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# SOME NEW INEQUALITIES FOR MOTZKIN NUMBERS

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(*communicated by J. Pečarić*)

*Abstract.* We prove some inequalities which follow from the log-convexity of the sequence of Motzkin numbers $M_n$ and from the log-concavity of the sequence $\frac{M_n}{n!}$.

## 1. Introduction

The Motzkin numbers $M_n$ were introduced for the first time in [5], as the number of ways of selecting $n$ points on a circle either singly or in pairs connected by non-crossing chords. Since then, many other combinatorial families counted by the same numbers have been found. In his recent book [6] Stanley lists some fifteen examples, relying mostly on the survey article [3]. The most popular objects enumerated by the Motzkin sequence are lattice paths from $(0,0)$ to $(n,0)$ with steps $(1,1)$, $(1,-1)$ and $(1,0)$ never falling below the $x$-axis. The $n$-th Motzkin number, $M_n$, is the number of such paths with exactly $n$ steps.

The property of log-convexity of the Motzkin sequence was first established algebraically in [1]. A combinatorial proof appeared little bit later [2], and recently two elementary proofs, based on simple "calculus" concepts were given ([4], [7]). In this paper we prove some inequalities for the Motzkin numbers which are consequences of their log-convexity.

For convenience of the reader, we list here the first few members of the Motzkin sequence:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_n$ | 1 | 1 | 2 | 4 | 9 | 21 | 51 | 127 | 323 | 835 |

## 2. The sequence $\frac{M_n}{n!}$.

DEFINITION. A sequence $(a_n)_{n\geqslant 0}$ of positive numbers is *logarithmically convex* (or log-convex for short) if $a_n^2 \leqslant a_{n-1}a_{n+1}$, for all $n \geqslant 1$. If the opposite inequality, $a_n^2 \geqslant a_{n-1}a_{n+1}$ holds for all $n \geqslant 1$, the sequence $(a_n)_{n\geqslant 0}$ is called *log-concave*.

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The property of log-convexity is equivalent with the condition that the sequence $x_n = \frac{a_n}{a_{n-1}}$ is non-decreasing for all $n \geqslant 1$. Similarly, the log-concavity of the sequence $(a_n)_{n\geqslant 0}$ is equivalent with non-increasing behavior of the sequence $\frac{a_n}{a_{n-1}}$.

PROPOSITION 1. *The sequence $a_n = \frac{M_n}{n!}$ is log-concave, for all $n \geqslant 0$.*

*Proof.* The Motzkin numbers satisfy the following two-term recursion [4]:

$$M_{n+1} = \frac{2n+3}{n+3}M_n + \frac{3n}{n+3}M_{n-1}.$$

Dividing this relation by $M_n$ and denoting $\frac{M_n}{M_{n-1}}$ by $x_n$, we get a recursion for the sequence $(x_n)_{n\geqslant 1}$:

$$x_{n+1} = \frac{2n+3}{n+3} + \frac{3n}{n+3}\frac{1}{x_n}.$$

From the log-convexity of the sequence $M_n$ we know that the sequence $x_n$ is non-decreasing. It is easy to see that the log-concavity of the sequence $\frac{M_n}{n!}$ is equivalent with the condition

$$x_{n+1} \leqslant \frac{n+1}{n}x_n,$$

for all $n \geqslant 1$. But,

$$x_{n+1} = \frac{2n+3}{n+3} + \frac{3n}{n+3}\frac{1}{x_n} \leqslant \frac{2n+3}{n+3} + \frac{3n}{n+3}\frac{1}{x_{n-1}}$$
$$= \frac{n+2}{n+3}\frac{n}{n-1}\left[\frac{(2n+3)(n-1)}{n(2n+1)}\frac{2n+1}{n+2} + \frac{3(n-1)}{n+2}\frac{1}{x_{n-1}}\right].$$

The term $\frac{(2n+3)(n-1)}{n(2n+1)}$ is clearly less than one for all $n \geqslant 1$, and the inequality

$$\frac{n+2}{n+3}\frac{n}{n-1} \leqslant \frac{n+1}{n}$$

is valid for all $n \geqslant 3$. Hence we get

$$x_{n+1} \leqslant \frac{n+1}{n}x_n,$$

for all $n \geqslant 3$. The validity of inequality

$$\left(\frac{M_n}{n!}\right)^2 \geqslant \frac{M_{n-1}}{(n-1)!}\frac{M_{n+1}}{(n+1)!}$$

for $1 \leqslant n \leqslant 3$ can be easily checked, and the claim follows. □

## 3. Consequences

The following two double inequalities follow now from log-convexity of Motzkin numbers and Proposition 1.

COROLLARY 2. $M_n^2 \leqslant M_{n-1}M_{n+1} \leqslant \left(1+\frac{1}{n}\right)M_n^2$, *for all* $n \geqslant 1$.

COROLLARY 3. $M_nM_m \leqslant M_{m+n} \leqslant \binom{m+n}{n}M_nM_m$, *for all* $n \geqslant 1$.

*Proof.* A simple combinatorial proof of the left inequality follows from the fact that concatenation of two Motzkin paths of lengths $m$ and $n$, respectively, gives a valid Motzkin path of length $m+n$.

To prove the right inequality, start from $x_n \geqslant \frac{n}{n+1}x_{n+1}$. By using this inequality repeatedly, we get

$$\frac{M_1}{M_0} \geqslant \frac{1}{2}\frac{M_2}{M_1} \geqslant \frac{1}{3}\frac{M_3}{M_2} \geqslant \ldots \geqslant \frac{1}{m+n}\frac{M_{m+n}}{M_{m+n-1}},$$

for all $n \geqslant 0$, $m \geqslant 1$.

Hence, for any $0 \leqslant j \leqslant m-1$, we have

$$\frac{M_{j+1}}{M_j} \geqslant \frac{j+1}{m+n}\frac{M_{m+n}}{M_{m+n-1}}.$$

From this we get

$$\frac{M_1}{M_0}\frac{M_2}{M_1}\frac{M_3}{M_2}\cdots\frac{M_m}{M_{m-1}} \geqslant \left(\frac{1}{n+1}\frac{M_{n+1}}{M_n}\right)\left(\frac{2}{n+2}\frac{M_{n+2}}{M_{n+1}}\right)\cdots\left(\frac{m}{m+n}\frac{M_{m+n}}{M_n}\right).$$

After the cancellations we get

$$\frac{M_m}{M_0} \geqslant \frac{n!m!}{(m+n)!}\frac{M_{m+n}}{M_n},$$

and, taking into account the fact that $M_0 = 1$, we finally get

$$M_{m+n} \leqslant \binom{m+n}{n}M_nM_m.$$

The case $m = 0$ is trivially valid for all $n \geqslant 0$. □

REMARK. The Motzkin sequence is a member of the family of the so called *secondary structure numbers* [8]. For some other members of this family, whose log-convexity has been recently established [4], similar results can be proved.

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