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ON THE MINIMUM NUMBER OF DISTINCT EIGENVALUES FOR A SYMMETRIC MATRIX WHOSE GRAPH IS A GIVEN TREE

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Abstract. It is shown that for any tree T the minimum number of distinct eigenvalues of an Hermitian matrix whose graph is T (diagonal entries free) is at least the number of vertices in a longest path of T. This is another step toward the general problem of characterizing the possible multiplicities for a given graph. Related observations are made and the result facilitates a table of multiplicities for trees on fewer than 8 vertices.

Let G be an undirected graph on n vertices throughout. If $A = (a_{ij})$ is an n–by–n Hermitian matrix, the graph of A, $G = G(A)$, is determined entirely by the off–diagonal entries of A, and $G(A)$ has an edge between distinct vertices i and j if and only if $a_{ij} \neq 0$. On the other hand, given G, $H(G) = \{A = A^* : G(A) = G\}$, the set of Hermitian matrices whose graph is G. We have been interested in the multiplicities that occur among the distinct eigenvalues of matrices $A \in H(G)$, and have thus far focused upon the case in which G is a tree. In this event, we have determined the maximum possible multiplicity in graph theoretic terms [5]. Here we are interested in the somewhat dual problem of the minimum number of distinct eigenvalues in a matrix in $H(G)$ in terms of the structure of the tree G. Armed with both results it is relatively easy to determine all possible lists of multiplicities for trees with modest numbers of vertices. In fact, for many trees any list allowed by both results is attained in $H(G)$. We present a table and some examples at the end.

Let $L(G) \equiv \{p = (p_1, \dots, p_q) : p_1 \geqslant \cdots \geqslant p_q$, there is an $A \in H(G)$ with distinct eigenvalues $\lambda_1, \dots, \lambda_q$, and λ_i has multiplicity $p_i\}$. Thus, each $p \in L(G)$ is a partition of n, and our general problem is to explicity describe $L(G)$ in terms of G. In [5] we characterized the maximum value of p_1 among partitions in $L(G)$ in a variety of ways, including showing that it is the path covering number of the tree G (which may be efficiently calculated). Here, we are interested in the minimum number of parts $q = q(G)$ in a partition in $L(G)$ (the minimum number of distinct eigenvalues over matrices in $H(G)$) and we call this number the *eigenwidth* of G. If we also denote the

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number of distinct eigenvalues of a matrix A by $q(A)$, then $q(G) = \min_{A \in H(G)} q(A)$. We shall relate $q(G)$ to the *diameter* $d(G)$, defined as follows for an arbitrary undirected graph G. Let l be the maximum, over pairs of vertices i and j of the minimum distance (measured in edges) in G between i and j. Then $d(G) = l + 1$. For a tree T, $d(T)$ is also then the number of vertices in a longest path in T.

We relate $d(G)$ and $q(G)$ by first showing that $q(A) \leqslant d(G(A))$ when A is an entry–wise nonnegative Hermitian matrix. A key fact to recall [4] is that for an Hermitian matrix, $q(A)$ is the degree of the minimal polynomial of A.

LEMMA 1. *If A is an n–by–n, entry–wise nonnegative, Hermitian matrix, then* $q(A) \geqslant d(G(A))$.

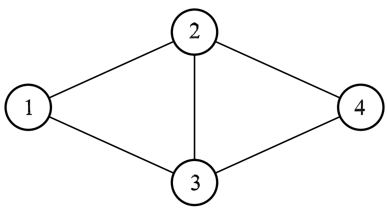
Proof. Let $d = d(G(A))$. We actually show that I, A, A^2, ..., A^{d-1} must be linearly independent, so that the degree of the minimal polynomial of A must be at least d, and, thus, that A has at least d distinct eigenvalues. Suppose, wlog, that the vertices of a path that attains $d(G(A))$ are numbered 1, 2, ..., d. Then, for $k = 2$, ..., d, A^{k-1} has a postive entry in the $1,k$ position, while A_j, $j < k-1$, does not. Thus, A^{k-1} cannot be a linear combination of lower powers, so that I, A, ..., A^{d-1} are linearly independent, as was to be shown. □

Arguments similar to the above have been used previously, e.g. [3,1], for the more limited objective of showing sufficientluy many distinct eigenvalues in a particular nonnegative matrix that is the adjacenct matrix of an undirected graph. Note that the entry–wise nonnegativity of A is important to the argument.

EXAMPLE. For

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$

$G(A)$ is



and $d(G(A)) = 3$. But $A^2 = 4A - I$, so that A has only the two distinct eigenvalues $2 \pm \sqrt{3}$, each with multiplicity 2.

Nonetheless, for trees the assumption of entry–wise nonnegativity is not important.

THEOREM 2. *For each tree T, $q(T) \geqslant d(T)$.*

Proof. Let $A \in H(T)$. Because T is a tree, there is a diagonal unitary similarity of A $(\hat{A} = U^*AU)$ that replaces the nonzero off–diagonal entries of A with their absolute values. Then $q(\hat{A}) = q(A)$. Now, choosing $t > 0$ sufficiently large, $B = tI + A$ is entry–wise nonnegative, lies in $H(T)$, and satisfies $q(B) = q(\hat{A}) = q(A)$. But, then lemma 1 applies and verifies that $q(A) \geqslant d(T)$. Since $A \in H(T)$ was arbitrary, the assertion of the theorem also follows.

CONJECTURE. We suspect that for each tree T, there is an $A \in H(T)$ with only $d(T)$ distinct eigenvalues, so that $q(T) = d(T)$ would follow.

Note that the classical fact that an irreducible, Hermitian, triagonal matrix has distinct eigenvalues is a special case of theorem 2. In this case T is a path, and we have $q(T) \geqslant d(T) = n$, implying n different eigenvalues. See also [2] for some related results for a cycle.

Using theorem 2 and the result in [5] to limit the number of cases to consider, we have determined $L(G)$ for each tree $G = T$ on fewer that 8 vertices. We omit details of the necessary constructions but list $L(G)$ with a depiction of the tree for all trees of 3 to 7 vertices. For simplicity, we omit the omnipresent all 1's vector from the list of partitions in $L(G)$ when there are other partitions present (non–paths). Such lists can be quite useful in making or dispensing with conjuctures. We also list with each graph the path cover number p (minimum number of vertex disjoint paths needed to cover all vertices) and the diameter q.

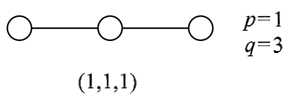


Figure 1. 3–Vertex Trees

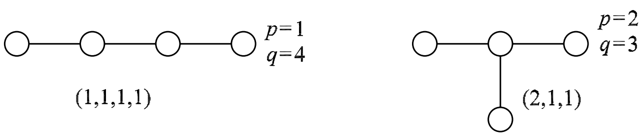


Figure 2. 4–Vertex Trees

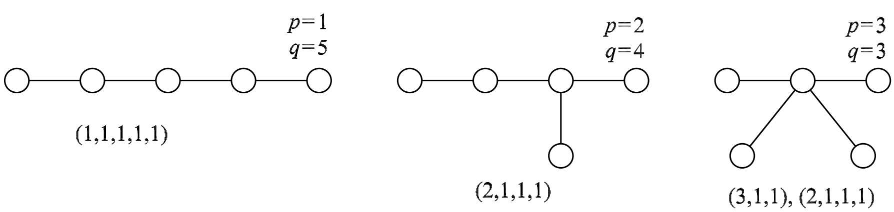


Figure 3. 5–Vertex Trees

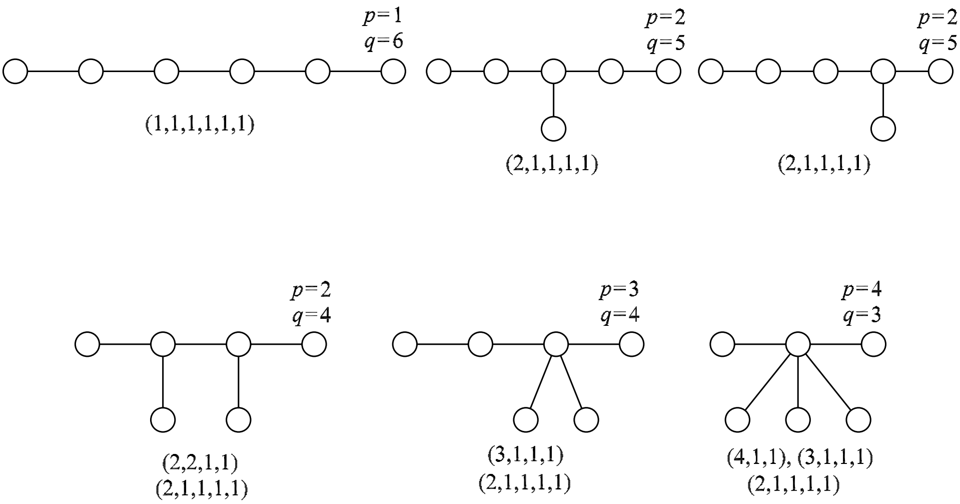
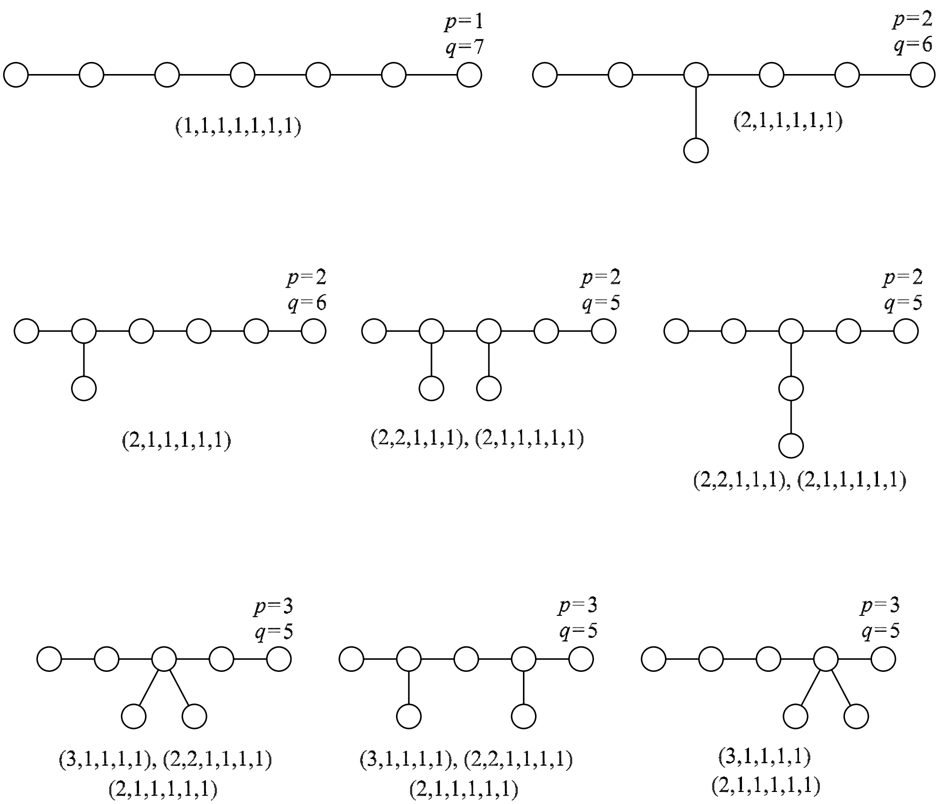


Figure 4. 6–Vertex Trees



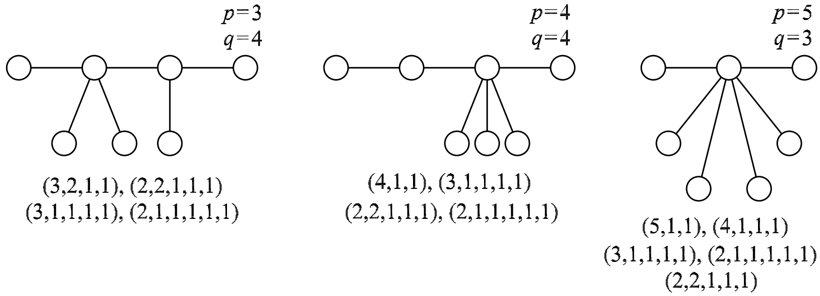


Figure 5. 7–Vertex Trees

Many observations may be made from the table; we note only a few. In all cases, the conjectured converse to our theorem is valid; in fact p and q are always realized together in a single element of $L(G)$. (The latter fails on a certain tree of 8 vertices.) In all but one case (contrast the 7^{th} and 8^{th} seven vertex trees), $L(G)$ is determined by the pair p, q; in the 8^{th} (7–vertex) graph, a partition allowed by p and q does not appear.

In spite of our example, it is possible to generalize theorem 2 to more general graphs. The difficulty in the example is that there are mulitple shortest paths that realize the longest distance (and thus the diameter), and, then it is possible to make the products of the entries (of a matrix with such a graph) along tied paths opposite in sign. The resulting possibility of cancellation renders invalid the proof of lemma 1 and its application to prove theorem 2. However, this may be repaired, and we close by modifying the definition of diameter to give generalizations of lemma 1 and then theorem 2.

A *path product* in either a numerical real symmetric matrix or a signed, undirected graph SG is defined in the natural way as the product of matrix entries, or signs, associated with the edges of the path. The *unambiguous diameter* of a symmetrically signed graph is then defined by $ud(SG) = 1 + \max d(i,j)$, in which $d(i,j)$ is the distance measured in edges from vertex i to vertex j in SG and the max is taken over all pairs i, j such that every path from i to j of length $d(i,j)$ has the same sign (as a path product). For an ordinary undirected graph G, we also define the "untied diameter" by $ud(G) = 1 + \max d(i,j)$, in which the max is taken over pairs i,j such that there is only one path of length $d(i,j)$ from i to j in G. Note that there is no ambiguity in using "ud" in both cases, because the function arguments are different. Also the two notions are naturally related, as for any symmetric signing SG of the undirected graph G, $ud(SG) \geqslant ud(G)$, because, when there is only one shortest path, it may be assigned only one path product. Of course, the signed graph of a real symmetric matrix A, $SG(A)$ is the signing of $G(A)$ in wich an edge gets the sing of the corresponding entry.

Analogous to our first lemma and theorem we have

LEMMA 3. *If A is an n–by–n real symmetric matrix, then $q(A) \geqslant ud(SG(A))$.*

and

THEOREM 4. *For each undirected graph G, $q(G) \geqslant ud(G)$.*

The proofs are similar to those of lemma 1 and theorem 2. Because all minimum length paths in a tree are unique, $ud(G) = d(G)$ when G is a tree, and it follows that theorem 2 is a special case of theorem 4. It is clear that lemma 1 is a special case of lemma 3.

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