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SOME INEQUALITIES AND EMBEDDINGS FOR WEIGHTED W_0 SPACES ON DOMAINS WITH FRACTAL BOUNDARIES

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Abstract. If Ω is a finite measure domain we show that several Poincaré, Hardy-type, or multiplicative inequalities as well as classical Sobolev embedding theorems on $W_0^{m,p}(\Omega)$ may be extended to versions with singular or degenerate weights involving powers of the distance to the boundary function provided that $\partial\Omega$ is "fractal" in the sense that $\partial\Omega$ has interior Minkowski dimension $\widetilde{M}_D(\partial\Omega) < n$. For unbounded non-finite measure domains such extensions may also often be made if $\partial\Omega$ satisfies a certain definition of "locally fractal".

1. Introduction and Notation

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, i.e., a nonempty open set, and for $p \in [1, \infty]$ and m a positive integer let $W^{m,p}(\Omega)$ denote the Sobolev space consisting of complex-valued measurable functions defined on Ω whose m-th order partial derivatives exist in the sense of distributions and which is equipped with the norm

$$\|u\|_{\Omega;m,p} := \|u\|_{\Omega;p} + \|\nabla^m u\|_{\Omega;p}. \tag{1.1}$$

Here the notation "$\|\nabla^m u\|_{\Omega;p}$" signifies $\sum_{|\alpha|=m} \|D^\alpha u\|_{\Omega;p}$ where the multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, the α_i being nonnegative integers, and $\|u\|_{\Omega;p}$ is the norm of the Lebesgue space $L^p(\Omega)$.

If $C_0^\infty(\Omega)$ consists of the infinitely differentiable compactly supported functions on Ω, we define $W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (1.1).

Let us note that the norm definition (1.1) is not universal. Ours is the same as given in Burenkov [9] and in Maz'ja [24], but much of the literature (see e.g. [1], [10], [16]) requires the lower order derivatives, i.e., $\sum_{|\alpha|\leqslant m} \|D^\alpha u\|_{\Omega;p}$ in (1.1). This makes certain inductive proofs easier, but the two definitions give equivalent norms on $W^{m,p}(\Omega)$ only under standard regularity assumptions on Ω. However they are always equivalent on $W_0^{m,p}(\Omega)$.

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The Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ enjoy many fundamental properties. We say that $W \equiv W^{m,p}(\Omega)$ or $W_0 \equiv W_0^{m,p}(\Omega)$ is continuously or compactly embedded in a space Z if W or $W_0 \subseteq Z$ and the natural maps $i : W \to Z$ or $i_0 : W_0 \to Z$ are continuous or compact. We express this in the continuous and compact cases respectively by the notations

$$W \text{ or } W_0 \hookrightarrow Z,$$
$$W \text{ or } W_0 \hookrightarrow\hookrightarrow Z.$$

For $W_0^{m,p}(\Omega)$ the following Poincaré inequality and associated embeddings are classical. Suppose $1 \leqslant p < \infty$ and $k \in \mathbb{N}$ such that $kp < n$. Define the *Sobolev conjugate* p_k^* of p by $p_k^* = np/(n-kp)$. Then (cf. [1, Theorem 6.2] or [10, Theorem V.3.6]) we have:

THEOREM A. *If Ω is an arbitrary domain there is a constant C, depending only upon p and n, such that for all all $u \in W_0$*

$$\|u\|_{\Omega,p_m^*} \leqslant C\|\nabla^m u\|_{\Omega,p}. \tag{1.2}$$

THEOREM B. *If Ω is bounded and $0 \leqslant j < m$ the following results hold.*

(i) *If $m-j < n/p$,*

$$W_0^{m,p}(\Omega) \hookrightarrow\hookrightarrow W_0^{j,q}(\Omega) \tag{1.3}$$

for $q \in [1, p_{m-j}^*)$. If $q = p_{m-j}^*$, the embedding (1.3) is continuous.

(ii) *If $m-j = n/p$, (1.5) holds for all $q \in [1,\infty)$.*

(iii) *If $m-j > n/p$, then*

$$W_0^{m,p}(\Omega) \hookrightarrow\hookrightarrow C_B^j(\Omega) \tag{1.4}$$

where C_B^j is the space of $j-$fold continuously differentiable functions u such that $D^\alpha u$ is bounded on Ω for $|\alpha| \leqslant j$. Since this space can be continuously embedded in $W^{j,q}(\Omega)$ it follows that (1.3) is also true in this case for all $q \in [1,\infty)$.

Other typical results include interpolation inequalities. We give an example from [10, Theorem V.3.8] which is a corollary of a more general result due to Nirenberg [25, p. 125].

THEOREM C. *Let m,j be nonnegative integers with $0 \leqslant j < m$ and $p,r \in [1,\infty)$. Define q by*

$$\frac{m}{q} = \frac{m-j}{r} + \frac{j}{p}.$$

Then the multiplicative inequality

$$\|\nabla^j u\|_{\Omega;q} \leqslant C\|u\|_{\Omega;r}^{\frac{m-j}{m}} \|\nabla^m u\|_{\Omega;p}^{\frac{j}{m}} \tag{1.5}$$

holds for all $u \in C_0^\infty(\Omega)$ *and any domain* Ω *where the constant* C *depends only on* m, j, p*, and* r*.*

Note that these results for $W_0^{m,p}(\Omega)$ are valid with no requirements on either $\partial\Omega$ or the interior of Ω. This is not the case for embeddings involving $W^{m,p}(\Omega)$. Here (1.3) and (1.4) of Theorem B in particular are *not* true (unless $q \in [1,p)$ and Ω has finite measure) without one of a bewildering array of additional assumptions holding on Ω; these include cone or twisted cone conditions [4], being star shaped or convex [16], having a "minimally smooth" boundary $\partial\Omega$ [10], being a Hölder or generalized ridge domain [27],[13], satisfying an extension property or Boman chain conditions [10], [28],[5–6], etc. Many of these properties are quite technical and the logical relations between them are unclear. A still useful survey of the situation in 1979 is Fraenkel [14]. More recent information can be found in [18] and [6].

In this paper we confine ourselves to W_0-type spaces. Our goal is to extend Theorems A and B along the following lines. Suppose $d(t) \equiv d(t, \partial\Omega)$ denotes the distance of $t \in \mathbb{R}^n$ to $\partial\Omega$. Then if $\partial\Omega$ is "fractal" in a sense defined below, there is a negative number β and positive numbers α, γ such that (1.2) and (1.5) become

$$\|d^{\beta/p_m^*} u\|_{\Omega;p_m^*} \leqslant C\|d^{\alpha/p}\nabla^m u\|_{\Omega;p},$$
$$\|d^{\beta/q}\nabla^j u\|_{\Omega;q} \leqslant C\|d^{\gamma/r}u\|^{\lambda}_{\Omega;r}\|d^{\alpha/p}\nabla^m u\|^{1-\lambda}_{\Omega;p} \tag{1.6}$$

where in (1.6) λ is a certain positive parameter. Also if $j = 0$ and $m < n/p$, then (1.6) may be replaced by

$$W_0^{m,p}(\Omega; d^\gamma, d^\alpha) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta) \tag{1.7}$$

for $q \in [1, p_m^*)$. Bounds on admissible α, β, and γ moreover can be expressed in terms of the Minkowski Dimension of $\partial\Omega$ relative to Ω.

These and other generalizations of Theorems A–C are presented in Sections 3–5. Section 2 consists of standard technical material concerning weighted Sobolev spaces, Minkowski dimension, covering theorems, and compact embeddings. However we do prove some new results here. The most important (Lemma 2.2) relates the Minkowski Dimension λ for finite measure domains to the finiteness of the integral $\int_\Omega d(s)^{-\mu}$ for $\mu \in (0, n-\lambda)$. This integrability property means that the proofs of the main results are applications of Hölder's inequality.

REMARK 1.1. Let us note at the outset that Theorems A, B, and C have trivial extensions of the form (1.6) or (1.7). Ω is at least quasicylindrical (1.6) and (1.7) are true for all $\alpha, \gamma \leqslant 0$ and $\beta \geqslant 0$. Since positive or negative powers of $d(t)$ are bounded above or below by positive constants on quasicylindrical Ω, we may combine this fact with the appropriate unweighted Theorem to derive a weighted inequality. So that for example if $u \in C_0^\infty(\Omega)$ in Theorem C we get

$$\|d^{\beta/q}\nabla^j u\|_{\Omega;q}K_1 \leqslant \|\nabla^j u\|_{\Omega;q} \leqslant K_2\|u\|^{\lambda}_{\Omega;r}\|\nabla^m u\|^{1-\lambda}_{\Omega;p} \leqslant K_3\|d^{\gamma/r}u\|^{\lambda}_{\Omega;r}\|d^{\alpha/p}\nabla^m u\|^{1-\lambda}_{\Omega;p}.$$

Similar extensions may be made in any of the inequalities demonstrated below.

The following notation and definitions will be used in the paper. Given a domain Ω and $\epsilon > 0$ let

$$\Omega_\epsilon := \{t \in \Omega : d(t) < \epsilon\},$$
$$\Omega^\epsilon := \{t \in \Omega : d(t) > \epsilon\}.$$

An unbounded Ω will be said to be *quasibounded* if $\lim_{t\in\Omega,|t|\to\infty} d(t) = 0$ and *quasicylindrical* if $\sup_{t\in\Omega} d(t) < \infty$. Further, given a set $E \subset \mathbb{R}^n$ let $|E|$ and $\overline{E}$ respectively denote its volume, i.e., n-dimensional Lebesgue measure, and closure.

Constants will be denoted by capital or small letters such as K, C, c, etc., whose value may change from line to line. If we wish to emphasize a change in the value of a constant we write $K_1, K_2 \ldots$. We sometimes indicate dependence of K on parameters (say α, β, p and q) by writing $K(\alpha, \beta, p, q)$, etc. If F, G are two functions, norms, or integrals we write $F \preccurlyeq G$ if $F \leqslant KG$ for some $K > 0$ and $F \approx G$ if $F \preccurlyeq G$ and $G \preccurlyeq F$. In the particular case $F \preccurlyeq \epsilon G$ for a small ϵ we write $F = \mathrm{O}(\epsilon)G$. $C^m(\Omega)$ is the space of m fold differentiable functions on Ω and C_0^m consists of those functions in $C^m(\Omega)$ of compact support. Finally, we denote balls of center t and radius r or cubes of center t and edge length L (edges assumed parallel to the axes) by $B_{t,r} \equiv B(t,r)$ or $Q_{t,L} \equiv Q(t,L)$. Context permitting, we often write B_t, Q_t, Q, and $d(t)$ in place of $d(t, \partial\Omega)$.

2. Preliminaries

Our results will be expressed in the language of *weighted Sobolev spaces*, a systematic study of which may be found in Kufner [19]. Suppose v_0, v_1, w are "weights", i.e., positive a.e. measurable functions defined on a domain $\Omega \subseteq \mathbb{R}^n$. Canonical examples of weights are powers or monotone functions of $d(t)$. For $1 \leqslant p, q, r < \infty$ we consider the spaces of complex-valued measurable functions $L^q(\Omega; w)$ and $W^{m,r,p}(\Omega; v_0, v_1)$ defined on Ω and equipped with the norms

$$\|u\|_{\Omega;w,q} := \left(\int_\Omega w|u|^q\right)^{\frac{1}{q}},$$
$$\|u\|_{\Omega;v_0,v_1,m,r,p} := \|u\|_{\Omega;v_0,r} + \|\nabla^m u\|_{\Omega;v_1,p}. \tag{2.1}$$

For simplicity in the unweighted case (when $v_0 = v_1 = w = 1$) we revert to notation like (1.1) and when $r = p$ write $W^{m,p}(\Omega; v_0, v_1)$ or $\|u\|_{\Omega;v_0,v_1,m,p}$. If $v_0^{-1/r}$, $v_1^{-1/p}$ are locally $L^{r'}, L^{p'}$ integrable where $1 < r', p' \leqslant \infty$ are the conjugate exponents of p or r (defined by $1/p + 1/p' = 1$, $1/r + 1/r' = 1$) one can show (cf. [19]) that $W^{m,r,p}(\Omega; v_0, v_1)$ is a Banach space. Likewise, if v_0, v_1 are locally integrable it is easily verified that C_0^∞ is dense in $W^{m,r,p}(\Omega; v_0, v_1)$. This implies that we can define $W_0^{m,r,p}(\Omega; v_0, v_1)$ in the same way as $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1).

We turn now to the well-known concept of the *Minkowski Dimension* of $\partial\Omega$ for a bounded domain Ω.

DEFINITION 2.1. Let $0 < \lambda \leqslant n$ and $r > 0$ and set

$$M_D^\lambda(\partial\Omega; r) := \frac{|(\partial\Omega + B(0,r))|}{r^{n-\lambda}},$$
$$M_D^\lambda(\partial\Omega) := \limsup_{r\to 0+} M_D^\lambda(\partial\Omega; r),$$
$$M_D(\partial\Omega) := \inf\{\lambda : M_D^\lambda(\partial\Omega) < \infty\}.$$

The last two of these quantities are called respectively the λ *-dimensional Minkowski content of* $\partial\Omega$ and *Minkowski dimension of* $\partial\Omega$. A consequence of the definition is that $M_D(\partial\Omega) \leqslant n$. However this dimension need not be strictly less than n. There exist Ω such that $M_D^\lambda(\partial\Omega) = \infty$ for all $\lambda \in (0,n)$ (see Edmunds and Hurri-Syrjären [11, Remark 4.3] and the associated references). Although the results of this paper will hold with trivial modifications in the hypotheses using $M_D(\partial\Omega)$, it is convenient to modify Definition 2.1 in the following way: $\widetilde{M}_D(\partial\Omega)$ is the *interior Minkowski dimension* or *Minkowski dimension of* $\partial\Omega$ *relative to* Ω if $|(\partial\Omega + B(0,r))|$ is replaced by $|\Omega_r| \equiv |(\partial\Omega + B(0,r)) \cap \Omega|$. Clearly $\widetilde{M}_D(\partial\Omega) \leqslant M_D(\partial\Omega)$. However $\widetilde{M}_D(\partial\Omega)$ and $M_D(\partial\Omega)$ are not equivalent concepts since they can differ for the same Ω. For examples see Tricot [29]. A consequence of the definition is that both $M_D(\partial\Omega)$ and $\widetilde{M}_D(\partial\Omega) \leqslant n$. Lapidus [21, Corollary 3.2] has shown that $M_D(\partial\Omega) \geqslant n-1$. For the equivalent statement about $\widetilde{M}_D(\partial\Omega)$ see Fleckinger-Pelle and Vassiliev [15]. We note additionally the useful fact that the definition of $\widetilde{M}_D(\partial\Omega)$ makes sense for finite measure Ω while $M_D(\partial\Omega)$ requires Ω to be bounded.

Either $\widetilde{M}_D(\partial\Omega)$ or $M_D(\partial\Omega)$ is a measure of the irregularity of $\partial\Omega$. Thus $M_D(\partial\Omega) = n-1$ if Ω satisfies a segment or cone condition, but the Koch curve in $\mathbb{R}^n$ has Minkowski dimension $\log 4/\log 3$ (see [21] or [29] for proofs and addtional examples). Because of the possibility of extreme irreguality Lapidus [20], [21] calls $\partial\Omega$ "fractal" if $M_D(\partial\Omega)$ or $\widetilde{M}_D(\partial\Omega) \in (n-1,n]$. However even though $M_D(\partial\Omega)$ or $\widetilde{M}_D(\partial\Omega) = n-1$ includes domains with smooth boundaries, as a shorthand we choose to use the word "fractal" for all possible dimensions of $\partial\Omega \leqslant n$.

Next let Ω be a (possibly unbounded) domain. Recall that a Whitney covering $\mathcal{W}$ of Ω is a family of closed cubes Q with edges parallel to the coordinate axes, each having edge length $L_Q = 2^{-k}$, $k \in \mathbb{N}$ and diameter $D_Q = L_Q\sqrt{n}$, such that the following five properties hold:

(i) $\Omega = \bigcup_{Q\in\mathcal{W}} Q$;

(ii) the interiors of distinct cubes are disjoint;

(iii) $1 \leqslant \operatorname{dist}(Q, \partial\Omega)/D_Q \leqslant 4$;

(iv) $\frac{1}{4} \leqslant \frac{\operatorname{diam}(Q_1)}{\operatorname{diam}(Q_2)} \leqslant 4$ if $Q_1 \cap Q_2 \neq \emptyset$;

(v) At most 12^n other cubes in $\mathcal{W}$ can touch a fixed $Q \in \mathcal{W}$; further for fixed $t \in (1, 5/4)$ each $x \in \Omega$ lies in at most 12^n of the dilated cubes tQ, $Q \in \mathcal{W}$.

It is known (Stein, [28, Chapter VI]) that such a covering exists for any Ω. Condition

(iii) in particular means that there are fixed constants c_1, c_2 such that

$$c_1 L_Q \leqslant d(s) \leqslant c_2 L_Q \tag{2.2}$$

for any $s \in Q$. Note that we can take $c_1 = \sqrt{n}$ and $c_2 = 5\sqrt{n}$ in (2.2).

Another useful covering result is due to Besicovitch. This says that if Ω is bounded and $\mathcal{C}$ is a collection of closed balls $\overline{B(t,r)}$ such that each $t \in \Omega$ is the center of some $\overline{B_{t,r}} \in \mathcal{B}$ we can extract a subcovering of Ω $\mathcal{S} \subset \mathcal{C}$ such that

(i) $\mathcal{S} = \bigcup_{i=1}^{\chi_n} \Gamma_i$ where each Γ_i is a finite or countable family of disjoint balls in $\mathcal{C}$;

(ii) each point in $\mathbb{R}^n$ belongs to at most λ_n balls;

(iii) the numbers χ_n and λ_n depend only on the dimension n.

The same is true for unbounded Ω provided the radii of the balls in $\mathcal{B}$ are uniformly bounded. Proofs of these assertions may be found in [10] or [17].

Returning now to Whitney coverings, let $n(k)$ denote the number of cubes in $\mathcal{W}_k$ where

$$\mathcal{W}_k := \{Q \in \mathcal{W} : L_Q = 2^{-k}\}$$

and k is a positive integer. The domain Ω is said to satisfy a Whitney cube #-condition if there is a continuous increasing function $h : (0,\infty) \to (0,\infty)$ such that $n(k) \leqslant h(k)$ for all $k \geqslant k_0 \geqslant 1$. It is obvious that Ω always satisfies a Whitney cube #-condition if we take $h(k) = K2^{nk}$ for a certain constant $K > 0$.

The following Lemma was proven for bounded Ω in [7, Lemma 6.1] using $\widetilde{M}_D(\partial\Omega)$ and in [23, Theorem 3.11] and [11, Lemma 2.2]) using $M_D(\partial\Omega)$. We now extend this result to the finite measure case.

LEMMA 2.1. *Let Ω have finite measure and $\lambda \in (0,n)$. Then $\widetilde{M}_D^{\lambda}(\partial\Omega) < \infty$ if, and only if, $n(k) \leqslant K2^{\lambda k}$ for all $k \geqslant k_0$ where K and k_0 are finite positive constants.*

Proof. If $\widetilde{M}_D^{\lambda}(\partial\Omega) < \infty$ then for some $K > 0$ and all $\delta < \delta_0$,

$$|\Omega_\delta| \leqslant K\delta^{n-\lambda}.$$

We choose k such that $\delta := 5\sqrt{n}2^{-k} < \delta_0$. By property (iii) of a Whitney covering $\mathcal{W}$ any cube of edge length 2^{-k} is contained in Ω_δ. The volume of such cubes cannot exceed $|\Omega_\delta|$. Therefore since the cubes have disjoint interiors

$$\begin{aligned} n_k 2^{-nk} &\leqslant |\Omega_\delta| \leqslant K(5\sqrt{n}2^{-k})^{n-\lambda} \\ &\leqslant K_1 2^{k(\lambda-n)} \end{aligned}$$

from which the assertion follows.

On the other hand, if $\widetilde{M}_D^{\lambda}(\partial\Omega) = \infty$ there exists a sequence $\{k_i\}$ such that $|\Omega_{2^{-k_i}}| \geqslant i2^{-k_i(n-\lambda)}$. Again by property (iii) the cubes in $\mathcal{W}$ that overlap $\Omega_{2^{-k_i}}$ have edge length $2^{-k} \leqslant 2^{-k_i}/\sqrt{n}$. This implies that $k > k_i$. Therefore

$$\sum_{k \geqslant k_i}^{\infty} n_k 2^{-nk} > |\Omega_{2^{-k_i}}| \geqslant i2^{-k_i(n-\lambda)}.$$

If $n_k \leqslant K2^{\lambda k}$ with $\lambda < n$, then the left side of this estimate is bounded by a geometric series with sum

$$\frac{2^{k_i(\lambda-n)}}{1-2^{\lambda-n}} = K(\lambda,n)2^{-k_i(n-\lambda)}.$$

Canceling, we see that $K(\lambda,n) \geqslant i$ for all i which is impossible. $\square$

A version for bounded Ω of the next result which connects the idea of Minkowski dimension to the possibility of integrating negative powers of $d(t)$ was shown in [7, Proposition 6.1].

LEMMA 2.2. *Suppose Ω has finite measure and that $\widetilde{M}_D(\partial\Omega) = \lambda$. Then the following conditions are equivalent:*

(i) $\lambda < n$.

(ii) $\int_\Omega d(s)^{-\mu}\,dx < \infty$ *for all* $\mu \in (0, n-\lambda)$.

Moreover $S(\mu,\Omega) := \{\mu : d^{-\mu} \in L(\Omega)\} = (0, n-\lambda)$.

Proof. We show that (i) $\Longrightarrow$ (ii): Let $\mathfrak{W}$ be a Whitney covering of Ω and $\mathfrak{W}_k \subset \mathfrak{W}$ the collection of cubes of edge length 2^{-k}. Then

$$\begin{aligned}
\int_\Omega d(s)^{-\mu} &= \sum_{Q\in\mathfrak{W}} \int_Q d(s)^{-\mu} \\
&= \sum_{k=0}^{\infty} \left(\sum_{Q\in\mathfrak{W}_k} \int_Q d(s)^{-\mu} \right) \\
&\approx \left\{ \sum_{k=0}^{\infty} n(k) 2^{-kn} (2^{-k})^{-\mu} \right\} \\
&\preccurlyeq K \left\{ \sum_{k=0}^{\infty} 2^{(\lambda-n+\mu)k} \right\}.
\end{aligned}$$

Since $\lambda < n$ it is evident that the final sum is finite if $\mu \in (0, n-\lambda)$.

Next we consider (ii) $\Longrightarrow$ (i): Assume (ii) and that $\widetilde{M}_D(\partial\Omega) = n$. Then according to Lemma 2.1, Ω does not satisfy the Whitney cube #-condition with $h(t) = K2^{\lambda t}$ for any $K > 0$ and $\lambda \in (0,n)$. Taking $\lambda' = n - \mu \in (\lambda, n)$, there exists a sequence of natural numbers $\{k_j\}$ such that $n(k_j) > 2^{\lambda' k_j}$. Then

$$\begin{aligned}
\int_\Omega d(s)^{-\mu} &= \sum_{k=0}^{\infty} \left(\sum_{Q\in\mathfrak{W}_k} \int_Q d(s)^{-\mu} \right) \succcurlyeq \sum_{k=0}^{\infty} n(k)(2^{-k})^{-\mu}|Q| \\
&\geqslant \sum_{j=1}^{\infty} n(k_j)(2^{-k_j})^{-\mu} 2^{-k_j n} \\
&> \sum_{j=1}^{\infty} 2^{k_j(\lambda'+\mu-n)} \\
&= \infty,
\end{aligned}$$

which is a contradiction.

Turning to last assertion, we first show that S is open. By Lemma 2.1 $n(k) \leqslant K2^{\lambda k}$. Let $\tilde{\lambda} \leqslant \lambda$ be the least number such that $n(k) \leqslant K2^{\tilde{\lambda} k}$ and let $\delta \in (0, \tilde{\lambda})$. Then by the definition of $\tilde{\lambda}$ there must exist a subsequence $\{k_j\}$ $n(k_j) > K2^{(\tilde{\lambda}-\delta)k_j}$. Consequently as in the proof that (ii) $\Longrightarrow$ (i) we have that

$$\sum_{j=1}^{\infty} 2^{k_j(\tilde{\lambda}-\delta+\mu-n)} \precsim \int_{\Omega} d(s)^{-\mu},$$

so that $\mu < n - (\tilde{\lambda} - \delta)$ is a necessary condition for $d^{-\mu} \in L(\Omega)$. Since δ is as small as we please, it suffices to choose any $\mu \in (0, n - \tilde{\lambda})$. On the other hand if this is so, we find as in the proof of (i) $\Longrightarrow$ (ii) above that

$$\begin{aligned} \int_{\Omega} d(s)^{-\mu} &\precsim \left\{ K \sum_{j=1}^{\infty} 2^{k(\tilde{\lambda}-n+\mu)} \right\} \\ &< \infty. \end{aligned}$$

It follows that $S(\mu, \Omega) = (0, n - \tilde{\lambda})$. It remains to verify that $\tilde{\lambda} = \lambda$. Let $\tilde{\lambda} < \hat{\lambda} < \lambda$. Put $\hat{\mu} = n - \hat{\lambda}$. Then $\hat{\mu} \in S(\mu, \Omega)$. Because $\hat{\lambda} < \widetilde{M}_D(\Omega)$, for every j there exists $k_j \in \mathbb{N}$ such that $|\Omega_{2^{-k_j}}| \geqslant j 2^{-k_j(n-\hat{\lambda})}$. Consider a Whitney covering of $\Omega_{2^{-k_j}}$. Then

$$\sum_{k=k_j}^{\infty} \left(\sum_{k \in \mathfrak{W}_k} |Q| \right) \geqslant j 2^{-k_j(n-\hat{\lambda})}.$$

Therefore,

$$\begin{aligned} \int_{\Omega_{2^{-k_j}}} d(s)^{-\hat{\mu}} &= \sum_{k=k_j}^{\infty} \left(\sum_{k \in \mathfrak{W}_k} \int_Q d(s)^{-\hat{\mu}} \right) \\ &\geqslant 2^{\hat{\mu} k_j} \sum_{k=k_j}^{\infty} \left(\sum_{k \in \mathfrak{W}_k} |Q| \right) \\ &\geqslant j 2^{k_j(\hat{\mu}+\hat{\lambda}-n)} \\ &= j. \end{aligned}$$

Consequently,

$$\lim_{j \to \infty} \int_{\Omega_{2^{-k_j}}} d(s)^{-\hat{\mu}} = \infty$$

which is impossible if $d^{-\hat{\mu}}$ is integrable on Ω. □

REMARK 2.1. We have recently become aware that Trotsenko [30] has given a result which is very close to to Lemma 2.2. For the class of bounded “1-John domains" which contain all bounded Ω such that the Hausdorff dimension of $\partial\Omega$ is less than n

he shows that the integrability condition (ii) holds for some $\mu \in (0,1)$. As we do he proves this by obtaining an upper bound to $n(k)$.

The last Lemma of this section is an unweighted Sobolev-type embedding to which we will frequently refer. The proof will be omitted but (i), (ii), or $(2.3')$ in the case $r = p$ follow by invoking the appropriate intermediate derivative estimate (cf. Burenkov [9, p. 160]) on the unit ball followed by a change of variables. For $r \neq p$ we work first with $W^{m,p}(B)$ if $r > p$ or $W^{m,r}(B)$ if $p > r$ and then use Hölder's inequality (with exponent r/p or p/r) on the first or second right-hand terms. For a detailed proof of (i)–(iii) see [8, Lemma 2.2].

LEMMA 2.3. *Let B be a ball of radius R, $1 \leqslant q,p,r < \infty$, $0 \leqslant j < m$, and $v := \min\{p,r\}$. Then the inequality*

$$\int_B |\nabla^j u|^q \leqslant K \left\{ R^{-q\left(j+\frac{n}{r}-\frac{n}{q}\right)} \left(\int_B |u|^r \right)^{\frac{q}{r}} + R^{q\left(m-j-\frac{n}{p}+\frac{n}{q}\right)} \left(\int_B |\nabla^m u|^p \right)^{\frac{q}{p}} \right\} \qquad (2.3)$$

holds for all $u \in C^m(B)$ and the constant K depends only on m and n if any one of the following conditions are true

(i) $m - j > n/v$,
(ii) $m - j \leqslant n/v$ *and* $m - j - n/v + n/q \geqslant 0$,
(iii) $q \leqslant p$.

Also, if $m > n/p$ the inequality (which implies (2.3))

$$\sup_{t \in B} |\nabla^j u(t)| \leqslant K \left\{ R^{-(j+\frac{n}{r})} \left(\int_B |u|^r \right)^{\frac{1}{r}} + R^{m-j-\frac{n}{p}} \left(\int_B |\nabla^m u|^p \right)^{\frac{1}{p}} \right\} \qquad (2.3')$$

is true. Finally, the mappings from $W^{m,r,p}(B)$ to $L^q(B)$ or to $L^\infty(B)$ defined by (2.3) or $(2.3')$ are compact in cases (i) and (ii) except when $m - n/v + n/q = 0$ in (ii).

In particular we note that if $r = p$ and $j = 0$ then (2.3) is true if $m - j \leqslant n/p$ and $q \leqslant p_m^* = np/(n - mp)$. It is also clear that if $r = p$ one of (i)–(ii) will hold if and only if $m - j - n/p + n/q \geqslant 0$. That (iii) is independent of (i) and (ii) is shown by the example $m = r = 1$, $j = 0$, $q = 3/2$, $p = 2$, and $n = 4$.

3. Weighted W_0-type Embeddings

In this section we show how the standard embeddings (1.3)–(1.4) of Theorem B may be improved by the addition of singular or degenerate weights involving powers of $d(t)$. Let $B_t := B(t,1)$. We are interested in the conditions

(C1) $\sup_{t \in \Omega} \widetilde{M}_D(\partial(B_t \cap \Omega)) := \bar{\lambda} < n$ (for arbitrary Ω).

(C2) $\widetilde{M}_D(\partial\Omega) := \lambda < n$ (for finite measure Ω).

LEMMA 3.1. *Suppose Ω is bounded. Then (C1) $\Leftrightarrow$ (C2) and $\bar{\lambda} = \lambda$.*

Proof. First we note that $d(t, \partial(B_t \cap \Omega)) \leqslant d(t, \partial\Omega)$ for all $t \in \Omega$. Assume that Ω satisfies (C1). Let $\mathcal{C} = \{B(t,1) : t \in \Omega\}$ be a covering of Ω. By the Besicovitch covering lemma there exists a subcovering $\mathcal{S}$ composed of families Γ_i, $i = 1, \ldots, \chi_n$, of disjoint balls in $\mathcal{C}$. Since Ω is bounded each of these families consists of a finite number n_i of balls $B_{t_{ij}}$. Let λ^* be the maximum of $\lambda_{ij} := \widetilde{M}_D(\partial(B_{t_{ij}} \cap \Omega))$, $i = 1, \ldots \chi_n$, $j = 1, \ldots, n_i$. Therefore for all $B_{t_{ij}} \cap \Omega$ we have the inequalities

$$\begin{aligned} &\lambda_{ij} \leqslant \lambda^* \leqslant \bar{\lambda} < n, \\ &\mu_0^{ij} := n - \lambda_{ij} \geqslant \mu_0^* := n - \lambda^* \geqslant \bar{\mu}_0 := n - \bar{\lambda}. \end{aligned}$$

It follows that if $\mu \in (0, \bar{\mu}_0) \subseteq (0, \mu_0^*) \subseteq (0, \mu_0^{ij})$ then

$$\int_{B_{t_{ij}} \cap \Omega} d(t, \partial\Omega)^{-\mu} \leqslant \int_{B_{t_{ij}} \cap \Omega} d(t, \partial B_t \cap \Omega)^{-\mu} < \infty$$

by (C1) and Lemma 2.2. A consequence of this and the finite mutual intersection property of elements of $\mathcal{S}$ is that

$$\int_{\Omega} d(t, \partial\Omega)^{-\mu} \leqslant \sum_{B_{t_{ij}} \in \mathcal{S}} \int_{B_{t_{ij}} \cap \Omega} d(t, \partial\Omega)^{-\mu} < \infty.$$

We conclude by Lemma 2.2 that $\lambda < n$. Also by Lemma 2.2 $\mu \in (0, n - \lambda)$ so that $(0, \bar{\mu}_0) \subseteq (0, n - \lambda)$ which implies that $\bar{\lambda} \geqslant \lambda$.

To complete the proof, suppose that Ω satisfies (C2). By Lemma 2.2 $d^{-u} \in L(\Omega)$ which means that $d^{-\mu} \in L(B_t \cap \Omega)$ for all $t \in \Omega$ if and only $\mu \in (0, n - \lambda)$. Again by Lemma 2.2 $\lambda_t := \widetilde{M}_D(\partial(B_t \cap \Omega)) < n$ and $\mu \in (0, n - \lambda_t)$. Hence $\lambda_t \leqslant \lambda$. This implies that $\bar{\lambda} \leqslant \lambda$. Therefore $\bar{\lambda} = \lambda$. $\square$

LEMMA 3.2. *Suppose that Ω satisfies (C1). Then if $\mu \in [0, n - \bar{\lambda})$*

$$\text{(C3)} \qquad \lim_{|t| \to \infty, t \in \Omega} \int_{B_t \cap \Omega} d^{-\mu} = 0$$

if Ω is quasibounded, and

$$\text{(C4)} \qquad \sup_{t \in \Omega} \int_{B_t \cap \Omega} d^{-\mu} < \infty$$

if Ω is unbounded.

Proof. The argument resembles that of Lemma 2.2. Let

$$n - 1 < \lambda_\infty := \limsup_{t \in \Omega, |t| \to \infty} \widetilde{M}_D(\partial(B_t \cap \Omega)) \leqslant \bar{\lambda} < n.$$

Choose $\mu \in [0, n - \bar{\lambda})$, then $\mu \in [0, n - \lambda_\infty)$ so that $\mu = \phi(n - \lambda_\infty)$ for some $\phi \in [0, 1)$. Since Ω is quasibounded, given $\epsilon > 0$, we can choose $|t|$ large enough that $B_t \cap \Omega$ has a Whitney covering $\mathfrak{W}_{t,k_0}$ by cubes of edge length less than 2^{-k_0} where k_0 satisfies

$$\left[1 - 2^{(\lambda_\infty - n)(1-\phi)}\right]^{-1} 2^{(\lambda_\infty - n)(1-\phi)k_0} < \epsilon \tag{3.1}$$

and $\widetilde{M}_D(\partial(B_t \cap \Omega)) > \lambda_\infty - \epsilon$. Then if $\mathfrak{W}_{t,k} \subset \mathfrak{W}_{t,k_0}$ denotes the cubes of $\mathfrak{W}_{t,k_0}$ of edge length 2^{-k}, $k > k_0$ we have that

$$\begin{aligned}
\int_{B_t \cap \Omega} d(s)^{-\mu} &= \sum_{Q \in \mathfrak{W}_{t,k_0}} \int_Q d(s)^{-\mu} \\
&= \sum_{k=k_0}^{\infty} \left(\sum_{Q \in \mathfrak{W}_{t,k}} \int_Q d(s)^{-\mu} \right) \\
&\approx \left\{ \sum_{k=k_0}^{\infty} n(k) 2^{-kn} (2^{-k})^{-\mu} \right\} \\
&\preccurlyeq K \left\{ \sum_{k=k_0}^{\infty} 2^{(\lambda_\infty - n + \mu)k} \right\} \\
&= K \left\{ \sum_{k=k_0}^{\infty} 2^{(\lambda_\infty - n)(1-\phi)k} \right\} \\
&= \mathrm{O}(\epsilon)
\end{aligned}$$

where the last step follows from the fact that $\lambda_\infty < n$ so that the geometric series with ratio $2^{(\lambda_\infty - n)(1-\phi)}$ converges to the value given in (3.1).

If Ω is not quasibounded, we can cover each $B_t \cap \Omega$ by cubes with edge length no greater than 2^{-k_t} where k_t is an appropriate integer such that $2^{-k_t} < 2/\sqrt{n}$ and set

$$M := \sup_{t \in \Omega} \left[1 - 2^{(\lambda_\infty - n)(1-\phi)}\right]^{-1} 2^{(\lambda_\infty - n)(1-\phi)k_t} < \infty,$$

then repeating the previous part of the proof will show that

$$\sup_{t \in \Omega} \int_{B_t \cap \Omega} d(s)^{-\mu} = \mathrm{O}(M),$$

proving (C4). □

Having disposed of these technicalities, our first main result shows that the assumption that $\widetilde{M}_D(\partial\Omega) < n$ allows a singular weight to be present in the target space of the embedding (1.3)–(1.4) of $W_0^{m,p}(\Omega)$ of Theorem B. For simplicity we consider the $j = 0$ case first.

THEOREM 3.1. *Suppose that $\partial\Omega$ is "fractal" in the sense that (C1) or (C2) is satisfied with λ or $\bar{\lambda} < n$. Set $\mu_0 = n - \lambda$ or $n - \bar{\lambda}$ and let $\beta \in (-\mu_0, 0]$. Assume $1 \leqslant p, q < \infty$ (with $q \geqslant p$ if Ω is quasibounded) and*

$$m - \frac{n}{p} + \frac{n}{q}\left(\frac{\beta}{\mu_0} + 1\right) > 0. \tag{3.2}$$

Then

$$W_0^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta). \tag{3.3}$$

If Ω is a non-quasibounded domain satisfying (C1) then the embedding is only continuous.

Proof. The case $\beta = 0$ is an instance of Theorem B. So we can assume $\beta < 0$. Since (3.2) is true we can find $\theta \geqslant (\beta/\mu_0+1)^{-1} > 1$ such that $m-n/p+n/(q\theta) > 0$. Let $u \in C_0^\infty(\Omega)$.

Case (1): Ω is bounded. We choose

$$\theta > \left(\frac{\beta}{\mu_0}+1\right)^{-1} \Leftrightarrow -\mu_0 < \frac{\beta\theta}{\theta-1} < 0$$

so that $m-n/p+n/(q\theta) > 0$. Then since Ω satisfies (C2) by Lemma 2.2 $\int_\Omega d^{\frac{\beta\theta}{\theta-1}} < \infty$, and by Theorem B

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^{q\theta}(\Omega). \tag{3.4}$$

Moreover, by Hölder's inequality

$$\begin{aligned}\int_\Omega d^\beta |u|^q &\leqslant \left(\int_\Omega d^{\frac{\beta\theta}{\theta-1}}\right)^{1-\frac{1}{\theta}} \left(\int_\Omega |u|^{q\theta}\right)^{\frac{1}{\theta}} \\ &\leqslant K\left(\int_\Omega |u|^{q\theta}\right)^{\frac{1}{\theta}}.\end{aligned} \tag{3.5}$$

(3.3) follows from (3.4) and (3.5).

Case (2): Ω is quasibounded. Set $-\mu := \beta\theta/(\theta-1) > -\mu_0$. By Lemma 3.1

$$\left(\int_{B_t\cap\Omega} d(s)^{-\mu}\right)^{1-\frac{1}{\theta}} \leqslant \epsilon \tag{3.6}$$

for all $t \in \Omega \backslash B(0,k)$ and sufficiently large k. Let $\mathcal{T}_n$ denote the the embedding of $W_0^{m,p}(\Omega)$ into $L^q(\Omega\cap B(0,k); d^\beta)$. By some elementary estimates and Case (1) we have that

$$W_0^{m,p}(\Omega) \hookrightarrow W_0^{m,p}(\Omega\cap B(0,k)) \hookrightarrow\hookrightarrow L^q(\Omega\cap B(0,k); d_{(k)}^\beta) \hookrightarrow L^q(\Omega\cap B(0,k); d^\beta). \tag{3.7}$$

Here

$$d_{(k)}(t) := d(t, \partial(\Omega\cap B(0,k))) \leqslant d(t), \quad t \in \Omega\cap B(0,k).$$

Just as in (3.4)–(3.5) we get from (3.6) and Lemma 2.3 (instead of Theorem B) that

$$\int_{B_t\cap\Omega} d^\beta |u|^q \preccurlyeq \epsilon^q \|u\|^q_{B_t\cap\Omega;m,p}, \quad t \in \Omega\cap\backslash B(0,k), \tag{3.8}$$

A Besicovitch argument on $\Omega\backslash B(0,k)$ using the elementary inequality $\sum A_i^r \leqslant (\sum A_i)^r$ for $r = q/p \geqslant 1$ to handle addition over disjoint families of unit balls on the right-hand side of (3.8)[1] yields

$$\int_{\Omega\cap\backslash B(0,k)} d^\beta |u|^q \preccurlyeq \epsilon^q \|u\|^q_{\Omega;m,p}. \tag{3.9}$$

If $\mathcal{T}$ denotes the embedding (3.3) and $\|u\|_{\Omega;m,p} = 1$

$$\begin{aligned}\|\mathcal{T} - \mathcal{T}_n\| &\leqslant \|d^{\frac{\beta}{q}} u\|_{\Omega\backslash B(0,k)} \\ &= \mathrm{O}(\epsilon).\end{aligned} \tag{3.10}$$

If Ω is unbounded but not quasibounded and (C1) applies, we can prove using (C4) of Lemma 3.2 and Hölder's inequality that

$$\|d^{\frac{\beta}{q}} u\|_{B_t\cap\Omega;q} \leqslant M\|u\|_{B_t\cap\Omega;m,p}$$

for all $t \in \Omega$ where M does not depend on t. The Besicovitch covering lemma then gives the required continuous embedding. $\square$

REMARK 3.1. To avoid repetitive arguments we will end future proofs of compact embeddings by just obtaining inequalities like (3.7) or (3.8). Lemma 2.3 to secure the needed inequalities on balls will be invoked without comment. Likewise, appeal to the principle that the uniform limit of compact operators is compact willl be automatic and it will be understood that we prove the inequalities we need first on $C_0^\infty(\Omega)$ and then extend them by closure arguments.

COROLLARY 3.1. *Suppose Ω and $\partial\Omega$ are as in Theorem 3.1. Let $\beta_i \in (-\mu_0, 0]$, $i = 1,2$, and $0 < j < m$. If Ω is quasibounded let $q \geqslant p$. Additionallly suppose that*

$$m - j - \frac{n}{p} + \frac{n}{q}\min\left\{\frac{\beta_1}{\mu_0} + 1, \frac{\beta_2}{\mu_0} + 1\right\} > 0.$$

Then

$$W_0^{m,p}(\Omega) \hookrightarrow\hookrightarrow W_0^{j,q}(\Omega; d^{\beta_1}, d^{\beta_2}). \tag{3.11}$$

If Ω is a non-quasibounded domain satisfying (C1) then the embedding (3.8) is only continuous.

Proof. We can assume that at least one of the $\beta_i < 0$, $i = 1,2$. If Ω is bounded

$$W_0^{m,p}(\Omega) \hookrightarrow\hookrightarrow W_0^{j,q\theta}(\Omega) \tag{3.12}$$

by Theorem B where

$$\begin{aligned}&\theta > \max\left\{\left(\frac{\beta_1}{\mu_0} + 1\right)^{-1}, \left(\frac{\beta_2}{\mu_0} + 1\right)^{-1}\right\} \\ &\implies 0 \geqslant \frac{\beta_i\theta}{\theta - 1} > -\mu_0, \quad i = 1,2,\end{aligned}$$

[1]The details of this chain of reasoning are given in the proof of Theorem 5.2 below.

is selected so that $m-j-n/p+n/(q\theta)>0$. Just as in the proof of Theorem 3.1 we obtain that

$$\begin{aligned}\int_\Omega d^{\beta_1}|u|^q &\leqslant \left(\int_\Omega d^{\frac{\beta_1\theta}{\theta-1}}\right)^{1-\frac{1}{\theta}}\left(\int_\Omega |u|^{q\theta}\right)^{\frac{1}{\theta}}\\ &\leqslant K_1\left(\int_\Omega |u|^{q\theta}\right)^{\frac{1}{\theta}},\\ \int_\Omega d^{\beta_2}|\nabla^j u|^q &\leqslant \left(\int_\Omega d^{\frac{\beta_2\theta}{\theta-1}}\right)^{1-\frac{1}{\theta}}\left(\int_\Omega |\nabla^j u|^{q\theta}\right)^{\frac{1}{\theta}}\\ &\leqslant K_2\left(\int_\Omega |\nabla^j u|^{q\theta}\right)^{\frac{1}{\theta}}.\end{aligned}$$

These inequalities and (3.12) imply (3.11). We omit the quasibounded case which follows Case 2 of Theorem 3.1. □

REMARK 3.2. Examination of the proof of Theorem 3.1 shows that the case $\beta=0$ in the quasibounded case actually uses only the condition

$$\text{(C5)} \qquad \lim_{|t|\to\infty,t\in\Omega}|\Omega\cap B(t,1)|=0.$$

When Ω satisfies (C1) this is implied by (C3) of Lemma 3.2. However we could take (C5) as an independent hypothesis. Berger and Schecter [3] used (C5) to show that $W_0^{m,p}(\Omega)$ embeds compactly into $L^q(\Omega)$ when $q\geqslant p$ and $m>\max\{n/p-n/q,n/p-1\}$. Reworking Theorem 3.1 using (C5) instead of (C1) when $\beta=0$ will give a more general result in that we only need that $m>n/p-n/q$. Note that (C5) implies quasiboundedness. The example of the “spiny urchin" [1, p. 151] however shows that the converse implication is not true.

COROLLARY 3.2. *Suppose Ω and $\partial\Omega$ are as in Theorem 3.1 while $\gamma\in[0,(r/r')\mu_0)$, $\alpha\in[0,(p/p')\mu_0)$, $\beta\in(-\mu_0,0]$, and*

$$m-n+\frac{n}{q}\left(\frac{\beta}{\mu_0}+1\right)>0, \tag{3.13}$$

then

$$W_0^{m,r,p}(\Omega;d^\gamma,d^\alpha)\hookrightarrow\hookrightarrow L^q(\Omega;d^\beta). \tag{3.14}$$

Proof. We proceed as in Cases (1) or (2) of Theorem 3.1 with $p=r=1$. If Ω is bounded or of finite measure (3.14) follows by the inequalities

$$\begin{aligned}\|u\|_{\Omega;1} &\leqslant \|d^{-\frac{\gamma}{r}}\|_{\Omega;r'}\|d^{\frac{\gamma}{r}}u\|_{\Omega;r},\\ \|\nabla^m u\|_{\Omega;1} &\leqslant \|d^{-\frac{\alpha}{p}}\|_{\Omega;p'}\|d^{\frac{\alpha}{p}}u\|_{\Omega;p}.\end{aligned} \tag{3.15}$$

If Ω is quasibounded, then as in (3.8)

$$\begin{aligned}\int_{B_t\cap\Omega} d^\beta|u|^q &\leqslant K\epsilon\|u\|^q_{B_t\cap\Omega;m,1}\\ &\leqslant K\epsilon\max\{\|d^{-\frac{\gamma}{r}}\|_{B_t\cap\Omega;r'},\|d^{-\frac{\alpha}{p}}\|_{B_t\cap\Omega;p'}\}^q\|u\|^q_{B_t\cap\Omega;d^\gamma,d^\alpha,m,r,p}\\ &= \mathrm{O}(\epsilon^2)\|u\|^q_{B_t\cap\Omega;d^\gamma,d^\alpha,m,r,p}\end{aligned} \tag{3.16}$$

by (C3) of Lemma 3.2 for large enough $|t|$. The rest of the argument parallels (3.7)–(3.10) of Theorem 3.1. $\square$

EXAMPLE 3.1. Suppose Ω and $\partial\Omega$ are as in Theorem 3.1, (3.13) holds and $\beta \in (-\mu_0, (q/q')\mu_0)$ and / or $\alpha \in (0, (p/p')\mu_0)$. Then

$$W_0^{m,q,p}(\Omega; d^\beta, d^\alpha) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta). \tag{3.17}$$

To see this if $\beta > 0$, let "β" play the role of "γ". Then Corollary 3.2 yields the embedding

$$W_0^{m,q,p}(\Omega; d^\beta, d^\alpha)) \hookrightarrow\hookrightarrow L^q(\Omega).$$

Since $\beta > 0$ by Remark 1.1 d^β may be introduced on the right to give (3.17). If $\beta \in (-\mu_0, 0]$ we apply Corollary 3.2 with $\gamma = 0$ and then introduce d^β on the left.

COROLLARY 3.3. *Suppose Ω and $\partial\Omega$ are as in Theorem 3.1 and that*

$$m - \frac{n}{v} + \frac{n}{q}\left(\frac{\beta}{\mu_0} + 1\right) > 0 \tag{3.18}$$

where $v = \min\{p, q\}$. *Then if* $\beta \in (-\mu_0, 0]$

$$W_0^{m,q,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta). \tag{3.19}$$

Proof. The proof is similar to that of Corollary 3.2. Suppose first that Ω has finite measure. In this case if $q > p$ (3.18) is (3.2) and Theorem 3.1 gives the embedding (3.3). But since

$$\|u\|_{\Omega;p} \leqslant |\Omega|^{\frac{q-p}{qp}} \|u\|_{\Omega;q}$$

the embedding (3.19) follows. If $q \leqslant p$ (3.18) and Theorem 3.1 gives

$$W_0^{m,q,}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta).$$

We get (3.19) as before via the inequality

$$\|\nabla^m u\|_{\Omega;q} \leqslant |\Omega|^{\frac{p-q}{qp}} \|\nabla^m u\|_{\Omega;p}.$$

We omit the quasibounded case as we can modify the proof of Case 2 Theorem 3.1 as in Corollary 3.2. $\square$

EXAMPLE 3.2. Suppose Ω and $\partial\Omega$ are as in Theorem 3.1, (3.18) holds and $\beta \in (-\mu_0, 0)$. Then

$$W_0^{m,q,p}(\Omega; d^\beta, 1) \hookrightarrow\hookrightarrow L^q(\Omega; d^\beta). \tag{3.20}$$

This is immediate from Corollary 3.3 and Remark 1.1.

4. Weak Hardy-type inequalities

Here we show that inequality (1.2) of Theorem A may be replaced by the weighted inequality

$$\int_\Omega d^\beta |u|^q \leqslant K\left(\int_\Omega d^\alpha |\nabla^m u|^p\right)^{\frac{q}{p}} \tag{4.1}$$

for $u \in W_0 \equiv W_0^{m,q,p}(\Omega; d^\beta, d^\alpha)$ and certain $\alpha > 0$, and $\beta < 0$ when (C1) or (C2) is satisfied. To do this we require a special case of [7, Theorem 5.1].

PROPOSITION 4.1. *Let $\mathcal{P}_{m-1}$ denote the polynomials on $\mathbb{R}^n$ of degree not exceeding $m-1$. Suppose $Z \supset W_0$ is a Banach space with norm $\|(\cdot)\|_Z$ and let $W(Z)$ be the space with norm*

$$\|u\|_{\Omega;d^\alpha,m,Z,p} := \|u\|_Z + \|\nabla^m u\|_{\Omega;d^\alpha,p}$$

where $u \in W_0$. Then inequality (4.1) holds for all $u \in W_0$ if

$$W_0 \hookrightarrow\hookrightarrow Z, \tag{4.2}$$

$$W(Z) \hookrightarrow L^q(\Omega; d^\gamma), \tag{4.3}$$

and $\mathcal{P}_{m-1} \cap W_0 = \{0\}$.

LEMMA 4.1. *If $\beta < 0$ let Ω be quasicylindrical, $1 \leqslant p, q < \infty$, and $m - n/p + n/q > 0$. If $\beta > 0$ suppose that Ω has finite measure and satisfies (C2). Then for any $\beta \in (-\infty, (q/q')\mu_0)$ the space $W_0^{m,q,p}(\Omega; d^\beta, 1)$ contains no nontrivial members of $\mathcal{P}_{m-1}$.*

Proof. Given $\epsilon > 0$, we consider the family of balls $\{B(t,\epsilon) : t \in \Omega\}$. Since $m - n/p + n/q > 0$ we can choose $q^* > \max\{q,p\}$ in either of the cases $q \geqslant p$ or $p > q$ such that $m - n/p + n/q^* > 0$. Then from Lemma 2.3

$$\int_{B_{t,\epsilon}} |u|^{q^*} \leqslant K\left\{\epsilon^{-q^*(\frac{n}{q}-\frac{n}{q^*})}\left(\int_{B_{t,\epsilon}} |u|^q\right)^{\frac{q*}{q}} + \epsilon^{q^*(m-\frac{n}{p}+\frac{n}{q^*})}\left(\int_{B_{t,\epsilon}} |\nabla^m u|^p\right)^{\frac{q^*}{p}}\right\}. \tag{4.4}$$

for $u \in C_0^\infty(\Omega)$. A Besicovitch argument as in Case (2) of Theorem 3.1 or Theorem 5.2 below gives the sum inequality

$$\int_\Omega |u|^{q^*} \leqslant K\left\{\epsilon^{-q^*\left(\frac{n}{q}-\frac{n}{q^*}\right)}\left(\int_\Omega |u|^q\right)^{\frac{q*}{q}} + \epsilon^{q^*\left(m-\frac{n}{p}+\frac{n}{q^*}\right)}\left(\int_\Omega |\nabla^m u|^p\right)^{\frac{q^*}{p}}\right\}. \tag{4.5}$$

If $\beta < 0$ we can introduce "d^β" freely into the first right-hand integral of (4.5) (cf. Remark 1.1). If $\beta \in [0,(q/q')\mu_0)$ we do this by first proving (4.4) with $q = 1$ and then using Hölder's inequality as in Corollary 3.2 and then noting that since $-\beta(q'/q) \in (-\mu_0, 0]$ Lemma 2.2 implies that the term $\int_{B_{t,\epsilon}\cap\Omega} d^{-\beta(q/q')}$ is uniformly bounded independently of t and ϵ.

Having incorporated d^β into (4.5), we note that the ϵ exponents in the first and second right-hand terms of the resulting inequality differ in sign. This means that the right-hand side considered as a function of ϵ has a unique minimum. Solving for the minimizing value of ϵ and substituting it back into the inequality gives

$$\int_\Omega |u|^{q^*} \leqslant K_1 \left(\int_\Omega d^\beta |u|^q\right)^{\left(\frac{q*}{q}\right)\lambda} \left(\int_\Omega |\nabla^m u|^p\right)^{\frac{q^*}{p}(1-\lambda)} \tag{4.6}$$

where

$$\lambda := \frac{m-n/p+n/q^*}{m-n/p+n/q},$$

$$K_1 = K\left[\left(\frac{\lambda}{1-\lambda}\right)^{1-\lambda} + \left(\frac{1-\lambda}{\lambda}\right)^{\lambda}\right].$$

A closure argument shows that this inequality continues to hold on $W_0^{m,q,p}(\Omega; d^\beta, 1)$. Therefore since it cannot be valid on $\mathcal{P}_{m-1}$, no member of $\mathcal{P}_{m-1}$ can belong to $W_0^{m,q,p}(\Omega; d^\beta, 1)$. □

THEOREM 4.1. *Let Ω and $\partial\Omega$ be as in Theorem 3.1,* $v = \min\{p,q\}$, $\beta \in (-\mu_0, 0)$, *and*

$$m - \frac{n}{v} + \frac{n}{q}\left(\frac{\beta}{\mu_0}+1\right) > 0 \quad \text{if } \alpha = 0,$$

$$m - n + \frac{n}{q}\left(\frac{\beta}{\mu_0}+1\right) > 0 \quad \text{if } \alpha \in (0,(p/p')\mu_0).$$

Then the Hardy-type inequality (4.1) is true on $W_0^{m,q,p}(\Omega; d^\beta, d^\alpha)$.

Proof. In Proposition 4.1 we take $Z = L^q(\Omega; d^\beta)$. Then (4.2) and (4.3) are the same embedding. Its continuity is trivial and compactness is guaranteed by (3.17) of (3.20) of Examples 3.1 or 3.2. Moreover by Lemma 4.1 W_0 contains no elements of $\mathcal{P}_{m-1}$. □

EXAMPLE 4.1. Let Ω be bounded or quasibounded, satisfy (C1) or (C2), and $\beta \in (-\mu_0 \min\{1, p/n\}, 0]$. Then

$$\int_\Omega d^\beta |u|^p \leqslant K \int_\Omega |\nabla u|^p \tag{4.7}$$

holds for all $u \in W_0^{1,p}(\Omega; d^\beta, 1)$.

REMARK 4.1. Under stronger assumptions on Ω (see e.g. [12], [22], [26], and [31]–[32]), we can prove (4.7) with $\beta = -p$. For $p > n$ nothing need be assumed about Ω for Lewis [22] has shown that (4.7) with $\beta = p$ is true on $W_0^{1,p}(\Omega; d^{-p}, 1)$ for any $\Omega \neq \mathbb{R}^n$.

5. Multiplicative Inequalities

By means of techniques like those of the previous sections we can show that $W_0^{m,r,p}(\Omega; d^\gamma, d^\alpha)$ satisfies some multiplicative inequalities similar to Theorem C.

THEOREM 5.1. *Suppose* $1 \leqslant p,q,r < \infty$, $0 \leqslant j < m$, Ω *has finite measure and satisfies (C2). Set* $\mu_0 := n - \lambda$ *and let* $\beta \in (-\mu_0, 0])$, $\alpha, \gamma \in [0,\infty)$, *and*

$$\left(\frac{q}{r}\right)\left[\frac{\gamma}{\mu_0}+1\right]\left(1-\frac{j}{m}\right)+\left(\frac{q}{p}\right)\left[\frac{\alpha}{\mu_0}+1\right]\left(\frac{j}{m}\right) < \frac{\beta}{\mu_0}+1. \tag{5.1}$$

Then the inequality

$$\int_\Omega d^\beta |\nabla^j u|^q \leqslant K\left(\int_\Omega d^\gamma |u|^r\right)^{\frac{q}{r}\left(\frac{m-j}{m}\right)}\left(\int_\Omega d^\alpha |\nabla^m u|^p\right)^{\frac{q}{p}\left(\frac{j}{m}\right)} \tag{5.2}$$

holds on $W_0^{m,r,p}(\Omega; d^\gamma, d^\alpha)$. *Here*

$$K \approx \left[\left(\int_\Omega d(s)^{-\frac{\beta\theta}{\theta-1}}\right)^{1-\frac{1}{\theta}}\left(\int_\Omega d(s)^{-\frac{\gamma}{\phi_1-1}}\right)^{\left(\frac{q}{r}\right)(\phi_1-1)\left(\frac{m-j}{m}\right)}\right.$$
$$\left.\times\left(\int_\Omega d(s)^{-\frac{\alpha}{\phi_2-1}}\right)^{\left(\frac{q}{p}\right)(\phi_2-1)\left(\frac{j}{m}\right)}\right] \tag{5.3}$$

where θ, ϕ_1, ϕ_2 *are parameters defined in (5.4) below.*

Proof. Choose

$$\phi_1 > \frac{\gamma}{\mu_0}+1, \quad \phi_2 > \frac{\alpha}{\mu_0}+1, \quad \theta > \left(\frac{\beta}{\mu_0}+1\right)^{-1} \tag{5.4}$$

so that

$$\theta\phi_1\left(\frac{q}{r}\right)\left(1-\frac{j}{m}\right)+\theta\phi_2\left(\frac{q}{p}\right)\left(\frac{j}{m}\right)=1.$$

In view of Theorem C we have the inequality

$$\int_\Omega |\nabla^j u|^{q\theta} \leqslant K\left(\int_\Omega |u|^{r/\phi_1}\right)^{\left(\frac{q}{r}\right)\theta\phi_1\left(\frac{m-j}{m}\right)}\left(\int_\Omega |\nabla^m u|^{p/\phi_2}\right)^{\left(\frac{q}{p}\right)\theta\phi_2\left(\frac{j}{m}\right)} \tag{5.5}$$

for $t \in \Omega$, $u \in C_0^\infty(\Omega)$. Three applications of Hölder's inequality give

$$\int_\Omega |\nabla^j u|^{q\theta} \geqslant \left(\int_\Omega d^\beta |\nabla^j u|^q\right)^\theta \left(\int_\Omega d(s)^{\frac{\beta\theta}{\theta-1}}\right)^{1-\theta},$$
$$\int_\Omega |u|^{\frac{r}{\phi_1}} \leqslant \left(\int_\Omega d^\gamma |u|^r\right)^{\frac{1}{\phi_1}}\left(\int_\Omega d(s)^{-\frac{\gamma}{\phi_1-1}}\right)^{1-\frac{1}{\phi_1}},$$
$$\int_\Omega |\nabla^m u|^{\frac{\alpha}{\phi_2}} \leqslant \left(\int_\Omega d^\alpha |\nabla^m u|^p\right)^{\frac{1}{\phi_2}}\left(\int_\Omega d(s)^{-\frac{\alpha}{\phi_2-1}}\right)^{1-\frac{1}{\phi_2}}. \tag{5.6}$$

By (5.4) the exponents on $d(s)$ are in $(-\mu_0, 0]$. Therefore by Lemma 2.2 the integrals involving $d(s)$ are finite. Substitution of the inequalities (5.6) into (5.5) gives (5.2) and (5.3). □

COROLLARY 5.1. *If* $1 \leqslant p,q,r < \infty$, $0 \leqslant j < m$, Ω *has finite measure and satisfies (C2),* $\beta \in (-\mu_0, 0)$, $\alpha, \gamma > 0$, *and*

$$\frac{r}{q} > \frac{\gamma + \mu_0}{\beta + \mu_0}, \quad \frac{p}{q} > \frac{\alpha + \mu_0}{\beta + \mu_0}.$$

Then (5.2) and (5.3) are true.

Proof. A calculation shows that the stated conditions imply (5.1). □

REMARK 5.1. If $1 \leqslant p,q,r < \infty$, $0 \leqslant j < m$, Ω has finite measure, $\alpha = \beta = \gamma = 0$, and

$$\frac{q}{r}\left(1 - \frac{j}{m}\right) + \frac{q}{p}\frac{j}{m} < 1 \Leftrightarrow \frac{m}{q} > \frac{m-j}{r} + \frac{j}{p}$$

we get the unweighted inequality (1.5) of Theorem C holding on $W_0^{m,r,p}(\Omega)$ with

$$K \approx |\Omega|^{1-\left(\frac{q}{r}\right)\left(\frac{m-j}{m}\right)-\left(\frac{q}{p}\right)\left(\frac{j}{m}\right)}.$$

REMARK 5.2. Suppose $q \geqslant \max\{p,q\}$, Then (5.1) cannot be satisfied for $\alpha, \gamma \geqslant 0$. But q can be small. For if $q = 1$, $p = r > 1$, and $\alpha = \gamma = 0$ (5.1) is satisfied if $\beta \in (-(1-1/p)\mu_0, 0]$.

The next result allows $q = p = r$.

THEOREM 5.2. *Suppose* $1 \leqslant p,r,q < \infty$, $q \geqslant \max\{p,r\}$, $0 \leqslant j < m$, $\beta \in (-\mu_0, 0]$, $\alpha, \gamma \in [0, \infty)$, Ω *is of finite measure and satisfies (C2), and*

$$m - j - \max\left\{\left(\frac{n}{r}\right)\left[\frac{\gamma}{\mu_0} + 1\right], \left(\frac{n}{p}\right)\left[\frac{\alpha}{\mu_0} + 1\right]\right\} + \left(\frac{n}{q}\right)\left[\frac{\beta}{\mu_0} + 1\right] > 0. \tag{5.7}$$

Then

$$\int_\Omega d^\beta |\nabla^j u|^q \leqslant K_1 \left(\int_\Omega d^\gamma |u|^r\right)^{\frac{q}{r}\lambda} \left(\int_\Omega d^\alpha |\nabla^m u|^p\right)^{\frac{q}{p}(1-\lambda)}. \tag{5.8}$$

holds on $W_0^{m,r,p}(\Omega; d^\gamma, d^\alpha)$ *with*

$$0 < \lambda \leqslant \frac{m-j}{m}. \tag{5.9}$$

Proof. It clear that

$$q\left(\frac{\beta}{\mu_0} + 1\right)^{-1} \geqslant \max\left\{\left(\frac{\gamma}{\mu_0} + 1\right)^{-1} r, \left(\frac{\alpha}{\mu_0} + 1\right)^{-1} p\right\},$$

and we can choose $\phi_1, \phi_2, \theta > 1$ when $\alpha, \beta, \gamma \neq 0$ as in (5.4) such that if $v := \min\{r/\phi_1, p/\phi_2\}$ then the inequalities

$$m - j - \frac{n}{v} + \frac{n}{q\theta} \geqslant 0, \tag{5.10}$$

$$q\theta \geqslant \max\left\{\frac{r}{\phi_1}, \frac{p}{\phi_2}\right\} \tag{5.11}$$

hold. (In the case that α, β, or $\gamma = 0$, the corresponding parameter ϕ_1, ϕ_2, or θ is taken to be 1.) >From (5.10) and Lemma 2.3 we have the inequality

$$\int_{B(t,\epsilon)} |\nabla^j u|^{q\theta} \leqslant K \left\{ \epsilon^{-q\theta[j + \frac{n\phi_1}{r} - \frac{n}{q\theta}]} \left(\int_{B(t,\epsilon)} |u|^{\frac{r}{\phi_1}} \right)^{\left(\frac{q}{r}\right)\theta\phi_1} + \epsilon^{q\theta[m - \frac{n\phi_2}{p} + \frac{n}{q\theta}]} \left(\int_{B(t,\epsilon)} |\nabla^m u|^{\frac{p}{\phi_2}} \right)^{\left(\frac{q}{p}\right)\theta\phi_2} \right\} \tag{5.12}$$

for $t \in \Omega$, $u \in C_0^\infty(\Omega)$, and $\epsilon > 0$. Next when α, β, or $\gamma \neq 0$ we use Hölder's inequality to get the same kind of inequalities as in (5.6) where $B(t, \epsilon)$ or $B(t, \epsilon) \cap \Omega$ replaces Ω in the integrals involving u or $d(s)$. The "$d(s)$ integrals" exist as in (5.6) and are uniformly bounded above independently of t and ϵ by Lemma 2.2.

We then combine these inequalities with (5.12), obtaining that

$$\int_{B(t,\epsilon)} d^\beta |\nabla^j u|^q \leqslant K \left\{ \epsilon^{-q[j + \frac{n\phi_1}{r} - \frac{n}{q\theta}]} \left(\int_{B(t,\epsilon)} d^\gamma |u|^r \right)^{\frac{q}{r}} + \epsilon^{q[m - \frac{n\phi_2}{p} + \frac{n}{q\theta}]} \left(\int_{B(t,\epsilon)} d^\alpha |\nabla^m u|^p \right)^{\frac{q}{p}} \right\}. \tag{5.13}$$

The Besicovitch covering lemma then gives that for any fixed $\epsilon > 0$

$$\begin{aligned} \int_{\Gamma_i} d^\beta |\nabla^j u|^q &\leqslant K \sum_{B(t_{ij},\epsilon) \in \Gamma_i} \left\{ \epsilon^{-q\theta[j + \frac{n\phi_1}{r} - \frac{n}{q\theta}]} \left(\int_{B(t_{ij},\epsilon)} d^\gamma |u|^r \right)^{\frac{q}{r}} + \epsilon^{q\theta[m - \frac{n\phi_2}{p} + \frac{n}{q\theta}]} \left(\int_{B(t_{ij},\epsilon)} d^\alpha |\nabla^m u|^p \right)^{\frac{q}{p}} \right\} \\ &\leqslant K \left\{ \epsilon^{-q\theta[j + \frac{n\phi_1}{r} - \frac{n}{q\theta}]} \left(\int_{\Gamma_i} d^\gamma |u|^r \right)^{\frac{q}{r}} + \epsilon^{q\theta[m - \frac{n\phi_2}{p} + \frac{n}{q\theta}]} \left(\int_{\Gamma_i} d^\alpha |\nabla^m u|^p \right)^{\frac{q}{p}} \right\} \end{aligned}$$

where Γ_i, $i = 1, \ldots, \chi_n$, consist of disjoint balls. (Recall that the final inequality requires that $q \geqslant \max\{p, r\}$) for we need to use the estimate $\sum A_i^r \leqslant (\sum A_i)^r$ for $r \geqslant 1$.) Summing over i, we conclude using the fact that u has support in Ω that

$$\int_\Omega d^\beta |\nabla^j u|^q \leqslant K\chi(n) \left\{ \epsilon^{-q\theta[j+\frac{n\phi_1}{r}-\frac{n}{q\theta}]} \left(\int_\Omega d^\gamma |u|^r \right)^{\frac{q}{r}} \right.$$
$$\left. + \epsilon^{q\theta[m-j-\frac{n\phi_2}{p}+\frac{n}{q\theta}]} \left(\int_\Omega d^\alpha |\nabla^m u|^p \right)^{\frac{q}{p}} \right\}. \tag{5.14}$$

By (5.10) and the fact that $q\theta \geqslant r/\phi_1$ in (5.11) the exponent of ϵ in the first right-hand term of (5.9) is negative while the second is positive. As in the derivation of (4.6) the right-hand term considered as a function of ϵ has a unique minimum. If we perform this minimization and simplify we will obtain (5.8) with

$$\lambda = \frac{m - j - (n\phi_2)/p + n/(q\theta)}{m - (n\phi_2)/p + n\phi_1/r},$$
$$1 - \lambda = \frac{j + (n\phi_1)/r - n/(q\theta)}{m - (n\phi_2)/p + n\phi_1/r},$$

and

$$K = K(\Omega, \alpha, \beta, \gamma, \phi_1, \phi_2, \theta, \mu_0) \left[\left(\frac{\lambda}{1-\lambda} \right)^{(1-\lambda)} + \left(\frac{1-\lambda}{\lambda} \right)^{\lambda} \right].$$

Set $c := -(n\phi_2)/p + n/(q\theta)$. By (5.11) $(n\phi_1)/r \geqslant n/(q\theta)$ and $c \leqslant 0$. Then using (5.5) we see that

$$0 < \lambda \leqslant \frac{m - j - (n\phi_2)/p + n/(q\theta)}{m - (n\phi_2)/p + n/(q\theta)} = \frac{m - j + c}{m + c} \leqslant \frac{m - j}{m}.$$

Depending on the choice of the parameters ϕ_1, ϕ_2, θ λ may take on any value in the interval $(0, (m-j)/m)$ which proves (5.8). □

EXAMPLE 5.1. If $p = q = r$, $\gamma = \alpha = 0$, and $m - j - n/p > 0$ (5.7) holds since $\beta/\mu_0 > -1$ so that (5.8) is valid for $\beta \in (-\mu_0, 0]$.

COROLLARY 5.2. *If Ω has finite measure and satisfies (C2), $q \geqslant p$, $\alpha \geqslant 0$, $\beta \in (-\mu_0, 0]$, and*

$$m - (n/p)[(\alpha/\mu_0) + 1] + (n/q)[(\beta/\mu_0) + 1] > 0,$$

then the Hardy-type inequality

$$\int_\Omega d^\beta |u|^q \leqslant K \left(\int_\Omega d^\alpha |\nabla^m u|^p \right)^{\frac{q}{p}} \tag{5.15}$$

is valid on $W_0^{m,q,p}(\Omega; d^\beta, d^\alpha)$.

Proof. Taking $\gamma = j = 0$, $p = q$, and $u \in C_0^\infty(\Omega)$, Theorem 5.2 gives the inequality

$$\int_\Omega d^\beta |u|^q \leqslant K_1 \left(\int_\Omega |u|^q \right)^{\frac{q}{p}\lambda} \left(\int_\Omega d^\alpha |\nabla^m u|^p \right)^{\frac{q}{p}(1-\lambda)}.$$

Since Ω is quasibounded, $\|u\|_{\Omega;q} \preccurlyeq \|d^{\beta/q}u\|_{\Omega;q}$ and (5.15) follows upon cancellation. $\square$

REMARK 5.3. Notice that although we have assumed that Ω is of finite measure, α and γ may be larger than in Sections 3 and 4 and that the results of this section imply at least continuous embeddings of $W^{m,p}(\Omega; d^{\gamma}, d^{\alpha})$ into $L^q(\Omega; d^{\beta})$.

REMARK 5.4. As we have remarked in an introductory section of this paper all the results of this paper could be reformulated using $M_D(\partial\Omega)$ instead of $\widetilde{M}_D(\partial\Omega)$ with very slight changes, e.g. in Lemma 2.2 and in (C2) Ω should be bounded. Besides applying to finite measure domains the choice of $\widetilde{M}_D(\partial\Omega)$ over $M_D(\partial\Omega)$ may on occasion allow α, β, and γ to be greater in absolute value since $\widetilde{M}_D(\partial\Omega) \leqslant M_D(\partial\Omega)$.

REFERENCES

[1] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] C. J. AMICK, *Some remarks on Rellich's theorem and the Poincaré inequality*, J. London Math. Soc. (2) **18** (1978), 81–93.

[3] M. S. BERGER AND M. SCHECHTER, *L_p embeddings and nonlinear eigenvalue problems for unbounded domains*, Bull. Amer. Math. Soc. **76** (1970), 1299-1302.

[4] O. V. BESOV, V. P. IL'IN, V. P. KUDRJAVCEV, L. D. LIZORKIN, AND S. M. NIKOL'SKII, *Integral Representations of Functions and Imbedding Theorems, Vols I and II*, V. H. Winston and Sons, Washington, 1978 and 1979.

[5] B. BOJARSKI, *Remarks on Sobolev embedding inequalities*, Complex analysis Joensuu 1987, Lecture notes in mathematics,no. 1351, Springer-Verlag, Berlin-Heidelberg-New York- Tokyo, 1988.

[6] J. BOMAN, *L_p -estimates for very strongly elliptic systems*, University of Stockholm, Sweden. Report No. 29 (1982).

[7] R. C. BROWN, D. E. EDMUNDS, AND J. RAKOSNIK, *Remarks on Poincaré inequalities*, Czech. Math. J. **45(120)** (1995), 351–377.

[8] ——— AND D. B. HINTON, *Weighted interpolation inequalities and embeddings in R^n*, Canad. J. Math. **47** (1990), 959–980.

[9] V. I. BURENKOV, *Sobolev spaces on domains*, Teubner-Texte zur Mathematik, Band 137, B. G. Teubner Stuggart and Leipzig, 1998.

[10] D. E. EDMUNDS AND W. D. EVANS, *Spectral theory and differential operators*, Oxford University Press, Oxford, UK, 1987.

[11] ——— AND R. HURRI-SYRJÄREN, *Weighted Poincaré inequalities and Minkowski content*, Proc. Roy. Soc. Edinburgh **125** (1995), 817–825.

[12] ———, *Remarks on the Hardy inequality*, J. Inequal. Appl. **1** (1997), 125–137.

[13] W. D. EVANS AND D. J. HARRIS, *Sobolev embeddings for generalized ridged domains*, Proc. Lond. Math. Soc. (3) **54** (1987), 141–175.

[14] L. E. FRAENKEL, *On the regularity of the boundary in the theory of Sobolev spaces*, Proc. Lond. Math. Soc. (3) **39** (1979), 385–427.

[15] J. FLECKINGER-PELLE AND D. VASSILIEV, *An example of two-term asympotics for the "counting function" of a fractal drum*, Trans. Amer. Math. Soc. **337** (1993), 99–116.

[16] D. GILBARG AND N. S. TRUDINGER, *Elliptic partial differential equations of second-order*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[17] M. DE GUZMAN, *Differentiation of integrals in $\mathbb{R}^n$*, Lecture Notes in Mathematics 481, Springer-Verlag, Berlin, 1975.

[18] R. HURRI-SYRJÄREN, *Poincaré domains in $\mathbb{R}^n$*, Ann. Acad. Sci. Fenn. Ser A, I Math. Diss. **71** (1988), 1–42.

[19] A. KUFNER, *Weighted Sobolev spaces*, J. Wiley and Sons, Chichester, New York, Brisbane, Toronto, Singapore, 1985.

[20] M. L. LAPIDUS, *Can one hear the shape of a fractal drum? Partial rsolution of the Weyl-Berry conjecture*, Geometric Analysis and Computer Graphics (P. Concus, *et al.,* eds.), Proc. Workshop Differential geometry, Calculus of Variations, and Computer Graphisc (MSRI, Berkeley, May 1988),

Mathematical Sciences Research Institute Publications, vol 17, Springer-Verlag, New York, 1990, pp. 119–126.
[21] ______, *Fractal drum inverse spectral problems for elliptic operators and a partial resolution of the Weyl-berry conjecture*, Trans. Amer. Math. Soc. **325** (1991), 465–529.
[22] J. L. LEWIS, *Uniformly fat sets*, Trans. Amer. Math. Soc. **308** (1988), 177–196.
[23] O. MARTIO AND M. VUORINEN, *Whitney cubes, p-capacity and Minkowski content*, Expo. Math. **5** (1987), 17–40.
[24] V. G. MAZ'YA, *Sobolev spaces*, Springer-Verlag, Berlin-Heidelberg-New York- Tokyo, 1985.
[25] L. NIRENBERG, *On elliptic partial differential equations*, Annali della Scuola Norm. Sup. Pisa **12** (1958), 115–162.
[26] B. OPIC AND A. KUFNER, *Hardy-type inequalities*, Longman Scientific and Technical, Harlow, Essex, UK, 1990.
[27] W. SMITH AND D. A. STEGENGA, *Hölder domains and Poincaré domains*, Trans. Amer. Math. Soc. **319** (1990), 67–100.
[28] E. M. STEIN, *Singular integrals and differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.
[29] C. TRICOT, *Dimensions aux bords d'un ouvert*, Ann. Sci. Math. Quebec **11** (1987), 205–35.
[30] D. A. TROTSENKO, *Properties of regions with a nonsmooth boundary (Russian)*, Sibirsk. Mat. Zh. **22** (1981), 221–224, 232.
[31] A. WANNEBO, *Hardy inequalities*, Proc. Amer. Math. Soc. **109** (1990), 85–95.
[32] ______, *Hardy inequalities and embeddings in domains generalising $C^{0,\alpha}$ domains*, Proc. Amer. Math. Soc. **122** (1994), 1181–1190.

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