{p_1} = \frac{1}{p} \qquad \text{and} \qquad \frac{1}{q_t} = \frac{1-t}{\infty} + \frac{t}{q_1} = \frac{1}{q}.$$

By Lemma 8.1 we obtain

$$\|U^t f\|_q \leqslant A\|f\|_p,$$

i.e.

$$\left(\int_{\frac{1}{N}\leqslant|x|\leqslant N}\left|\int_{\frac{1}{N}\leqslant|y|\leqslant N} e^{i|x|^a|y|^b}K(x,y)|f(y)||y|^{-\gamma/p}dy\right|^q|x|^\sigma dx\right)^{\frac{1}{q}} \leqslant A\|f\|_p \tag{59}$$

for all $f \in L^p(\mathbf{R})$ where A is independent of $N \in \mathbf{N}$. For $f \in L^p(\mathbf{R}, |x|^\gamma)$ by applying (59) to $f(y)|y|^{\gamma/p}\chi_{I_N}(y)$ we obtain that

$$\left(\int_{\frac{1}{N}\leqslant|x|\leqslant N}\left|\int_{\frac{1}{N}\leqslant|y|\leqslant N} e^{i|x|^a|y|^b}K(x,y)f(y)dy\right|^q|x|^\sigma dx\right)^{1/q} \leqslant A\|f\|_{p,|x|^\gamma}. \tag{60}$$

By Lebesgue's Dominated Convergence Theorem we have

$$\lim_{N\to\infty}\int_{\frac{1}{N}\leqslant|y|\leqslant N} e^{i|x|^a|y|^b}K(x,y)f(y)dy = T_{a,b}f(x)$$

for $f \in L^1(\mathbf{R})$. Therefore by (60), (61), and Fatou's lemma we obtain

$$\left(\int_{\mathbf{R}} |T_{a,b}f(x)|^q |x|^\sigma dx\right)^{\frac{1}{q}} \leqslant A\|f\|_{p,|x|^\gamma} \tag{61}$$

whenever $f \in L^p(\mathbf{R}, |x|^\gamma) \cap L^1(\mathbf{R})$. Since $L^p(\mathbf{R}, |x|^\gamma) \cap L^1(\mathbf{R})$ is a dense subspace of $L^p(\mathbf{R}, |x|^\gamma)$, $T_{a,b}$ extends to be a bounded operator from $L^p(\mathbf{R}, |x|^\gamma)$ to $L^q(\mathbf{R}, |x|^\sigma)$, which proves this theorem.

9. Concluding remarks

We shall end the paper by providing explanations for two remarks made in Section 1. By letting $K(x,y) \equiv 1$, we have

$$(T_{a,b}f)(x) = \int_{\mathbf{R}} e^{i|x|^a|y|^b} f(y)dy.$$

(i) For $\delta > 0$ let $(D_\delta f)(x) = f(\delta x)$. Then $\|D_\delta f\|_{p,|x|^\gamma} = \delta^{-\frac{(1+\gamma)}{p}} \|f\|_{p,|x|^\gamma}$. Suppose that $T_{a,b}$ is a bounded operator from $L^p(\mathbf{R}, |x|^\gamma)$ to $L^q(\mathbf{R}, |x|^\sigma)$. Then by $\delta^{\frac{1}{b}}(D_{\delta^{\frac{1}{a}}} T_{a,b} D_{\delta^{\frac{1}{b}}} f)(x) = (T_{a,b}f)(x)$, one obtains $\delta^{\frac{1}{b}-(\frac{1+\sigma}{aq}+\frac{1+\gamma}{bp})} \geqslant 1$ for all $\delta > 0$, which implies that

$$\frac{1}{b} - \frac{1+\gamma}{bp} - \frac{1+\sigma}{aq} = 0.$$

(ii) Let $f(y) = \frac{1}{2\sqrt{y}}\chi_{[1,9]}(y) \in L^2(\mathbf{R})$. Then by a simple calculation one obtains that

$$T_{\frac{1}{2},\frac{1}{2}}f(x) = \left(2e^{2i|x|^{\frac{1}{2}}}\right)\frac{\sin(|x|^{\frac{1}{2}})}{|x|^{\frac{1}{2}}} \notin L^2(\mathbf{R}).$$

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