Mathematical
Inequalities
& Applications
Volume 5, Number 1 (2002), 155–162

COMPUTING THE FIXED–POINTS OF GENERAL MIXED VARIATIONAL INEQUALITIES

MUHAMMAD ASLAM NOOR

(*communicated by Th. Rassias*)

Abstract. In this paper, we suggest and analyze a class of predictor-corrector methods for computing the fixed-points of general mixed variational inequalities. The convergence of the proposed methods only requires the partially relaxed strongly monotonicity of the operator, which is weaker than co-coercivity. As special cases, we obtain a number of known and new results for solving various classes of variational inequalities and related problems.

1. Introduction

In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative technique to study a wide class of problem in pure and applied sciences. An important and useful generalization of variational inequalities is called the mixed variational inequality (or the variational inequality of the second kind) involving the nonlinear term. Such type of variational inequalities arise as a result of minization of the extremal mappings describing the equilibrium problems in economics and engineering sciences. For the applications and numerical methods, see [1–18] and the references therein. Due to the presence of the nonlinear term, projection method and its variant forms, Wiener-Hopf equations, descent methods cannot be extended and modified to suggest iterative methods for solving the mixed variational inequalities.

If the nonlinear term involving the general mixed variational inequalities is a proper, convex and lower-semicontinuous, then it has been shown that the general mixed variational inequalities are equivalent to the fixed point and resolvent equations. These alternative formulations have been used to develop a number of iterative type methods for solving mixed variational inequalities. In this approach, one has to evaluate the resolvent operator, which is itself a difficult problem. To overcome this drawback, the auxiliary principle technique has been developed, the origin of which can be traced back to Lions and Stampacchia [5]. In recent years, this technique has been used to suggest and analyze various iterative methods for solving various classes of variational inequalities. It can be shown that a several numerical methods including the projection , extragragient,

Mathematics subject classification (2000): 49J40, 90C33.

Key words and phrases: Variational inequalities, auxiliary principle, iterative methods, convergence, fixed points.

and Newton, can be obtained as special cases from this technique, see [4, 7–15, 17,18] and references therein. In this paper, we again use the auxiliary principle to suggest a class of predictor-corrector methods for solving general mixed variational inequalities. The convergence of these methods requires only that the operator is partially relaxed strongly monotone, which is weaker than co-coercive. Consequently, we improve the convergence results of previously known methods, which can be obtained as special cases from our results. Our results can be considered an extension of the results of Noor [7] for solving general variational inequalities and complementarity problems.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $< \cdot,\cdot >$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $\varphi : H \to R \cup \{+\infty\}$ be a nondifferentiable nonlinear function.

For given nonlinear operators $N(.,.) : H \times H \to H$ and $g : H \longrightarrow H$, consider the problem of finding $u \in H$ such that

$$\langle N(u,u), g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geqslant 0, \qquad \text{for all } g(v) \in Hn. \quad (2.1)$$

The inequality of type (2.1) is called the general mixed variational inequality or the general variational inequality of the second kind. If the function φ is proper, convex and lower-semicontinuous, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in N(u,u) + \partial\varphi(g(u)), \quad (2.2)$$

where $\partial\varphi$ is the subdifferential, which is a maximal monotone operator. It can be shown that a wide class of linear and nonlinear equilibrium problems arising in pure and applied sciences can be studied via the general mixed variational inequalities (2.1) and (2.2).

EXAMPLE 2.1. For simplicity and to convey an idea of applications of the general mixed variational inequalities (2.1), we consider the problem of computing a fixed-point of the extremal inclusion

$$u \in \text{argmin } \{ F(u,v) + \varphi(g(u)) : g(v) \in H\}, \quad (2.3)$$

where the function $F(u,v)$ is defined on the product space $H \times H$,. If $F(u,v)$ is convex with respect to v for every $u \in H$, then it can be shown that the minimum u of the extremal mapping defined by (2.3) can be characterized by a class of mixed variational inequalities of the type

$$\langle F_v(u,u), g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)), \quad \text{for all } g(v) \in H, \quad (2.4)$$

where $F_v(u,v)$ is the differential of $F(u,v)$ with respect to v. It has been shown in [1] that the quadratic and inverse parametric linear programming equilibrium problems arising in economics and engineering sciences can be reformulated in term of (2.3) and (2.4).

Special Cases

We remark that if $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle N(u,u), v-u \rangle + \varphi(v) - \varphi(u) \geqslant 0, \qquad \text{for all } v \in H, \tag{2.5}$$

which are called the mixed variational inequalities.

We note that if φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.1) is equivalent to finding $u \in H$, $g(u) \in K$ such that

$$\langle N(u,u), g(v) - g(u) \rangle \geqslant 0, \qquad \text{for all } g(v) \in K. \tag{2.6}$$

The inequality of the type (2.6) is known as the general variational inequality. For $N(u,u) \equiv Tu$, Problem (2.6) was introduced and studied by Noor [9] in 1988. It turned out that the odd-order and nonsymmetric free, unilateral, obstacle and equilibrium problems can be studied by the general variational inequality, see [9, 10, 12, 13].

If $K^* = \{u \in H : \langle u, v \rangle \geqslant 0, \text{for all } v \in K\}$ is a polar cone of a convex cone K in H and g is onto K, then problem (2.6) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^*, \text{and} \quad \langle N(u,u), g(u) \rangle = 0, \tag{2.7}$$

which is known as the general complementarity problem. We note that if $g(u) = u - m(u)$, where m is a point -to-point mapping, then problem(2.7) is called the quasi(implicit) complementarity problem. For $g \equiv I$, problem (2.7) is known as the generalized complementarity problem. For the formulation and numerical methods of complementarity problems, see [2, 3, 6, 12, 13].

For $g \equiv I$, the identity operator, problem (2.6) collapses to: find $u \in K$ such that

$$\langle N(u,u), v-u \rangle \geqslant 0, \qquad \text{for all } v \in K, \tag{2.8}$$

which is called the standard variational inequality. For the recent state-of-the art, see [1–18].

It is clear that problems (2.4)–(2.8) are special cases of the general mixed variational inequality (2.1). In brief, for a suitable and appropriate choice of the operators $N(.,.)$, g, φ and the space H, on can obtain a wide class of variational inequalities and complementarity problems. This clearly shows that problem (2.1) is quite general and unifying one. Furthermore, problem (2.1) has important applications in various branches of pure and applied sciences.

We also need the following concepts.

LEMMA 2.3. *For all $u, v \in H$, we have*

$$2\langle u, v \rangle = ||u+v||^2 - ||u||^2 - ||v||^2. \tag{2.9}$$

Proof. It is trivial.

DEFINITION 2.1. For all $u, v, z \in H$, an operator $N(.,.) : H \times H \to H$ is said to be:

(i) g**-partially relaxed strongly monotone**, if there exists a constant $\alpha > 0$ such that

$$\langle N(u,u) - N(v,v), g(z) - g(v)\rangle \geqslant -\alpha||g(u) - g(z)||^2$$

(ii) g**-co-coercive**, if there exists a constant $\mu > 0$ such that

$$\langle N(u,u) - N(v,v), g(u) - g(v)\rangle \geqslant \mu||N(u,u) - N(v,v)||^2.$$

We remark that if $z = u$, then g-partially relaxed strongly monotonicity is exactly g-monotonicity of the operator $N(.,.)$. For $N(u,u) \equiv Tu$, Definition 2.1 reduces to the standard definition of g-partially relaxed strongly monotonicity, and g-co-coercivity of the operator, see Noor [7]. Using the technique of Noor [7], it can be shown that g-co-coercivity implies g-partially relaxed strongly monotonicity. This shows that partially relaxed strongly monotonicity is a weaker condition than co-coercivity.

3. Main Results

In this section, we suggest and analyze a new iterative method for solving the problem (2.1) by using the auxiliary principle technique of Glowinski, Lions and Tremolieres [4] as developed by Noor [9, 12, 13, 15].

For a given $u \in H$ such that $g(u) \in H$, consider the problem of finding a unique $w \in H$ such $g(w) \in H$ satisfying the auxiliary variational inequality

$$\langle \rho N(u,u) + g(w) - g(u), g(v) - g(w)\rangle + \rho\varphi(g(v)) - \rho\varphi(g(u)) \geqslant 0, \quad \text{for all } v \in H, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is a solution of the general mixed variational inequality (2.1). This observation enables us to suggest the following iterative method for solving the general mixed variational inequalities (2.1).

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho N(w_n, w_n) + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle + \rho\varphi(g(v)) - \rho\varphi(g(u_{n+1})) \geqslant 0, \quad \text{for all } v \in H \tag{3.2}$$

and

$$\langle \beta N(u_n, u_n) + g(w_n) - g(u_n), g(v) - g(w_n)\rangle + \beta\varphi(g(v)) - \beta\varphi(g(w_n)) \geqslant 0, \quad \text{for all } v \in H, \tag{3.3}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Note that if $g \equiv I$, the identity operator, then Algorithm 3.1 reduces to:

ALGORITHM 3.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\langle \rho N(w_n, w_n) + u_{n+1} - w_n, v - u_{n+1}\rangle + \rho\varphi(v) - \rho\varphi(u_{n+1}) \geqslant 0, \quad \text{for all } v \in H,$$

and

$$\langle \beta N(u_n, u_n) + w_n - u_n, v - w_n\rangle + \beta\varphi(v) - \beta\varphi(w_n) \geqslant 0, \quad \text{for all } v \in H.$$

If φ is a proper, convex and lower-semicontinuous function, then Algorithm 3.1 collapses to:

ALGORITHM 3.3. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} g(u_{n+1}) &= J_\varphi[g(w_n) - \rho N(w_n, w_n)], \\ g(w_n) &= J_\varphi[g(u_n) - \beta N(u_n, u_n)], \quad n = 0, 1, 2 \ldots \end{aligned}$$

where J_φ is the resolvent operator associated with the maximal monotone operator $\partial\varphi$.

If the function φ is the indicator function of a closed convex set K in H, then Algorithm 3.1 reduces to the following method for solving general variational inequalities (2.6) and complementarity problems (2.7).

ALGORITHM 3.4. For a given $u_0 \in H$ such that $g(u_0) \in K$, compute u_{n+1} by the iterative schemes

$$\langle \rho N(w_n, w_n) + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle \geqslant 0, \quad \text{for all } g(v) \in K$$

and

$$\langle \beta N(u_n, u_n) + g(w_n) - g(u_n), g(v) - g(w_n)\rangle \geqslant 0, \quad \text{for all } g(v) \in K.$$

For a suitable choice of the operators $N(.,.), g$ and the space H, one can obtain various new and known methods for solving variational inequalities.

For the convergence analysis of Algorithm 3.1, we need the following result, which is proved by using the techniques of Noor [7].

LEMMA 3.1. *Let $\bar{u} \in H$ be the exact solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If the operator $N(.,.) : H \times H \longrightarrow H$ is g-partially relaxed strongly monotone operator with constant $\alpha > 0$, then*

$$||g(u_{n+1}) - g(\bar{u})||^2 \leqslant ||g(u_n) - g(\bar{u})||^2 - (1 - 2\rho\alpha)||g(u_{n+1}) - g(u_n)||^2. \tag{3.4}$$

Proof. Let $\bar{u} \in H$ be solution of (2.1). Then

$$\langle \rho N(\bar{u}, \bar{u})g(v) - g(\bar{u})\rangle + \rho\varphi(g(v)) - \rho\varphi(g(\bar{u})) \geqslant 0, \quad \text{for all } v \in H, \tag{3.5}$$

and

$$\langle \beta N(\bar{u}, \bar{u}), g(v) - g(\bar{u})\rangle + \beta\varphi(g(v)) - \beta\varphi(g(\bar{u})) \geqslant 0, \quad \text{for all } v \in H, \tag{3.6}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.5) and $v = \bar{u}$ in (3.2), we have

$$\langle \rho N(\bar{u}, \bar{u}), g(u_{n+1}) - g(\bar{u})\rangle + \rho\varphi(g(u_{n+1})) - \rho\varphi(g(\bar{u})) \geqslant 0 \tag{3.7}$$

and

$$\langle \rho N(w_n,w_n)+g(u_{n+1})-g(u_n), g(\bar{u})-g(u_{n+1})\rangle +\rho\varphi(g(\bar{u}))-\rho\varphi(g(u_{n+1})) \geqslant 0. \quad (3.8)$$

Adding (3.7) and (3.8), we have

$$\begin{aligned}\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle &\geqslant \rho\langle N(w_n,w_n) - N(\bar{u},\bar{u}), g(u_{n+1}) - g(\bar{u})\rangle \\ &\geqslant -\alpha\rho||g(u_{n+1}) - g(w_n)||^2, \qquad (3.9)\end{aligned}$$

where we have used the fact that $N(.,.)$ is g-partially relaxed strongly monotone with constant $\alpha > 0$.

Setting $u = g(\bar{u}) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(u_n)$ in (2.9), we obtain

$$\begin{aligned}\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle &= \frac{1}{2}\{||g(\bar{u}) - g(u_n)||^2 - ||g(\bar{u}) - g(u_{n+1})||^2 \\ &\quad - ||g(u_{n+1}) - g(u_n)||^2\}. \qquad (3.10)\end{aligned}$$

Combining (3.9) and (3.10), we have

$$||g(u_{n+1}) - g(\bar{u})||^2 \leqslant ||g(u_n) - g(\bar{u})||^2 - (1 - 2\alpha\rho)||g(u_{n+1}) - g(w_n)||^2. \quad (3.11)$$

Taking $v = \bar{u}$ in (3.3) and $v = w_n$ in (3.6), we have

$$\langle \beta N(\bar{u},\bar{u}), g(w_n) - g(\bar{u})\rangle + \beta\varphi(g(w_n)) - \beta\varphi(g(\bar{u})) \geqslant 0 \quad (3.12)$$

and

$$\langle \beta N(u_n,u_n) + g(w_n) - g(u_n), g(\bar{u}) - g(w_n)\rangle + \beta\varphi(g(\bar{u})) - \beta\varphi(g(w_n)) \geqslant 0. \quad (3.13)$$

Adding (3.12) and (3.13) and rearranging the terms, we have

$$\begin{aligned}\langle g(w_n) - g(u_n), g(\bar{u}) - g(w_n)\rangle &\geqslant \beta\langle N(u_n,u_n) - N(\bar{u},\bar{u}), g(w_n) - g(\bar{u})\rangle \\ &\geqslant -\beta\alpha||g(u_n) - g(w_n)||^2, \qquad (3.14)\end{aligned}$$

since $N(.,.)$ is g-partially relaxed strongly monotone operator with constant $\alpha > 0$.

Now taking $v = g(w_n) - g(u_n)$ and $u = g(\bar{u}) - g(w_n)$ in (2.9), (3.14) can be written as

$$\begin{aligned}||g(\bar{u}) - g(w_n)||^2 &\leqslant ||g(\bar{u}) - g(u_n)||^2 - (1 - 2\beta\alpha)||g(u_n) - g(w_n)||^2 \\ &\leqslant ||g(\bar{u})) - g(u_n)||^2, \quad \text{for } 0 < \beta < 1/2\alpha. \qquad (3.15)\end{aligned}$$

Consider

$$\begin{aligned}||g(u_{n+1}) - g(w_n)||^2 &= ||g(u_{n+1}) - g(u_n) + g(u_n) - g(w_n)||^2 \\ &= ||g(u_{n+1}) - g(u_n)||^2 + ||g(u_n) - g(w_n)||^2 \\ &\quad + 2\langle g(u_{n+1}) - g(u_n), g(u_n) - g(w_n)\rangle. \qquad (3.16)\end{aligned}$$

Combining (3.11), (3.15) and (3.16), we obtain

$$||g(u_{n+1}) - g(\bar{u})||^2 \leqslant ||g(u_n) - g(\bar{u})||^2 - (1 - 2\rho\alpha)||g(u_{n+1}) - g(u_n)||^2,$$

the required result. □

THEOREM 3.1. *Let* $g : H \longrightarrow H$ *be invertible and* $0 < \rho < \frac{1}{2\alpha}$. *Let* u_{n+1} *be the approximate solution obtained from Algorithm 3.1 and* $\bar{u} \in H$ *be the exact solution of (2.1), then* $\lim_{n\to\infty} u_n = \bar{u}$.

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Since $0 < \rho < \frac{1}{2\alpha}$. From (3.4) ,it follows that the sequence $\{||g(\bar{u}) - g(u_n)||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1-2\alpha\rho)||g(u_{n+1}) - g(u_n)||^2 \leqslant ||g(u_0) - g(\bar{u})||^2,$$

which implies that

$$\lim_{n\to\infty} ||g(u_{n+1}) - g(u_n)|| = 0. \tag{3.17}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (3.2) and (3.3), taking the limit $n_j \longrightarrow \infty$ and using (3.17), we have

$$\langle N(\hat{u},\hat{u}), g(v) - g(\hat{u})\rangle + \varphi(g(v)) - \varphi(g(\hat{u})) \geqslant 0, \quad \text{for all } v \in H,$$

which implies that $\hat{u}$ solves the general mixed variational inequality (2.1) and

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leqslant \|g(u_n) - g(\bar{u})\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} g(u_n) = g(\hat{u}).$$

Since g is invertible, so

$$\lim_{n\to\infty} (u_n) = \hat{u},$$

the required result. □

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(Received December 16, 2000)

Department of Mathematics and Statistics
Dalhousie University
Halifax, Nova Scotia, Canada B3H 3J5
e-mail: noor@mscs.dal.ca