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# NEW ESTIMATIONS OF THE SOLUTIONS OF MULTIDIMENSIONAL INTEGRAL INEQUALITIES

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*Abstract.* In this paper new general estimations of integral inequalities of Volterra's type as some special estimations based on the Bessel function are given.

## 1. Introduction

In several fundamental papers from the theory of differential and integral equations (the existence of the unique solution, stability, etc) the well-known Bellman-Gronwall lemma [1] or its generalizations play an important role. To these generalizations voluminous references are dedicated, among which we draw a special attention to paper [2]. Considering some classes of multidimensional integral or partial differential equations, the multidimensional integral inequalities appear.

Let us consider, for instance, an $n$-dimensional integral equation of Volterra type of the first order

$$\int_0^{x_1} \cdots \int_0^{x_n} K(x_1,\ldots,x_n;s_1,\ldots,s_n)\varphi(s_1,\ldots,s_n)ds_1\ldots ds_n = f(x_1,\ldots,x_n), \tag{1}$$

$$(0 \leqslant x_i \leqslant b_i;\ 1 \leqslant i \leqslant n)$$

or in the operator form

$$V\varphi = f. \tag{1'}$$

Let

$$K(x_1,\ldots,x_n;s_1,\ldots,s_n) \in C^{(n)}_{\Omega_n}, \tag{2}$$

$$f(x_1,\ldots,x_n) \in C^{(n)}_{\Pi_n}, \tag{3}$$

where

$$\Omega_n = \{(\mathbf{x},\mathbf{s}) \mid 0 \leqslant s_i \leqslant x_i \leqslant b_i, \quad 1 \leqslant i \leqslant n\},$$
$$\Pi_n = \{\mathbf{x} \mid 0 \leqslant x_i \leqslant b_i; \quad 1 \leqslant i \leqslant n\}$$

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and

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{s} = (s_1, \dots, s_n).$$

From (2) we obtain

$$\max_{(x,s)\in\Omega_n} \left| \frac{\partial^m K(x_1, \dots, x_n; s_1, \dots, s_n)}{\partial x_{i_1} \dots \partial x_{i_m}} \right| = \alpha_{i_1} \dots i_m < \infty, \tag{4}$$
$$(1 \leqslant i_1 < \cdots < i_m \leqslant n, \quad 1 \leqslant m \leqslant n)$$
$$\max_{x\in\Pi_n} \left| \frac{\partial^n f(x_1, \dots, x_n)}{\partial x_1 \dots \partial x_n} \right| = F < \infty. \tag{5}$$

Let us assume subsequently that

$$\min_{\mathbf{x}\in\Pi_n} |K(x_1, \dots, x_n; x_1, \dots, x_n)| = k \neq 0. \tag{6}$$

By differentiating (1) with respect to $x_1, \dots, x_n$, we pass to the estimation by modulus and taking into account (4) – (6), we get the inequality

$$\Psi \leqslant \frac{F}{k} + k^{-1}\Big( \sum_{1\leqslant i_1 \leqslant n} \alpha_{i_1} V_{i_1}\Psi + \sum_{1\leqslant i_1<i_2\leqslant n} \alpha_{i_1 i_2} V_{i_1 i_2}\Psi + \dots$$
$$+ \sum_{1\leqslant i_1<\cdots<i_m\leqslant n} \alpha_{i_1 \dots\, i_m} V_{i_1 \dots\, i_m}\Psi + \cdots + \alpha_{12\dots n} V_{12\dots n}\Psi \Big), \tag{7}$$

where

$$V_{i_1\dots i_m}\Psi = \int_0^{x_{i_1}} \dots \int_0^{x_{i_m}} \Psi(x_1, \dots, x_{i_1-1}, s_1, x_{i_1+1}, \dots, x_{i_m-1}, s_m, x_{i_m+1}, \dots, x_n) ds_1 \dots ds_m,$$
$$\Psi(x_1, \dots, x_n) = |\varphi(x_1, \dots, x_n)|.$$

The first part of inequality (7) contains $n$-ordinary integrals, $C_n^2$ - double etc, and hence the general number of summands is $2^n$. Let us assume that using (7), the estimation of the following inequality will be of form

$$\Psi \leqslant KF, \quad K < \infty. \tag{8}$$

Therefore it would be

$$\|\varphi\|_{C_{\Pi_n}} \leqslant K\|f\|_{C^{(n)}_{\Pi_n}}, \tag{9}$$

and from (9) the correctness of equations (1) and (1') stems in expression $(C_{\Pi_n}, V, C^{(n)}_{\Pi_n})$ and the estimation of

$$\|V^{-1}\|_{C^{(n)}_{\Pi_n}\to C_{\Pi_n}} \leqslant K. \tag{10}$$

Vendorff [1] got some two-dimensional analogies of the Bellman - Gronwall inequality, but in [3] a general approach of the obtaining of the estimation of the solution of integral inequalities of Volterra's type operators is proposed.

If we use the results from [3], we will get the estimation of (10)

$$\|V^{-1}\|_{C^{(n)}_{\Pi_n}\to C_{\Pi_n}} \leqslant k^{-1}\cdot \exp\Big[\sum_{m=1}^{n}\sum_{1\leqslant i_1<\dots<i_m\leqslant n}(\alpha_{i_1}\dots i_m k^{-1}b_{i_1}\dots b_{i_m})^{1/2}\Big]^2. \qquad (11)$$

We wish to stress, that for the existence of the correctness of equation (1), it is sufficient to get in (10) each constant $k<\infty$, then when investigating the error problem by numerical methods for solving (10) it is desirable to have a maximal accurate estimation of the inverse operator norm. Further, the problem appears naturally to get the estimation of the solution of inequality of type (7). Therefore, this problem has an interest of its own. The goal of the present paper is the occurrence and obtaining of similar assessments.

## 2. New estimations of two-dimensional inequalities

Further, without loss of generality, for the sake of simplicity we assume the norming condition $k=1$. By changing of "$\leqslant$" by "$=$" in (7), we consider the equation

$$\eta = F + \sum_{1\leqslant i_1\leqslant n}\alpha_{i_1}V_{i_1}\eta + \sum_{1\leqslant i_1<i_2\leqslant n}\alpha_{i_1i_2}V_{i_1i_2}\eta + \dots + \alpha_{12\dots n}V_{12\dots n}\eta. \qquad (12)$$

The solution $\eta$ of equation (12) appears as an assessment of the solution of inequality (7), i.e. $\Psi\leqslant\eta$. The search of the solution of equality (12) is based on the following simple idea.

Let the following equation be given

$$\xi = F + M\xi + L\xi, \qquad (13)$$

where

$$M\xi = \alpha_1\int_0^{x_1}\xi(s,x_2)ds, \quad L\xi = \alpha_2\int_0^{x_2}\xi(x_1,s)ds$$

and let the functions $\hat{\epsilon}_i = M^iF$, $(i=1,2,\dots)$ be found earlier.

If

$$\xi = \sum_{k=0}^{\infty}(M+L)^kF = \sum_{k=0}^{\infty}\sum_{i=0}^{k}C_k^iL^iM^{k-i}F = \sum_{k=0}^{\infty}\sum_{i=0}^{k}C_k^iL^i\hat{\epsilon}_{k-i},$$

then the search of the solution (13) reduces on the calculation of the function

$$\epsilon_k = \sum_{i=0}^{k}C_k^iL^i\hat{\epsilon}_{k-i}. \qquad (14)$$

In the given paragraph we will illustrate the idea for solving (12) in the case when $n=2$.

Let

$$\eta(x_1,x_2) = F + \alpha_1\int_0^{x_1}\eta(s,x_2)ds + \alpha_2\int_0^{x_2}\eta(x_1,s)ds + \alpha_{12}\int_0^{x_1}\int_0^{x_2}\eta(s_1,s_2)ds_1ds_2. \qquad (15)$$

Let us assume at the beginning that $\alpha_2 = \alpha_{12} = 0$ and let us consider the simple equation

$$\xi = F + M\xi.$$

Obviously

$$M^i F = F\frac{(\alpha_1 x_1)^i}{i!}$$

and

$$\xi(x_1, x_2) = F\sum_{k=0}^{\infty}\frac{(\alpha_1 x_1)^k}{i!} = Fe^{\alpha_1 x_1}.$$

Hence it follows that *one-dimensional* estimation of Bellman–Gronwall for the adequate inequality appears as unimproved.

Let $\alpha_2 \neq 0$, i.e.

$$\xi(x_1, x_2) = F + \alpha_1\int_0^{x_1}\xi(s, x_2)ds + \alpha_2\int_0^{x_2}\xi(x_1, s)ds. \qquad (16)$$

By putting

$$\hat{\epsilon}_{k-i} = F\frac{(\alpha_1 x_1)^{k-i}}{(k-i)!},$$

we find $\epsilon_k$ with respect to the equation (14)

$$\epsilon_k = F\sum_{i=0}^{k} C_k^i\frac{(\alpha_1 x_1)^{k-i}(\alpha_2 x_2)^i}{(k-i)!i!} = F\sum_{i_1+i_2=k}\frac{k!(\alpha_1 x_1)^{i_1}(\alpha_2 x_2)^{i_2}}{(i_1!)^2(i_2!)^2}. \qquad (17)$$

Besides,

$$\xi(x_1, x_2) = F\sum_{k=0}^{\infty}\sum_{i_1+i_2=k}\frac{k!(\alpha_1 x_1)^{i_1}(\alpha_2 x_2)^{i_2}}{(i_1!)^2(i_2!)^2}. \qquad (18)$$

However, the form (18) appears as an estimation of the inequality

$$\Psi(x_1, x_2) \leqslant F + \alpha_1\int_0^{x_1}\Psi(s, x_2)ds + \alpha_2\int_0^{x_2}\Psi(x_1, s)ds. \qquad (19)$$

For solving (15), the last step remains. Let us assume that $\hat{\epsilon}_k$ is equal to the right-hand side of (17) and

$$L\eta = \alpha_{12}\int_0^{x_1}\int_0^{x_2}\eta(s_1, s_2)ds_1 ds_2.$$

Then

$$L^i\hat{\epsilon}_{k-i} = F\sum_{i_1+i_2=k-i}\frac{(k-i)!(\alpha_1 x_1)^{i_1}(\alpha_2 x_2)^{i_2}(\alpha_{12}x_1 x_2)^i}{(i_1!)^2(i_2!)^2 A^i_{i_1+i}A^i_{i_2+i}},$$

where

$$A_p^t = \frac{p!}{(p-t)!}.$$

From (14) it yields

$$\epsilon_k = F\sum_{i=0}^{k}\sum_{i_1+i_2=k-i}\frac{C_k^i(x-i)!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}(\alpha_{12}x_1x_2)^i}{(i_1!)^2(i_2!)^2A^i_{i_1+i}A^i_{i_2+i}}$$
$$= F\sum_{i_1+i_2+i_{12}=k}\frac{k!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}(\alpha_{12}x_1x_2)^i}{i_1!i_2!i_{12}!(i_1+i_{12})!(i_2+i_{12})!}$$

and apart from that, the solution of the equation (15) is presented in the following form

$$\eta(x_1,x_2) = F\sum_{k=0}^{\infty}\sum_{i_1+i_2+i_{12}=k}\frac{k!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}(\alpha_{12}x_1x_2)^{i_{12}}}{i_1!i_2!i_{12}!(i_1+i_{12})!(i_2+i_{12})!}. \tag{20}$$

Thereby the estimation of the above solution is given by

$$\Psi(x_1,x_2) \leqslant F+\alpha_1\int_0^{x_1}\Psi(s,x_2)ds+\alpha_2\int_0^{x_2}\Psi(x_1,s)ds+\alpha_{12}\int_0^{x_1}\int_0^{x_2}\Psi(s_1,s_2)ds_1ds_2. \tag{21}$$

Now, we will present another way of getting the estimation of inequalities (19) and (21). As we have seen this method yields to the possibility to assume (18) and (20) in a more clear form. Differentiating (16) with respect to $x_1$ and $x_2$, we pass from the integral equation (16) to its equivalent problem of Goursa

$$\xi''_{x_1x_2} = \alpha_1\xi'_{x_2}+\alpha_2\xi'_{x_1}, \tag{22}$$
$$\xi(x_1,0) = Fe^{\alpha_1x_1}, \tag{23}$$
$$\xi(0,x_2) = Fe^{\alpha_2x_2}. \tag{24}$$

The standard substitution

$$\xi(x_1,x_2) = e^{\alpha_1x_1+\alpha_2x_2}\zeta(x_1,x_2),$$

regarding the function $\zeta(x_1,x_2)$ yields to the equation

$$\zeta''_{x_1x_2} = \alpha_1\alpha_2\zeta, \tag{22'}$$

with boundary conditions

$$\zeta(x_1,0) = F, \tag{23'}$$
$$\zeta(0,x_2) = F. \tag{24'}$$

The generally known solution of the equation (22') has the form

$$\zeta(x_1,x_2) = \int_0^{x_1}f_1(s)J_0(2i\sqrt{\alpha_1\alpha_2x_2(x_1-s)})ds$$
$$+\int_0^{x_2}f_2(s)J_0(2i\sqrt{\alpha_1\alpha_2x_2(x_1-s)})ds+[f_1(0)+f_2(0)]\cdot J_0(2i\sqrt{\alpha_1\alpha_2x_1x_2}),$$

where $J_0(z)$ is the Bessel function of the zero order, $i$-imaginary unit, and $f_1$ and $f_2$ are arbitrary functions of $C^{(1)}$ class. With reference to (23') and (24') we find that for solving equation (22'), the following function appears

$$\zeta(x_1,x_2) = J_0(2i\sqrt{\alpha_1\alpha_1 x_2}),$$

such that

$$\xi(x_1,x_2) = Fe^{\alpha_1 x_1+\alpha_2 x_2} J_0(2i\sqrt{\alpha_1\alpha_2 x_1 x_2}). \tag{25}$$

We get the second form of the estimation of inequality (19).

If $\alpha_{12} \neq 0$, instead of (22') we have

$$\zeta''_{x_1x_2} = (\alpha_1\alpha_2 + \alpha_{12})\zeta,$$

further, analogously to the previous that for the solution of (15) the following function occurs

$$\eta(x_1,x_2) = Fe^{\alpha_1 x_1+\alpha_2 x_2} J_0(2i\sqrt{(\alpha_1\alpha_2+\alpha_{12})x_1 x_2}), \tag{26}$$

yielding, the second presentation of (20) for the estimation of inequality (21).

Let us compare the obtained estimations with those given in references. As far as inequality (19) is concerned, the following is known:

a) the estimation of Vendorff [1]

$$\Psi(x_1,x_2) \leqslant Fe^{\alpha_1 x_1+\alpha_2 x_2+\alpha_1\alpha_2 x_1 x_2} \tag{27}$$

b) the estimation from [3]

$$\Psi(x_1,x_2) \leqslant Fe^{2(\alpha_1 x_1+\alpha_2 x_2)}. \tag{28}$$

All estimations (25), (27) and (28) contain the multiplier $Fe^{\alpha_1 x_1+\alpha_2 x_2}$, further it remains to compare the functions

$$J_0(2i\sqrt{\alpha_1\alpha_2 x_1 x_2}), \quad e^{\alpha_1\alpha_2 x_1 x_2}, \quad e^{2\sqrt{\alpha_1\alpha_2 x_1 x_2}}, \quad \text{and} \quad e^{\alpha_1 x_1+\alpha_2 x_2}.$$

We will denote them respectively by $\Psi_1$, $\Psi_2$, $\Psi_3$ and $\Psi_4$. Obviously, $\Psi_2 \leqslant \Psi_3$ if $\alpha_1\alpha_2 x_1 x_2 \leqslant 4$, and $\Psi_2 > \Psi_3$ if $\alpha_1\alpha_2 x_1 x_2 > 4$. From the known inequality between the geometric and arithmetic means of $\alpha_1$, $\alpha_2$, $x_1$, $x_2$, we obtain $\Psi_3 \leqslant \Psi_4$.

Since

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^2}{2^{2k}(k!)^2},$$

$$\Psi_1 = J_0(2i\sqrt{\alpha_1\alpha_2 x_1 x_2}) = \sum_{k=0}^{\infty} \frac{(\alpha_1\alpha_2 x_1 x_2)^k}{(k!)^2},$$

and developing $\Psi_2$ and $\Psi_3$ into the Taylor series, it is easily seen that for each $\alpha_1$, $\alpha_2$, $x_1$, $x_2$

$$\Psi_1 \leqslant \min \Psi_i, \quad (i = 2,3,4).$$

The minimality $\Psi_1$ is a natural consequence of (25). Analogously, we can verify that (26) decreases the estimations from [3] obtained for inequality (21).

## 3. New estimations of three-dimensional and $n$-dimensional inequalities

Let $n = 3$ in (12). Let us assume (12) into a developed form

$$\eta(x_1,x_2,x_3)=F+\alpha_1\int_0^{x_1}\eta(s,x_2,x_3)ds+\alpha_2\int_0^{x_1}\eta(x_1,s,x_3)ds+\alpha_3\int_0^{x_3}\eta(x_1,x_2,s)ds$$
$$+\alpha_{12}\int_0^{x_1}\int_0^{x_2}\eta(s_1,s_2,x_3)ds_1ds_2+\alpha_{13}\int_0^{x_1}\int_0^{x_3}\eta(s_1,x_2,s_2)ds_1ds_2$$
$$+\alpha_{23}\int_0^{x_2}\int_0^{x_3}\eta(x_1,s_1,s_2)ds_1ds_2+\alpha_{123}\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\eta(s_1,s_2,s_3)ds_1ds_2ds_3. \quad (29)$$

If all coefficients, except $\alpha_1$, $\alpha_2$ and $\alpha_{12}$, in (29) are equal to zero, the solution of (29) overlaps with already found solution to equality (15), determined by formula (20). Further, in accordance with the general idea, exposed in the previous section, the solution of equation (29) reduces to four subproblems.

Let $t$ be the number of subproblems. For (14) a series of calculations is determined by means of the formula

$$\epsilon_k^{(t)}\sum_{i=0}^{k}C_k^iL_t^i\hat{\epsilon}_{k-1}^{(t-1)}, \quad (t=1,2,3,4) \quad (30)$$
$$\hat{\epsilon}_k^{(t)}=\epsilon_k^{(t)}, \quad (t=1,2,3)$$

where

$$\hat{\epsilon}_k^{(0)}=F\sum_{i_1+i_2+i_{12}=k}\frac{k!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}(\alpha_{12}x_1x_2)^{i_{12}}}{i_1!i_2!i_{12}!(i_1+i_{12})!(i_2+i_{12})!},$$

and $L_t$ be the integral operator, completed accordingly, accepting a series into a subproblem t.

Let us take, for example, a natural order where the integral operators are written in (29) and the technique of obtaining function $\epsilon_k$, by formula (14), which is considered in detail in the previous section by conclusion (20), we find the final formulas for $\epsilon_k^{(t)}$, $(t = 1,2,3,4)$

$$\epsilon_k^{(1)}=F\sum_{i_1+i_2+i_3+i_{12}=k}\frac{k!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}(\alpha_3x_3)^{i_3}}{i_1!i_2!i_{12}!(i_1+i_{12})!(i_2+i_{12})!}(\alpha_{12}x_1x_2)^{i_{12}},$$
$$\epsilon_k^{(2)}=F\sum_{i_1+i_2+i_3+i_{12}+i_{13}=k}\frac{k!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}(\alpha_3x_3)^{i_3}}{i_1!i_2!i_3!}$$
$$\times\frac{(\alpha_{12}x_1x_2)^{i_{12}}(\alpha_{13}x_1x_3)^{i_3}}{i_{12}!i_{13}!(i_1+i_{12}+i_{13})!(i_2+i_{12})!(i_3+i_{13})!},$$
$$\epsilon_k^{(3)}=F\sum_{i_1+i_2+i_3+i_{12}+i_{13}+i_{23}=k}\frac{k!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}}{i_1!i_2!i_3!} \quad (31)$$
$$\times\frac{(\alpha_3x_3)^{i_3}(\alpha_{12}x_1x_2)^{i_{12}}(\alpha_{13}x_1x_3)^{i_3}(\alpha_{23}x_2x_3)^{i_{23}}}{i_{12}!i_{13}!i_{23}!(i_1+i_{12}+i_{13})!(i_2+i_{12}+i_{23})!(i_3+i_{13}+i_{23})!},$$

$$\epsilon_k^{(4)} = F \sum_{i_1+i_2+i_3+i_{12}+i_{13}+i_{23}+i_{123}=k} \frac{k!(\alpha_1 x_1)^{i_1}(\alpha_2 x_2)^{i_2}}{i_1!i_2!i_3!}$$
$$\times \frac{(\alpha_3 x_3)^{i_3}(\alpha_{12}x_1x_2)^{i_{12}}(\alpha_{13}x_1x_3)^{i_{13}}(\alpha_{23}x_2x_3)^{i_{23}}(\alpha_{123}x_1x_2x_3)^{i_{123}}}{i_{12}!i_{13}!i_{23}!i_{123}!(i_1+i_{12}+i_{13}+i_{123})!(i_2+i_{12}+i_{23}+i_{123})!(i_3+i_{13}+i_{23}+i_{123})!}.$$

Thereby the solution of equation (29) was found in the form

$$\eta(x_1,x_2,x_3) = \sum_{k=0}^{\infty} \epsilon_k^{(4)}, \tag{32}$$

where $\epsilon_k^{(4)}$ is determined by formula (31).

From (31) and (32) it is obviously possible to get the solution to each arbitrary equation of the form (29), occurring as an estimation of adequate integral inequalities.

At the and we can find the solution of equation (12) with respect to arbitrary n. Then we consider $2^3+2^4+\cdots+2^{n-1}$ of the subproblem in the form (30), and we get

$$\eta(x_1,\dots,x_n) = F\sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_{12\dots n}=k} \frac{k!}{i_1!\dots i_n!}$$
$$\times \frac{(\alpha_1x_1)^{i_1}\dots(\alpha_nx_n)^{i_n}(\alpha_{12}x_1x_2)^{i_{12}}\dots(\alpha_{n-1,n}x_{n-1}x_n)^{i_{n-1,n}}\dots(\alpha_{12\dots n}x_1x_2\dots x_n)^{i_{12\dots n}}}{i_{12}!\dots i_{n-1,n}!\dots i_{12\dots n}!\beta_1!\beta_2!\dots\beta_n!} \tag{33}$$

where

$$\begin{aligned} \beta_1 &= i_1+i_{12}+\cdots+i_{1n}+i_{123}+\cdots+i_{1,n-1,n}+\cdots+i_{12\dots n},\\ \beta_2 &= i_2+i_{12}+\cdots+i_{2n}+i_{123}+\cdots+i_{2,n-1,n}+\cdots+i_{12\dots n},\\ &\vdots\\ \beta_n &= i_n+i_{1n}+\cdots+i_{n-1,n}+i_{12n}+\cdots+i_{n-2,n-1,n}+\cdots+i_{12\dots n}. \end{aligned} \tag{34}$$

So, the problem of obtaining the estimation of the solution of inequality (7) is solved.

In the previous paragraph for the case $n=2$, another form is given for representing the estimation, which, evidently, has a better form. However, even at $n=3$ we have a similar representation, with exception of some cases which do not have their place.

In the next paragraph we will consider these cases.

## 4. New estimations of special cases

Let all coefficients except $\alpha_1$, $\alpha_2$, $\alpha_3$ in (29) be equal to zero, i.e. let us consider the equation

$$\xi(x_1,x_2,x_3) = F+\alpha_1\int_0^{x_1}\xi(s,x_2,x_3)ds+\alpha_2\int_0^{x_2}\xi(x_1,s,x_3)ds+\alpha_3\int_0^{x_3}\xi(x_1,x_2,s)ds. \tag{35}$$

Then from (31) it follows

$$\xi(x_1,x_2,x_3) = F\sum_{k=0}^{\infty}\sum_{i_1+i_2+i_3=k}\frac{k!(\alpha_1x_1)^{i_1}(\alpha_2x_2)^{i_2}(\alpha_3x_3)^{i_3}}{(i_1!)^2(i_2!)^2(i_3!)^2}. \tag{36}$$

Again, as in two-dimensional cases we pass from (35) to a partial differential equation. We differentiate equation (35) with respect to $x_1$, $x_2$, $x_3$ and we get

$$\xi'''_{x_1x_2x_3} = \alpha_1\xi''_{x_2x_3}+\alpha_2\xi''_{x_1x_3}+\alpha_3\xi''_{x_1x_2}. \tag{37}$$

The limiting conditions taking into account (25) have the form

$$\begin{aligned}\xi(x_1,x_2,0) &= Fe^{\alpha_1x_1+\alpha_2x_2}J_0(2i\sqrt{\alpha_1\alpha_2x_1x_2}),\\ \xi(x_1,0,x_3) &= Fe^{\alpha_1x_1+\alpha_3x_3}J_0(2i\sqrt{\alpha_1\alpha_3x_1x_3}),\\ \xi(0,x_2,x_3) &= Fe^{\alpha_2x_2+\alpha_3x_3}J_0(2i\sqrt{\alpha_2\alpha_3x_2x_3}).\end{aligned} \tag{38}$$

Obviously, problems (35), (37) and (38) are equivalent. Again we will apply the substitution

$$\xi(x_1,x_2,x_3) = e^{\alpha_1x_1+\alpha_2x_2+\alpha_3x_3}\zeta(x_1,x_2,x_3). \tag{39}$$

Then whatever regards till $\eta(x_1,x_2,x_3)$ we have

$$\zeta'''_{x_1x_2x_3} = 2\alpha_1\alpha_2\alpha_3\zeta+\alpha_2\alpha_3\zeta'_{x_1}+\alpha_1\alpha_3\zeta'_{x_2}+\alpha_1\alpha_2\zeta'_{x_3}, \tag{40}$$

$$\begin{aligned}\zeta(x_1,x_2,0) &= FJ_0(2i\sqrt{\alpha_1\alpha_2x_1x_2}),\\ \zeta(x_1,0,x_3) &= FJ_0(2i\sqrt{\alpha_1\alpha_3x_1x_3}),\\ \zeta(0,x_2,x_3) &= FJ_0(2i\sqrt{\alpha_2\alpha_3x_2x_3}).\end{aligned} \tag{41}$$

Problems (40) and (41) are equivalent with the integral of the equation

$$\begin{aligned}\zeta(x_1,x_2,x_3) &= F+\alpha_1\alpha_2\int_0^{x_1}\int_0^{x_2}\zeta(\cdot)ds_1ds_2+\alpha_1\alpha_3\int_0^{x_1}\int_0^{x_3}\zeta(\cdot)ds_1ds_2\\ &+\alpha_2\alpha_3\int_0^{x_2}\int_0^{x_3}\zeta(\cdot)ds_1ds_2+2\alpha_1\alpha_2\alpha_3\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\zeta(\cdot)ds_1ds_2ds_3,\end{aligned} \tag{42}$$

whose solution could be written in accordance with (31) and (32), and thereby the second representation to (36) with condition (39) is obtained

$$\begin{aligned}&\xi(x_1,x_2,x_3) = Fe^{\alpha_1x_1+\alpha_2x_2+\alpha_3x_3}\\ &\times\sum_{k=0}^{\infty}\sum_{i_1+i_2+i_3=k}\frac{k!(\alpha_1\alpha_2x_1x_2)^{i_1}(\alpha_1\alpha_3x_1x_3)^{i_2}(\alpha_2\alpha_3x_2x_3)^{i_3}(2\alpha_1\alpha_2\alpha_3x_1x_2x_3)^{i_4}}{i_1!i_2!i_3!i_4!(i_1+i_2+i_4)!(i_1+i_3+i_4)!(i_2+i_3+i_4)!}.\end{aligned} \tag{43}$$

If we separate the exponents from (36) we arrive at the occurrence of the series with complex structure. Instead of (38) it is natural to assume that this series contains the product of the Bessel functions.

We will calculate these problems in detail. That is why we will consider at the beginning of (42) a simple equation

$$\hat{\zeta}(x_1,x_2,x_3) = F + \gamma_{12}\int_0^{x_1}\int_0^{x_2}\hat{\zeta}(\cdot)ds_1ds_2 + \gamma_{13}\int_0^{x_1}\int_0^{x_3}\hat{\zeta}(\cdot)ds_1ds_2. \qquad (44)$$

Applying the general scheme we get

$$\hat{\zeta}(x_1,x_2,x_3) = F\sum_{k=0}^{\infty}\sum_{i_1+i_2=k}\frac{(\gamma_{12}x_1x_2)^{i_1}(\gamma_{13}x_1x_3)^{i_2}}{(i_1!)^2(i_2!)^2} = FJ_0(2i\sqrt{\gamma_{12}x_1x_2})J_0(2i\sqrt{\gamma_{13}x_1x_3}).$$

In general, for each $n$ in the solution of the integral equation

$$\hat{\zeta}(x_1,\ldots,x_k,\ldots,x_n) = F + \sum_{1\leqslant j\leqslant n, j\neq k}\gamma_{jk}\int_0^{x_i}\int_0^{x_k}\hat{\zeta}(\cdot)ds_1ds_2 \qquad (45)$$

the following function appears

$$\hat{\zeta}(x_1,\ldots,x_n) = F\prod_{1\leqslant j\leqslant n, j\neq k}J_0(2i\sqrt{\gamma_{jk}x_jx_k}), \qquad (46)$$

i.e. (46) is the estimation for inequality, obtained from (45) with the substitution " $=$ " by " $\leqslant$ ". We stress, that each of the operators in (45) involves the integration with respect to $x_k$.

At present we will give another form to equation (44)

$$\hat{\zeta}(x_1,x_2,x_3) = F + \gamma_{12}\int_0^{x_1}\int_0^{x_2}\hat{\zeta}(\cdot)ds_1ds_2$$
$$+\gamma_{13}\int_0^{x_1}\int_0^{x_3}\hat{\zeta}(\cdot)ds_1ds_2 + \gamma_{23}\int_0^{x_2}\int_0^{x_3}\hat{\zeta}(\cdot)ds_1ds_2. \qquad (47)$$

We prove that the solution (47) majorities the product of three Bessel's functions

$$\hat{\zeta}(x_1,x_2,x_3) \leqslant FJ_0(2i\sqrt{\gamma_{12}x_1x_2})J_0(2i\sqrt{\gamma_{13}x_1x_3})J_0(2i\sqrt{\gamma_{23}x_2x_3}) = B(x_1,x_2,x_3). \quad (48)$$

The general scheme yields to

$$\hat{\zeta}(x_1,x_2,x_3) = F\sum_{k=0}^{\infty}\sum_{i_1+i_2+i_3=k}\frac{k!(\gamma_{12}x_1x_2)^{i_1}(\gamma_{13}x_1x_3)^{i_2}(\gamma_{23}x_2x_3)^{i_3}}{i_1!i_2!i_3!(i_1+i_2)!(i_1+i_3)!(i_2+i_3)!}. \qquad (49)$$

On the other hand

$$B(x_1,x_2,x_3) = F\sum_{k=0}^{\infty}\sum_{i_1+i_2+i_3=k}\frac{k!(\gamma_{12}x_1x_2)^{i_1}(\gamma_{13}x_1x_3)^{i_2}(\gamma_{23}x_2x_3)^{i_3}}{(i_1!)^2(i_2!)^2(i_3!)^2(i_1+i_2+i_3)!}. \qquad (50)$$

From (49) and (50), it is indispensable to confirm that

$$(i_1+i_2)!(i_1+i_3)!(i_2+i_3)! \geqslant i_1!i_2!i_3!(i_1+i_2+i_3)!. \tag{51}$$

Also it can be shown that (51) is equivalent to the inequality

$$C^{i_3}_{i_1+i_3}C^{i_3}_{i_2+i_3} \geqslant C^{i_3}_k, \quad k=i_1+i_2+i_3. \tag{52}$$

Inequality (52) will be proven, if we show that

$$C^{j}_{j+p}C^{j}_{k-p} \geqslant C^{j}_k, \quad (0 \leqslant p \leqslant k-j). \tag{53}$$

Now we will give the proof of inequality (53).

Let us denote

$$f(p) = C^{j}_{j+p}C^{j}_{k-p}, \quad (0 \leqslant p \leqslant k-j).$$

We will show that

$$\ln f(p) \geqslant \ln C^{j}_k, \quad (0 \leqslant p \leqslant k-j) \tag{54}$$

and hence (53) will follow.

Since

$$C^{j}_{j+p} = \frac{1}{j!}\prod_{i=0}^{j-1}(j+p-i), \quad C^{j}_{k-p} = \frac{1}{j!}\prod_{i=0}^{j-1}(k-p-i),$$

$$\ln f(p) = \sum_{i=0}^{j-1}\ln(j+p-i) + \sum_{i=0}^{j-1}\ln(k-p-i) - 2\ln(j!),$$

it follows

$$(\ln f(p))'' = -\sum_{i=0}^{j-1}\frac{1}{(j+p-i)^2} - \sum_{i=0}^{j-1}\frac{1}{(k-p-i)^2} < 0,$$

i.e. the function $\ln f(p)$ is convex for $0 \leqslant p \leqslant k-j$.

From

$$\min_{0\leqslant p\leqslant k-j}[\ln f(p)] = \min[\ln f(0), \ln f(k-j)] = \ln C^{j}_k,$$

which proves (54). Thereby the validity on (48) is proven. Now we can return to the equation (42), which is different from (47) with the presence of the triple integral.

By means of techniques analogous to those applied above, we obtain that

$$\begin{aligned}\zeta(x_1,x_2,x_3) \leqslant {} & FJ_0(2i\sqrt{\alpha_1\alpha_2x_1x_2})J_0(2i\sqrt{\alpha_1\alpha_3x_1x_3}) \\ & \times J_0(2i\sqrt{\alpha_2\alpha_3x_2x_3})\sum_{k=0}^{\infty}\frac{(2\alpha_1\alpha_2\alpha_3x_1x_2x_3)^k}{(k!)^3}.\end{aligned} \tag{55}$$

Let us denote the first part of (56) (without multiplier F) by $B_1(\alpha_1x_1;\alpha_2x_2;\alpha_3x_3)$. From (39) for solving of the equation (35) $\xi(x_1,x_2,x_3)$ the estimation is obtained

$$\xi(x_1,x_2,x_3) \leqslant Fe^{\alpha_1x_1+\alpha_2x_2+\alpha_3x_3}B_1(\alpha_1x_1;\alpha_2x_2;\alpha_3x_3). \tag{56}$$

If one of the coefficients $\alpha_i$, $(i = 1,2,3)$ in $(35)$ is equal to zero, then $(56)$ yields to the estimation obtained in section 1.

Using the property for addition of numbers, it is possible on the basis of the general scheme to get the majored estimation, similar to $(56)$ for each $n$-dimensional equation of type $(12)$.

We bring into the conclusion such an estimation for solving equation $(29)$.

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