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INEQUALITIES FOR ALMOST PERIODIC MEASURES

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Abstract. Some inequalities for the mean of certain almost periodic measures are presented. We give inequalities like Hölder, Minkowski or Jensen, for the mean of almost periodic measures defined by densities.

1. Introduction

We try to find out if the classical inequalities for integrals can be adapted to the mean of some almost periodic measures. In our paper we consider almost periodic measures defined by densities and we give inequalities for the mean of these measures. There are alternative inequalities for those of Hölder, Minkowski, Jensen, which can be found in [3], [6], [9], [11].

J. Lamadrid and L. Argabright defined the almost periodic measures on a locally compact abelian group G and the mean of theirs ([2], [8]). They proved that the measure $f\mu$ which is defined by an almost periodic function f as density and an almost periodic measure μ as base, is also an almost periodic measure. We take the mean of such measures and we establish some inequalities. First we prove a Cauchy type inequality and some applications. Next we give similar results with the Hölder, Minkowski and Jensen inequalities for integrals and some corollaries. So, we present an alternative for the weighted arithmetic mean - geometric mean - harmonic mean inequalities (weighted A-G-H inequalities). Finally we construct an almost periodic function which can be defined like a mean with parameter and we establish an inequality regarding this function.

2. Preliminaries

Consider a Hausdorff locally compact abelian group G and let λ be the Haar measure on G. Let us denote by $\mathscr{C}(G)$ the set of all bounded continuous complex-valued functions on G, and by $\mathscr{C}_U(G)$ the subset of $\mathscr{C}(G)$ containing the uniformly continuous functions. The sets $\mathscr{C}(G)$ and $\mathscr{C}_U(G)$ are Banach algebras endowed with the supremum norm.

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Throughout this paper, $\|\cdot\|$ denotes the supremum norm on $\mathscr{C}(G)$. For $f \in \mathscr{C}(G)$ and $a \in G$, the translate of f by a is the function $f_a(x) = f(xa)$ for all $x \in G$. Denote by $K(G)$ the linear space of all continuous complex-valued functions on G, having a compact support. We denote by $m(G)$ the space of complex Radon measures on G. That is, the space of all complex linear functionals μ on $K(G)$ satisfying the following: for each compact subset A of G there exists a positive number $m_{\mu,A}$ such that $|\mu(f)| \leqslant m_{\mu,A}\|f\|$ whenever $f \in K(G)$ and the support of f is contained in A. The action of a measure $\mu \in m(G)$ on a function $f \in K(G)$ will be denoted either $\mu(f)$ or $\int_G f(x)d\mu(x)$.

For the Borel function f and the measures $\mu, \nu \in m(G)$, we define their convolutions $f * \mu$, $\nu * \mu$, when that is possible ([5]). The Dirac measure at $x \in G$ is written δ_x. In [1] L. Argabright and J. Lamadrid defined the concept of the translation-bounded measure as being a measure $\mu \in m(G)$ with the property that for every compact set $A \subseteq G$, $sup_{x\in G} |\mu|(xA) < \infty$, where $|\mu|(xA)$ is the variation measure of μ on xA. The linear space of the translation-bounded measures will be denoted by $m_B(G)$. We identify an arbitrary measure $\mu \in m_B(G)$ with an element of the space $[\mathscr{C}_U(G)]^{K(G)}$ in the following way: $\mu \equiv \{f * \mu\}_{f\in K(G)}$. From this identification we have the inclusion $m_B(G) \subset [\mathscr{C}_U(G)]^{K(G)}$. The space $[\mathscr{C}_U(G)]^{K(G)}$ has the *product topology* defined by the Banach space structure on $\mathscr{C}_U(G)$, hence, $m_B(G)$ is a locally convex space of measures with the relative topology. A system of seminorms for the product topology on $m_B(G)$ is given by the family $\{\|\cdot\|_f\}_{f\in K(G)}$, where, for a function $f \in K(G)$, $\|\mu\|_f = \|f * \mu\|$, for all $\mu \in m_B(G)$. In [7] there are defined the almost periodic functions.

DEFINITION 2.1. ([7]) A function $g \in \mathscr{C}(G)$ is called an almost periodic function on G, if the family of translates of g, $\{g_a : a \in G\}$ is relatively compact in the sense of uniform convergence on G.

The set $AP(G)$ of all almost periodic functions on G is a Banach algebra with respect to the supremum norm, closed to conjugation. In [7] it is proved that there exists a unique positive linear functional $M : AP(G) \to C$ such that $M(f_a) = M(f)$, for all $a \in G$, $f \in AP(G)$. If $f \in AP(G)$ we define the mean value of f as being the above complex number $M(f)$. The almost periodic measures are introduced and studied by L. Argabright and J. Lamadrid.

DEFINITION 2.2. ([2], [8]) The measure $\mu \in m_B(G)$ is said to be an almost periodic measure, if for every $f \in K(G)$, $f * \mu \in AP(G)$.

The set $ap(G)$ of all almost periodic measures is a locally convex space with respect to the product topology. If $f \in AP(G)$ and $\mu \in ap(G)$, then the measure $f\mu$, defined by $f\mu(g) = \mu(gf)$, $g \in K(G)$ is an almost periodic measure ([8]).

In [8] it is proved that there exists a unique positive linear functional $M : ap(G) \to C$ such that M is continuous on $ap(G)$, $M(\mu) = M(\delta_x * \mu)$, for all $x \in G$, $\mu \in ap(G)$ and $M(\lambda) = 1$. If $\mu \in ap(G)$ we define the mean value of μ as being the above complex number $M(\mu)$.

3. Inequalities for the mean of almost periodic measures

PROPOSITION 3.1. (Cauchy inequality) *Let μ be a positive almost periodic measure. Then for all $f, g \in AP(G)$ we have*

$$| M(fg\mu) |^2 \leqslant M(| f |^2 \mu) M(| g |^2 \mu).$$

Proof. Denote $\langle f\mu, g\mu \rangle = M(f\overline{g}\mu)$, $f, g \in AP(G)$. Considering $\alpha \in C$, $| \alpha | = 1$ and $a \in \boldsymbol{R}$, we obtain

$$0 \leqslant \langle af\mu + \alpha g\mu, af\mu + \alpha g\mu \rangle = a^2 \langle f\mu, f\mu \rangle + \langle g\mu, g\mu \rangle + 2aRe[\overline{\alpha}\langle f\mu, g\mu \rangle]. \tag{1}$$

We choose α such that $\overline{\alpha}\langle f\mu, g\mu \rangle = | \langle f\mu, g\mu \rangle |$. So (1) leads to

$$a^2 \langle f\mu, f\mu \rangle + 2a | \langle f\mu, g\mu \rangle | + \langle g\mu, g\mu \rangle \geqslant 0,$$

for all $a \in \boldsymbol{R}$. Since the discriminant is negative, we get

$$| \langle f\mu, g\mu \rangle |^2 \leqslant \langle f\mu, f\mu \rangle \langle g\mu, g\mu \rangle. \quad \square$$

COROLLARY 3.1. *Let μ be a positive almost periodic measure. For all $f, g \in AP(G)$ we have*

i) $| M(f\mu) |^2 \leqslant M(| f |^2 \mu) M(\mu)$;

ii) $| M(f\overline{g}\mu)M(\mu) - M(\overline{f}\mu)M(g\mu) |^2 \leqslant \left(M(| f |^2 \mu)M(\mu) - | M(f\mu) |^2\right)\left(M(| g |^2 \mu)M(\mu) - | M(g\mu) |^2\right)$.

Proof. i) This is a simple consequence of Proposition 3.1.

ii) From i) we infer that for all $\alpha \in C$

$$| M(f\mu + \alpha g\mu) |^2 \leqslant M(| f + \alpha g |^2 \mu) M(\mu).$$

This means that

$$[M(f\mu) + \alpha M(g\mu)][\overline{M(f\mu)} + \overline{\alpha M(g\mu)}] \leqslant M[(f + \alpha g)(\overline{f} + \overline{\alpha g})\mu] M(\mu)$$

and further

$$M(| f |^2 \mu)M(\mu) - | M(f\mu) |^2 + | \alpha |^2 [M(| g |^2 \mu)M(\mu) - | M(g\mu) |^2] + \alpha[M(\overline{f}g\mu)M(\mu) - M(g\mu)M(\overline{f}\mu)] + \overline{\alpha}[M(f\overline{g}\mu)M(\mu) - M(\overline{g}\mu)M(f\mu)] \geqslant 0.$$

Consider

$$\alpha = \alpha_1 + i\alpha_2,$$

$$M(| g |^2 \mu)M(\mu) - | M(g\mu) |^2 = u$$

$$M(\overline{f}g\mu)M(\mu) - M(g\mu)M(\overline{f}\mu) = v_1 + iv_2$$

$$M(| f |^2 \mu)M(\mu) - | M(f\mu) |^2 = w$$

Clearly $\alpha_1, \alpha_2, v_1, v_2 \in \boldsymbol{R}$ and $u, w \geqslant 0$. So

$$(\alpha_1^2 + \alpha_2^2)u + w + (\alpha_1 + i\alpha_2)(v_1 + iv_2) + (\alpha_1 - i\alpha_2)(v_1 - iv_2) \geqslant 0,$$

for all $\alpha_1, \alpha_2 \in \boldsymbol{R}$. This implies

$$(\alpha_1^2 + \alpha_2^2)u + 2\alpha_1 v_1 - 2\alpha_2 v_2 + w \geqslant 0 \text{ for all } \alpha_1, \alpha_2 \in \boldsymbol{R}. \tag{2}$$

Next we prove that

$$v_1^2 + v_2^2 \leqslant uw. \tag{3}$$

If $u = 0$ then (2) yields $2\alpha_1 v_1 - 2\alpha_2 v_2 + w \geqslant 0$ for all $\alpha_1, \alpha_2 \in \boldsymbol{R}$ hence $v_1 = v_2 = 0$ and (3) is proved. If $u > 0$ we may take in (2) $\alpha_1 = -\frac{v_1}{u}$ and $\alpha_2 = -\frac{v_2}{u}$. This yields (3), giving the proof of (ii). □

LEMMA 3.1. *Let $f \in AP(G), \mu \in ap(G), \mu \geqslant 0$. Then*

$$| M(f\mu) | \leqslant M(| f | \mu).$$

Proof. The conclusion follows from the fact that M is a positive linear functional. □

THEOREM 3.1. (Hölder inequality) *Let μ be a positive almost periodic measure and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For all $f, g \in AP(G)$ we have*

$$| M(fg\mu) | \leqslant \{M(| f |^p \mu)\}^{\frac{1}{p}} \{M(| g |^q \mu)\}^{\frac{1}{q}}.$$

Proof. Young's inequality (see e.g. [9]) gives $| ab | \leqslant \frac{|a|^p}{p} + \frac{|b|^q}{q}$ for all $a, b \in \boldsymbol{R}$. We obtain

$$\frac{| fg |}{\{M(| f |^p \mu)\}^{\frac{1}{p}} \{M(| g |^q \mu)\}^{\frac{1}{q}}} \leqslant \frac{| f |^p}{pM(| f |^p \mu)} + \frac{| g |^q}{qM(| g |^q \mu)}.$$

Hence,

$$\frac{| fg | \mu}{\{M(| f |^p \mu)\}^{\frac{1}{p}} \{M(| g |^q \mu)\}^{\frac{1}{q}}} \leqslant \frac{| f |^p \mu}{pM(| f |^p \mu)} + \frac{| g |^q}{qM(| g |^q \mu)}. \tag{4}$$

Applying the mean in (4) and using Lemma 3.1 it results

$$| M(fg\mu) | \leqslant \{M(| f |^p \mu)\}^{\frac{1}{p}} \{M(| g |^q \mu)\}^{\frac{1}{q}}. \quad \square$$

COROLLARY 3.2. (Minkowski inequality) *Let $f_1, \cdots, f_n$ be a finite set of almost periodic functions on G, μ a positive almost periodic measure such that $M(\mu) = 1$, and $k \in (1, \infty)$. Then we have*

$$\{M[| \sum_{m=1}^{n} f_m |^k]\mu\}^{\frac{1}{k}} \leqslant \sum_{m=1}^{n} [M(| f_m |^k \mu)]^{\frac{1}{k}}.$$

COROLLARY 3.3. *Let* μ *be a positive almost periodic measure such that* $M(\mu) = 1$. *For all* $f \in AP(G)$ *and* $k \in (1,\infty)$ *we have*

$$| M(f\mu) |^k \leqslant M(| f |^k \mu).$$

THEOREM 3.2. (Jensen inequality) *Let* f *be an almost periodic function such that* $f : G \to [a,b]$, $a \in \boldsymbol{R}$; $\Phi : \boldsymbol{R} \to \boldsymbol{R}$ *a non-decreasing convex function and* μ *a positive almost periodic measure,* $\mu \neq 0$. *Then*

$$\Phi\{\frac{M(f\mu)}{M(\mu)}\} \leqslant \frac{M[(\Phi \circ f)\mu]}{M(\mu)}.$$

Proof. First we observe that $\Phi \circ f$ is an almost periodic function and therefore $(\Phi \circ f)\mu$ is an almost periodic measure. Indeed, since Φ is continuous on $[a,b]$, for $\varepsilon > 0$ there exists, (by Weierstrass approximation theorem), a polynomial P_ε such that

$$sup_{y\in[a,b]} \mid P_\varepsilon(y) - \Phi(y) \mid < \varepsilon.$$

Hence

$$sup_{x\in G} \mid P_\varepsilon(f(x)) - \Phi(f(x)) \mid < \varepsilon,$$

and further $\|P_\varepsilon \circ f - \Phi \circ f\| < \varepsilon$. Taking into account that $AP(G)$ is an algebra of functions we obtain that $P_\varepsilon \circ f \in AP(G)$. Hence, the above considerations show us that $\Phi \circ f$ is in $AP(G)$.

Let

$$\gamma = \frac{M(f\mu)}{M(\mu)}, \tag{5}$$

and we remark that $a \leqslant \gamma \leqslant b$. If $a < \gamma < b$, and k is the slope of a supporting line of Φ through the point $(\gamma, \Phi(\gamma))$ we obtain $\Phi(f(x)) - \Phi(\gamma) \geqslant k[f(x) - \gamma]$, for all $x \in G$. Therefore we can write

$$(\Phi \circ f - \Phi(\gamma))\mu \geqslant k(f - \gamma)\mu,$$

so,

$$M[(\Phi \circ f - \Phi(\gamma))\mu] \geqslant k[M(f\mu) - \gamma M(\mu)].$$

Taking into account of (5) we obtain $[M(f\mu) - \gamma M(\mu)] = 0$ and further

$$M[(\Phi \circ f)\mu] \geqslant \Phi\{\frac{M(f\mu)}{M(\mu)}\}M(\mu).$$

If $\gamma = b$ then (5) can be written as $M[(b - f)\mu] = 0$, hence $f = b$. Similarly for $\gamma = a$. $\square$

COROLLARY 3.4. (Weighted A-G-H inequalities) *Let* f *be an almost periodic function such that* $f : G \to [a,b]$, $a > 0$; μ *a positive almost periodic measure such that* $M(\mu) = 1$. *Then we have*

$$M(f\mu) \geqslant e^{M[(lnf)\mu]} \geqslant \frac{1}{M[\frac{1}{f}\mu]}.$$

Proof. For the first inequality we select in Theorem 3.2, $\Phi(x) = e^x$ and the second inequality follows from the first. □

REMARK 3.1. If $f,g \in AP(G), \mu \in ap(G)$, then for a fixed $x \in G$, the notation $M_y[f(xy^{-1})g(y)\mu(y)] \in C$ means the mean of the almost periodic measure $\tilde{f}g\mu$, where $\tilde{f}(y) = f(xy^{-1})$, $y \in G$.

LEMMA 3.2. *Let* $f,g \in AP(G), \mu \in ap(G), \mu \geqslant 0$. *Then the function* $x \in G \to M_y[f(xy^{-1})g(y)\mu(y)] \in C$ *is in* $AP(G)$.

Proof. Consider a sequence (u_n) in G. Since f is an almost periodic function, it results that there exists a subsequence (v_n) of (u_n) and an almost periodic function f_0 such that $\|f_{v_n} - f_0\| \to 0$.

We obtain that

$$\begin{aligned} &| M_y[f(xv_ny^{-1})g(y)\mu(y)] - M_y[f_0(xy^{-1})g(y)\mu(y)] | \\ &\leqslant \|f_{v_n} - f_0\|\|g\|M(\mu). \end{aligned}$$

Hence the function $x \in G \to M_y[f(xy^{-1})g(y)\mu(y)] \in C$ is in $AP(G)$. □

THEOREM 3.3. *Let* $f,g \in AP(G), k > 1$ *and consider* $\mu \in ap(G), \mu \geqslant 0$ *such that* $M(\mu) = 1$. *Then*

$$\begin{aligned} &M_x\{M_y[| f(xy^{-1})g(y) | \mu(y)]\mu(x)\} \\ &\leqslant \left(M_x\{M_y[| f |^k (xy^{-1}) | g |^k (y)\mu(y)]\mu(x)\}\right)^{\frac{1}{k}}. \end{aligned}$$

Proof. Using Theorem 3.1 we obtain

$$M_y[| f(xy^{-1})g(y) | \mu(y)] \leqslant \{M_y[| f |^k (xy^{-1}) | g |^k (y)\mu(y)]\}^{\frac{1}{k}},$$

for all $x \in G$. Hence

$$\begin{aligned} &M_x\{M_y[| f(xy^{-1})g(y) | \mu(y)]\mu(x)\} \\ &\leqslant \left(M_x\{\{M_y[| f | (xy^{-1}) | g | (y)\mu(y)]\}^k\mu(x)\}\right)^{\frac{1}{k}} \\ &\leqslant \left(M_x\{M_y[| f |^k (xy^{-1}) | g |^k (y)\mu(y)]\mu(x)\}\right)^{\frac{1}{k}}. \quad \square \end{aligned}$$

REMARK 3.2. Theorem 3.3 may be considered as a generalization of Corollary 3.3. Indeed if in Theorem 3.3 we take $f = 1$ and $M(\mu) = 1$ we obtain

$$M(| g | \mu) \leqslant [M(| g |^k \mu)]^{\frac{1}{k}}.$$

REFERENCES

[1] L. N. ARGABRIGHT AND J. GIL DE LAMADRID, *Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups*, Mem. Amer. Math. Soc. No. 145 (1974).
[2] L. N. ARGABRIGHT AND J. GIL DE LAMADRID, *Almost Periodic Measures*, Mem. Amer. Math. Soc. No. 428 (1990).
[3] E. BECKENBACH, R. BELLMAN, *Inequalities*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.
[4] N. DINCULEANU, *Vector Measures*, Veb Deutscher Verlag Der Wissenschaften, Berlin, 1966.
[5] N. DINCULEANU, *Integrarea pe Spaţii Local Compacte*, Editura Academiei R. P. R., Bucureşti, 1965 (Romanian).
[6] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, Cambridge Univ. Press, 1934.
[7] E. HEWITT AND K. A. ROSS, *Abstract Harmonic Analysis* Vol. I, Springer–Verlag, Berlin, Göttingen, Heidelberg, 1963.
[8] J. GIL DE LAMADRID, *Sur les Mesures Presque Périodiques*, Séminaire KGB sur les marches aléatoires, Astérisque 4, 1973.
[9] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer–Verlag, Berlin, Heidelberg, New York, 1970.
[10] W. RUDIN, *Fourier Analysis on Groups*, Interscience Tracts in Pure and Applied Mathematics, Number 12, Interscience Publishers – John Wiley and Sons, New York, London, 1962.
[11] A. ZYGMUND, *Trigonometric Series*, Vol. I, Cambridge, 1959.

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