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IMBEDDINGS OF ANISOTROPIC ORLICZ–SOBOLEV SPACES AND APPLICATIONS

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(*communicated by A. M. Fink*)

1. Introduction

Generalized Young functions (Young functions of several variables) were introduced and studied by S. Wang [26] and M. S. Skaff [20], [21]. The so called G-functions were studied by N. S. Trudinger [25] who introduced the space $L_G(\Omega)$ of such G-functions of n variables $(n \in \mathbf{Z}_+)$. This type of construction is very important as it enables us to describe the different integral behaviour of the derivatives in different directions. In [25], an imbedding theorem was proved for the completion of $C_0^1(\Omega)$ with respect to the norm $\|Du\|_{G,\Omega}$. A variant of this theorem is given in [11] for the space $W^1L_G(\Omega)$ of weakly differentiable functions u of $(n+1)$ variables with the norm $\|(u,Du)\|_{G,\Omega}$. In this paper we generalize this result to the traces on Ω^k $(k \leqslant n-1)$, where Ω^k is the k-dimensional hyperplane. This means that we prove some new inequalities involving Orlicz-Sobolev norms. Moreover, we present an application of these inequalities to variational problems.

The paper is organized as follows. In Section 2, we give some notations and terminology which we shall be using in the sequel. In Section 3 we discuss some imbedding properties of Orlicz-Sobolev spaces. Section 4 contains the continuous imbedding and two compact imbeddings. Some applications are given in Section 4. Finally, Section 5 is left for some concluding remarks.

2. Preliminaries

A Young function $A : [0,\infty) \to [0,\infty]$ is a function defined by

$$A(t) = \int_0^t a(x)dx$$

where $a : [0,\infty) \to [0,\infty]$ is an increasing, left continuous function which is neither identically zero nor identically infinity on $(0,\infty)$.

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The Orlicz space $L_A(\Omega)$, $\Omega \subset \mathbf{R}^n$, is defined as the set of all (equivalance classes of) measurable functions f on Ω such that $\|f\|_{A,\Omega} < \infty$, where $\|\cdot\|_{A,\Omega}$ denotes the Luxemburg norm on $L_A(\Omega)$ given by

$$\|\cdot\|_{A,\Omega} = \inf\left\{\theta > 0 : \int_\Omega A\left(\frac{|f(x)|}{\theta}\right) dx \leqslant 1\right\}.$$

A G-function of n variables $G : \mathbf{R}^n \to [0,\infty]$ is a function satisfying the following properties :

(i) $G(0) = 0$;
(ii) $\lim_{|x|\to\infty} G(x) = \infty$, $\left[x \in R^n : |x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}\right]$;
(iii) G is convex i.e.

$$G(\lambda x + (1-\lambda)y) \leqslant \lambda G(x) + (1-\lambda)G(y)$$

for all $0 \leqslant \lambda \leqslant 1$, $x, y \in \mathbf{R}^n$;
(iv) G is symmetric i.e. $G(-x) = G(x)$, $x \in \mathbf{R}^n$;
(v) the set $G^{-1}(\infty) = \{x \in R^n; G(x) = \infty\}$ is separated from 0;
(vi) G is lower semi-continuous.

Clearly, G function of 1 variable is a Young function.

We shall be assuming, in addition, that (vii) G is monotonically increasing in each variable separately. The vector valued Orlicz-space $L_G(\Omega)$ is defined as follows :

Let G be a G-function and let Ω be a domain in $\mathbf{R}^n$. Further, let $u_1, u_2, ..., u_n$ be real valued measurable functions defined on Ω and let $u = (u_1, u_2, ..., u_n)$ be a vector valued function. Then, u is said to belong to $L_G(\Omega)$ if there exists a $\lambda > 0$ such that

$$\int_\Omega G(\lambda u(x)) < \infty.$$

The space $L_G(\Omega)$ is equipped with a norm corresponding to the Luxemburg norm given by

$$\|u\|_{G,\Omega} = \inf\left\{\theta > 0 : \int_\Omega G\left(\frac{|u|}{\theta}\right) dx \leqslant 1\right\}.$$

It is noted that the space $L_G(\Omega)$ so defined is a Banach space. Let us point out that there should not be any ambiguity for the same notations $L_A(\Omega)$ and $L_G(\Omega)$ (also $\|\cdot\|_{A,\Omega}$ and $\|\cdot\|_{G,\Omega}$) used, respectively, for Young function and G-function. Moreover, we have used the symbols A, B, C for Young functions and G, H for G-functions. For a G-function G, the complementary function G_+^* is defined by

$$G_+^*(u) = \sup_{\substack{v_i \geqslant 0 \\ i=1,2,...,n}} (u.v - G(v)),$$

where $u.v = \sum_{i=1}^{n} u_i v_i$. For $u \in L_G(\Omega)$ and $v \in L_{G_+^*}(\Omega)$, the following Hölder's inequality holds:

$$\int_{\Omega} u.v dx \leqslant 2 \|u\|_{G,\Omega} \|v\|_{G_+^*,\Omega} . \tag{1}$$

Let $G : \mathbf{R}^{n+1} \to [0,\infty]$ be a G-function. The anisotropic Orlicz-Sobolev space, denoted by $W^1L_G(\Omega)$, is defined to be the space of weakly differentiable functions u for which $(u, Du) = (u, D_1u, D_2u, ..., D_nu)$ belongs to $L_G(\Omega)$. A norm for the space $W^1L_G(\Omega)$ is given by

$$\|u\|_{1,G,\Omega} = \|(u, Du)\|_{G,\Omega} .$$

A domain $\Omega \subset \mathbf{R}^n$ is said to be admissible if there exists a constant α (depending only upon n) such that

$$\|u\|_{\frac{n}{n-1},\Omega} \leqslant \alpha \|u\|_{1,1,\Omega} \quad , \qquad u \in W^{1,1}(\Omega)$$

where $\|\cdot\|_{\frac{n}{n-1},\Omega}$ and $\|\cdot\|_{1,1,\Omega}$ denote, respectively, the norms in the Lebesgue space $L^{\frac{n}{n-1}}(\Omega)$ and in the Sobolev space $W^{1,1}(\Omega)$. A domain Ω is said to have the cone property if there exists a finite cone K such that each point $x \in \Omega$ is the vertex of a finite cone K_x contained in Ω and congruent to K. Finally, we shall be using the symbols $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$ for, respectively, continuous and compact imbeddings. For further details regarding the concepts given in this section, one may refer to the monographs [1] and [17].

3. On some anisotropic Orlicz-Sobolev spaces

Let Ω be a domain in $\mathbf{R}^n$ and let G be a G-function of n variables (equal to the dimension of the space). For $u \in C_0^1(\Omega)$, define the space $H^0(G,\Omega)$ as the completion of the space $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{H^0(G,\Omega)} = \|Du\|_G ,$$

where Du is treated as a vector valued function in $L_G(\Omega)$. This gives the anisotropic character of the function $u \in H^0(G,\Omega)$. The space $H^0(G,\Omega)$ was introduced by Trudinger [25] where he proved the following famous imbedding theorem :

THEOREM A. *Let $\Omega \subset \mathbf{R}^n$ be a domain, $f_1, f_2, ..., f_n$ be continuous non-negative non-decreasing functions on $[0,\infty)$ and let Let $G : \mathbf{R}^{n+1} \to [0,\infty]$ be a G-function of $n+1$ variables such that*

$$G_+^* \left(0, f_1(s), f_2(s), ..f_n(s)\right) \leqslant s.$$

Also assume that $\int_0^1 \frac{ds}{m(s)} < \infty$, where

$$m(s) = s \left(\prod_{i=1}^{n} f_i(s) \right)^{\frac{1}{n}} .$$

Then, the continuous imbedding

$$H^0(G,\Omega) \hookrightarrow L_A(\Omega)$$

holds for any Young function A *satisfying*

$$\int_0^t \frac{ds}{m(s)} \leqslant kA^{-1}(t).$$

Let us mention that following Trudinger, people have worked with Orlicz-Sobolev spaces of anisotropic nature e.g. one can see [13], [16] and very recently Cianchi [6] derived an anisotropic Sobolev inequality which is more general than those discussed above.

In the literature, a different kind of anisotropy has been considered which is in terms of so called "mixed norms". These mixed norms in Lebesgue spaces were first considered in [5] and then many people followed e.g. see [14], [18] and [19]. In the setting of Orlicz spaces, mixed norms were initiated by Firlej and Matuszewska [10] (see also [9]). Our results have no concern with mixed norms but since this norm gives rise an anisotropic space, a few lines have been mentioned.

In [11], the following theorem was proved which is a variant of the Trudinger's Theorem A :

THEOREM B. *Let* Ω *be a bounded admissible domain in* $\mathbf{R}^n$, f *be a continuous non-negative function on* $[0,\infty)$ *and* G *be a* G*-function of* $(n+1)$ *variables on* $[0,\infty)$ *such that*

$$G_+^*\left(0,f(s),f(s),...f(s)\right) \leqslant s.$$

Further, let A *be a Young function given by*

$$A^{-1}\left(|t|\right) = \frac{1}{\eta}\int_0^{|t|} \frac{ds}{s^{1/n}f(s)}ds$$

for some constant $\eta > 0$. *Then, the continuous imbedding*

$$W^1L_G(\Omega) \hookrightarrow L_A(\Omega)$$

holds.

In Theorems A and B (and also in all the results obtained by others mentioned above in this section), the spaces are considered in which the functions are defined on Ω. From the application point of view it seems also reasonable to deal with spaces where the functions are defined on the boundary $\partial\Omega$ of Ω. Such boundary values (or traces) can even be defined on the intersection of a $k-$ dimensional hyperplane with Ω (this intersection is denoted by Ω^k). A good account of results concerning Lebesgue spaces and Sobolev spaces with traces is given in [17].

Our aim, in this paper, is to estabilish the imbedding in Theorem B for traces on Ω^k. We also give compactness of this imbedding and also for the imbedding in Theorem B.

4. Imbedding properties of Orlicz-Sobolev spaces

We begin with the following theorem :

THEOREM 1. *Let Ω be a bounded domain in $\mathbf{R}^n$ having the cone property and let Ω^k ($1 \leqslant k \leqslant n$) denotes the intersection of Ω with a k-dimensional hyperplane in $\mathbf{R}^n$. Let $G : \mathbf{R}^{n+1} \to [0,\infty]$ be a $G-$function and suppose that f is a continuous, non-negative function on $[0,\infty)$ such that*

$$G_+^*\left(0,f(s),f(s),..f(s)\right) \leqslant s \tag{2}$$

holds. Let A be a Young function such that

$$A^{-1}\left(|t|\right) = \frac{1}{\eta}\int_0^{|t|}\frac{ds}{s^{\frac{1}{n}-\frac{1}{p}+1}f(s)} \tag{3}$$

for some constant $\eta > 0$ and $p \in [1,n)$, where p is such that

$$H(t_1,t_2,...,t_{n+1}) = G(t_1^{1/p},t_2^{1/p},...,t_{n+1}^{1/p}) \tag{4}$$

is also a G-function. If either $n-p < k \leqslant n$ or $p = 1$ and $n-1 \leqslant k \leqslant n$, then the imbedding

$$W^1L_G(\Omega) \hookrightarrow L_{A^{k/n}}(\Omega^k)$$

holds, where $A^{k/n}(t) = [A(t)]^{k/n}$

REMARK. When Ω has the cone property Theorem 1 contains Theorem B which can be obtained by taking $p = 1$ and $k = n$ and using the fact that a domain having cone property is admissible (see [8]).

For proving Theorem 1, we need the following:

LEMMA 1. [1, Lemma 5.19] *Let Ω be a domain in $\mathbf{R}^n$ having the cone property and let Ω^k denote the intersection of Ω with some k-dimensional plane in $\mathbf{R}^n$, where $1 \leqslant k \leqslant n$ $(\Omega^n \equiv \Omega)$. If $n \geqslant mp$ and $n - mp < k \leqslant n$, then the imbedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega^k)$$

holds for $p \leqslant q \leqslant kp/(n-mp)$ if $n > mp$, or for $p \leqslant q < \infty$ if $n = mp$. If $p = 1$, $n > m$ and $n - m \leqslant k \leqslant n$, then the above imbedding holds for $1 \leqslant q \leqslant k/(n-m)$.

Proof of Theorem 1. It can be verified that $A^{k/n}$ is a Young function. We shall first prove the assertion for a bounded function $u \in W^1L_G(\Omega)$.

If we take $\theta = \|u\|_{A^{k/n},\Omega^k}$, then

$$\int_{\Omega^k} A^{k/n}\left(\frac{|u(t)|}{\theta}\right)dt = 1. \tag{5}$$

Set

$$h(t) = [A(t)]^{\frac{1}{p}-\frac{1}{n}}.$$

Then, (5) and Lemma 1 give

$$\begin{aligned} 1 &= \left(\int_{\Omega^k}\left[A\left(\frac{|u(t)|}{\theta}\right)\right]^{k/n}dt\right)^{\frac{n-p}{k}} \\ &= \left\|h\left(\frac{|u(t)|}{\theta}\right)\right\|^p_{\frac{kp}{k-p},\Omega^k} \\ &\leqslant K_1\left[\sum_{i=1}^n\int_\Omega\left|D_ih\left(\frac{|u(t)|}{\theta}\right)\right|^p dx+\left\|h\left(\frac{|u(t)|}{\theta}\right)\right\|^p_{p,\Omega}\right] \\ &= \frac{K_1}{\theta^p}\sum_{i=1}^n\int_\Omega\left|h'\left(\frac{|u|}{\theta}\right)D_iu\right|^p dx+K_1\left\|h\left(\frac{|u(t)|}{\theta}\right)\right\|^p_{p,\Omega} \qquad (6)\end{aligned}$$

for some constant K_1.

In view of the Hölder's inequality (1), we have

$$\sum_{i=1}^n\int_\Omega\left|h'\left(\frac{|u|}{\theta}\right)D_iu\right|^p\leqslant 2\left\|\left(0,h'\left(\frac{|u|}{\theta}\right),...,h'\left(\frac{|u|}{\theta}\right)\right)^p\right\|_{H_+^*,\Omega}\left\|(u,D_1u,...,D_nu)^p\right\|_{H,\Omega}, \tag{7}$$

where we use the symbol

$$(t_1,t_2,...,t_n)^p=\left(t_1^p,t_2^p,...,t_n^p\right).$$

Now, in view of (4), we note that

$$\left\|(u,D_1u,...,D_nu)^p\right\|_H\leqslant\|(u,D_1u,...,D_nu)\|_G^p=\|u\|^p_{1,G,\Omega}. \tag{8}$$

Thus, from (6), (7) and (8), we obtain

$$1\leqslant\frac{2K_1}{\theta^p}\left\|\left(0,h'\left(\frac{|u|}{\theta}\right),...,h'\left(\frac{|u|}{\theta}\right)\right)^p\right\|_{H_+^*,\Omega}\|u\|^p_{1,G,\Omega}+K_1\left\|h\left(\frac{|u(t)|}{\theta}\right)\right\|^p_{p,\Omega} \tag{9}$$

Further, by the definition of h and (3), we have

$$h'(y)=\left[A^{\frac{1}{p}-\frac{1}{n}}(y)\right]'=\eta\left(\frac{1}{p}-\frac{1}{n}\right)f(A).$$

Using this along with (4), we obtain from (9)

$$1\leqslant\frac{2K_1}{\theta^p}(\frac{1}{p}-\frac{1}{n})\left\|0,f\left[A\left(\frac{|u|}{\theta}\right)\right],...,f\left[A\left(\frac{|u|}{\theta}\right)\right]\right\|^p_{G_+^*}\|u\|^p_{1,G,\Omega}+K_1\left\|h\left(\frac{|u(t)|}{\theta}\right)\right\|^p_{p,\Omega}. \tag{10}$$

Now, recall that $\theta=\|u\|_{A^{k/n},\Omega^k}$. The aim is to show that there exists a constant $K_2>0$ such that

$$\theta\leqslant K_2\|u\|_{1,G,\Omega}$$

but in view of Theorem B, the last estimate holds for the special case $k = n$ and $p = 1$ and thus without any loss of genarality, we may assume that

$$\|u\|_{A^{n/n},\Omega^n} = \|u\|_{A,\Omega} \leqslant \theta. \tag{11}$$

From (2) and (11), we get

$$\int_\Omega G_+^* \left(0, f\left[A\left(\frac{|u|}{\theta}\right)\right], ..., f\left[A\left(\frac{|u|}{\theta}\right)\right]\right) dx \leqslant \int_\Omega A\left(\frac{|u|}{\theta}\right) dx \leqslant 1$$

and so

$$\left\|0, f\left[A\left(\frac{|u|}{\theta}\right)\right], ..., f\left[A\left(\frac{|u|}{\theta}\right)\right]\right\|_{G_+^*}^p \leqslant 1.$$

Using this in (10), we obtain

$$1 \leqslant \frac{2K_1}{\theta^p}(\frac{1}{p} - \frac{1}{n}) \|u\|_{1,G,\Omega}^p + K_1 \left\|h\left(\frac{|u(t)|}{\theta}\right)\right\|_{p,\Omega}^p. \tag{12}$$

Setting $\phi(t) = \frac{A(t)}{t^p}$ and $\psi(t) = \left(\frac{h(t)}{t}\right)^p$, we observe that

$$\frac{\phi(t)}{\psi(t)} = A^{p/n}(t) \to \infty \text{ as } t \to \infty$$

and therefore for each $\epsilon > 0$, there exists a constant K_3 (depending only upon ϵ) such that

$$\psi(t) \leqslant \epsilon\phi(t) + K_3$$

or

$$(h(t))^p \leqslant \epsilon A(t) + K_3 t^p$$

which along with (11) gives

$$\begin{aligned}\left\|h\left(\frac{|u|}{\theta}\right)\right\|_{p,\Omega}^p &\leqslant \epsilon \int_\Omega A\left(\frac{|u|}{\theta}\right) dx + \frac{K_3}{\theta^p}\int_\Omega |u|^p\, dx \\ &\leqslant \epsilon + \frac{K_3}{\theta^p}\left\||u|^p\right\|_{1,\Omega}. \end{aligned}\tag{13}$$

An application of Hölder's inequality (1) and (4) yield

$$\begin{aligned}\left\||u|^p\right\|_{1,\Omega} &\leqslant 2\|(1,0,...,0)\|_{H_+^*,\Omega} \left\|(|u|, |D_1 u|, ..., |D_n u|)^p\right\|_{H,\Omega} \\ &\leqslant 2\|(|u|, |D_1 u|, ..., |D_n u|)\|_{G,\Omega}^p = 2\|u\|_{1,G,\Omega}^p.\end{aligned}$$

Now, using the last estimate and (13) in (12), we get

$$1 \leqslant \frac{2K_1}{\theta^p}(\frac{1}{p} - \frac{1}{n}) \|u\|_{1,G,\Omega}^p + \epsilon K_1 + \frac{2K_1 K_3}{\theta^p} \|u\|_{1,G,\Omega}^p.$$

Choosing $\epsilon = \frac{1}{2K_1}$ and using the definition of θ, we obtain

$$\|u\|_{A^{k/n},\Omega^k}^p \leqslant K_2 \|u\|_{1,G,\Omega}^p,$$

where $K_2 = 4K_1\left(\frac{1}{p} - \frac{1}{n} + K_3\right)$ which depends only upon n. This establishes the theorem for bounded $u \in W^1L_G(\Omega)$.

In the case of arbitrary function $u \in W^1L_G(\Omega)$, define

$$u_\beta(t) = \begin{cases} u(t) & , |u(t)| \leqslant \beta \\ \beta \cdot sgn\, u(t) & , |u(t)| > \beta. \end{cases}$$

Then, u_β is bounded and by so called "Chain Rule" ([1], Lemma 8.31) belongs to $W^1L_G(\Omega)$. Also, $\|u_\beta\|_{A,\Omega}$ increases with β but bounded by $K_2\|u\|_{1,G,\Omega}$ and therefore $\lim_{\beta\to\infty}\|u_\beta\|_{A,\Omega} = \theta$ exists. By Fatou's lemma

$$\int_\Omega A\left(\frac{|u|}{\theta}\right)dt \leqslant \lim_{\beta\to\infty}\int_\Omega A\left(\frac{|u_\beta|}{\theta}\right)dt \leqslant 1$$

and consequently $u \in L_A(\Omega)$. Thus , the theorem is proved for arbitrary u too.

Now, we proceed to establish the compactness of the imbeddings given in Theorems B and 1. For that we need a notation and a lemma which we give below:

A notation: For two functions A and B, we shall write $A \prec\prec B$, if for every $\lambda > 0$

$$\lim_{t\to\infty}\frac{A(t)}{B(\lambda t)} = 0$$

and for this situation, we usually say that A increases essentially more slowly than B near infinity.

LEMMA 2. ([1], Theorem 8.23.) *Let Ω be a domain in $\mathbf{R}^n$ with finite volume. Let A and B be Young functions such that $B \prec\prec A$. Then, any bounded subset S of $L_A(\Omega)$ which is precompact in $L^1(\Omega)$ is also precompact in $L_B(\Omega)$.*

THEOREM 2. *Assume that all the hypothesis in Theorem B hold. If B is a Young function such that $B \prec\prec A$, then the compact imbedding*

$$W^1L_G(\Omega) \hookrightarrow\hookrightarrow L_B(\Omega)$$

holds.

Proof. Let $u \in W^1L_G(\Omega)$. By Hölder's inequality (1), we have

$$\|u\|_{1,\Omega} \leqslant 2\,\|(1,0,...,0)\|_{G_+^*,\Omega}\,\|(u,D_1u,...,D_nu)\|_{G,\Omega} \tag{14}$$

and

$$\|Du\|_{1,\Omega} \leqslant 2\,\|(0,1,...,1)\|_{G_+^*,\Omega}\,\|(u,D_1u,...,D_nu)\|_{G,\Omega}\,. \tag{15}$$

Now, (14) and (15) give the imbedding

$$W^1L_G(\Omega) \hookrightarrow W^{1,1}(\Omega).$$

Further, we have the trivial imbedding

$$W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$$

which, by Rellich-Kondrachov Theorem, is compact. Hence, if S is any bounded subset in $W^1L_G(\Omega)$, it is bounded in $L_A(\Omega)$ and precompact in $L_B(\Omega)$ by Lemma 1. This proves the result.

THEOREM 3. *Let all the assumptions from Theorem 1 hold. If $p > 1$ and C is any Young function such that $C \prec\prec A^{k/n}$, then we have the imbedding*

$$W^1L_G(\Omega) \hookrightarrow\hookrightarrow L_C(\Omega).$$

Proof. Since $H(t_1, t_2, ..., t_n) = G(t_1^{1/p}, t_2^{1/p}, ..., t_n^{1/p})$ is a Young function, we can apply Hölder's inequality as in Theorem 2 to get the imbedding

$$W^1L_G(\Omega) \hookrightarrow W^{1,p}(\Omega),$$

Ω being bounded. Already, the imbedding

$$W^{1,p}(\Omega) \hookrightarrow L^1(\Omega^k)$$

is known to exists which is compact as well, again, by the Rellich-Kondrachov Theorem. By using the same argument as in Theorem 2 the result now follows.

5. Application to variational problems

Let all the hypotesis of Theorem 2 hold with the following additional assumption

$$G(u, x_1, x_2, ..., x_n) = B(|u|) + G_0(x_1, x_2, ..., x_n),$$

where $G_0(x_1, x_2, ..., x_n) = G(0, x_1, x_2, ..., x_n)$. Moreover we assume that G and G^* are continuous and satisfy the Δ_2 condition (so that $W^1L_G(\Omega)$ is reflexive). Let $f : \Omega \times R^n \to \overline{R}$ be a Caratheodory function, i.e.

$$\begin{cases} f(x, \cdot) \text{ is continuous for a.e. } x \in \Omega \\ f(\cdot, \xi) \text{ is Lebesgue measurable for each } \xi \in R^n \end{cases}.$$

We assume that

1. f is convex in the second variable
2. There exist $a_f \in L^1(\Omega)$ and a constant $c_f > 0$ such that

$$f(x, \xi) \geqslant c_f G(0, \xi_1, ..., \xi_n) - a_f(x)$$

   for a.e. $x \in \Omega$ and every $\xi \in R^n$.

Let $g : \Omega \times R \to \overline{R}$ be a Caratheodory function and assume that

1. g is lower semicontinuous in the second variable
2. There exist $a_g \in L^1(\Omega)$ and a constant $c_g > 0$ such that

$$g(x, u) \geqslant c_g B(|u|) - a_g(x).$$

PROPOSITION 1. *The functional F defined by $F(u) = \int_\Omega f(x, Du(x))dx$ is lower semicontinuous in the weak topology of $W^1L_G(\Omega)$, i.e. if $u_h \rightharpoonup u$ weakly in $W^1L_G(\Omega)$ then the following inequality holds*

$$F(u) \leqslant \liminf_{h\to\infty} F(u_h).$$

Proof. We first prove that the functional $F_1(w) = \int_\Omega f(x, w(x))dx$ is lower semicontinuous in $L_{G_0}(\Omega)$. Let w_h converge to w in $L_{G_0}(\Omega)$ such that $\lim_{h\to\infty} F_1(w_h)$ exists. If we can prove that $F_1(w) \leqslant \lim_{h\to\infty} F_1(w_h)$ we are done. It is possible to prove that there exists a subsequence (still denoted (w_h)) such that $w_h \to w$ and that $\int_\Omega (G_0(w_h(x)))\,dx \to \int_\Omega (G_0(w(x)))\,dx$. Indeed, by Fatou's Lemma and the definition of the norm in $L_{G_0}(\Omega)$ it follows that

$$\int_\Omega G_0(\frac{u}{\|u\|})dx \leqslant 1.$$

(choose a decreasing sequence (k), $k \to \|u\|$). Thus, using Fatou's Lemma again we obtain that

$$\int_\Omega \liminf_{h\to\infty} G_0(\frac{w_h - w}{\theta_h})dx \leqslant 1$$

where $\theta_h = \|w_h - w\|$. Hence by the lower semicontinuity of G_0 it follows that

$$G_0(\liminf_{h\to\infty} \frac{w_h - w}{\theta_h}) \leqslant \liminf_{h\to\infty} G_0(\frac{w_h - w}{\theta_h}) < \infty \text{ a.e.},$$

and by property (ii) and (iv) of the G-function we obtain that

$$\liminf_{h\to\infty} \frac{|w_h - w|}{\theta_h} < \infty \text{ a.e.}$$

This shows that there exists a subsequence (still denoted (w_h)) such that $w_h \to w$ a.e.. By the convexity we have

$$\begin{aligned} G_0(w_h(x)) &= G_0(\theta_h\left(\frac{w_h(x) - w(x)}{\theta_h}\right) + (1-\theta_h)\frac{w(x)}{1-\theta_h}) \\ &\leqslant \theta_h G_0\left(\frac{w_h(x) - w(x)}{\theta_h}\right) + (1-\theta_h)\,G_0(\frac{w(x)}{1-\theta_h}), \end{aligned}$$

i.e.

$$\int_\Omega G_0(w_h(x))dx \leqslant \theta_h + (1-\theta_h)\int_\Omega G_0(\frac{w(x)}{1-\theta_h})dx. \tag{16}$$

Moreover, by assuming that $\theta_h < \frac{1}{2}$ and by letting m be such that $\|w\| \leqslant 2^m$, the Δ_2-condition yields that

$$G_0(\frac{w(x)}{1-\theta_h}) \leqslant G_0(2w(x)) \leqslant kG_0(w(x))$$

where

$$\begin{aligned}\int_\Omega G_0(w)dx &= \int_\Omega G_0(\|w\|\left(\frac{w}{\|w\|}\right))dx \\ &\leqslant \int_\Omega G_0(2^m\left(\frac{w}{\|w\|}\right))dx \leqslant k^m \int_\Omega G_0(\frac{w}{\|w\|})dx \leqslant k^m < \infty.\end{aligned}$$

Thus the Lebesgue Dominated convergence theorem gives that

$$\lim_{h\to\infty} \int_\Omega G_0(\frac{w(x)}{1-\theta_h})dx = \int_\Omega G_0(w(x))dx$$

and therefore, by (16), we have that

$$\limsup_{h\to\infty} \int_\Omega G_0(w_h(x))dx \leqslant \int_\Omega G_0(w)dx.$$

Moreover, Fatou's Lemma yields

$$\int_\Omega G_0(w_h(x))dx \leqslant \liminf_{h\to\infty} \int_\Omega G_0(w_h(x))dx,$$

so we obtain the convergence

$$\int_\Omega (G_0(w_h(x)))\, dx \to \int_\Omega (G_0(w(x)))\, dx.$$

Because $f(x,\cdot)$ and G_0 are continuous, it holds that

$$f(x,w(x)) - c_f G_0(w(x)) + a_f(x) = \lim_{h\to\infty} (f(x,w_h(x)) - c_f G_0(w_h(x)) + a_f(x)) \text{ a.e.}$$

Thus, Fatou's Lemma, applied on the sequence

$$f(x,w_h(x)) - c_f G_0(w_h(x)) + a_f(x) \geqslant 0,$$

gives that

$$\int_\Omega f(x,w(x))dx \leqslant \lim_{h\to\infty} \int_\Omega f(x,w_h(x))dx$$

and it follows that $F_1(w) = \int_\Omega f(x,w(x))dx$ is lower semicontinuous in $L_{G_0}(\Omega)$. This implies lower semicontinuity of the functional $F(u) = \int_\Omega f(x,Du(x))dx$ in $W^1L_G(\Omega)$) (only using the fact that $u_h \to u$ in $W^1L_G(\Omega)$ implies that $Du_h \to Du$ in $L_{G_0}(\Omega)$). Hence, since F trivially is convex and since $W^1L_G(\Omega)$ is a locally convex Hausdorff topological vector space it follows that F is lower semicontinuous in the weak topology (this result is classical, see e.g. [7] p. 14).

PROPOSITION 2. *The functional F_2 defined by $F_2(u) = \int_\Omega g(x,u(x))dx$ is sequentially lower semicontinuous in the weak topology of $W^1L_G(\Omega)$.*

Proof. If $u_h \rightharpoonup u$ weakly in $W^1L_G(\Omega)$ then u_h is norm bounded by the Banach Steinhaus Theorem. Thus Theorem 2 yields that u_h contains a subsequence (still denoted u_h) such that $u_h \to u$ strongly in $L_B(\Omega)$. Now the lower semicontinuity on $W^1L_G(\Omega)$ follows by the fact that F_2 is lower semicontinuous on $L_B(\Omega)$ (which is seen by using the functional F_1 in the previous proof with f replaced by g and G_0 by B).

THEOREM 4. *Assume that K is a sequentially weakly closed subset of $W^1L_G(\Omega)$. Then there exists a solution to the minimum problem*

$$\min_{u\in K}\left(\int_\Omega f(x,Du)dx+\int_\Omega g(x,u(x))dx\right) \tag{17}$$

Moreover, if in addition K is convex and g is strictly convex in the second variable, then the solution is unique.

Proof. The minimum problem (17) can be written in the following equivalent form:

$$\min_{u\in W^1L_G(\Omega)}\left(F(u)+F_2(u)+\chi_K(u)\right),$$

where F and F_2 are defined as in Proposition 1 and Proposition 2 above, respectively, and where χ_K is the indicatior function on K ($\chi_K = 0$ on K and ∞ elsewhere). It is easy to see that χ_K is sequentially lower semicontinuous in the weak topology of $W^1L_G(\Omega)$ (since K is a sequentially weakly closed) and, hence, by the previous propositions, so is the sum $F+F_2+\chi_K$ (since the sum of lower semicontinuous functions are lower semicontinuous). By the properties of F and F_2 we find that the inequality

$$F+F_2+\chi_K \geqslant k_1\Psi - k_2 \tag{18}$$

holds for some positive constants k_1 and k_2, where $\Psi(u) = \int_\Omega G(u(x))dx$. The functional Ψ is sequentially coercive in the weak topology of $W^1L_G(\Omega)$. In order to see this we observe that $\Psi(u) \leqslant \|u\|$ if $\|u\| \leqslant 1$ and that $\|u\| < \Psi(u)$ if $1 < \|u\|$. Thus the set $\{u : \Psi(u) \leqslant t\}$ is bounded in $W^1L_G(\Omega)$, and, hence, $\overline{\{u : \Psi(u) \leqslant t\}}$ is sequencially compact in the weak topology of $W^1L_G(\Omega)$ since this space is reflexive. Therefore we obtain that also $F+F_2+\chi_K$ is sequentially coercive in the weak topology of $W^1L_G(\Omega)$. The existence of a minimizer of (18) now follows from the "direct method in the Calculus of Variation" [which states that a sequentially coercive and sequentially lower semicontinuous functional on a topological vector space has a minimum].

If K is convex and g is strictly convex in the second variable then the functional $\Phi = F+F_2+\chi_K$ is strictly convex and thus the minimum is unique. Indeed, assume on the contrary that u_1 and u_2 are minimum points and $u_1 \neq u_2$, then the strictly convexity would imply the inequality

$$\Phi\left(\frac{1}{2}u_1+\frac{1}{2}u_2\right) < \frac{1}{2}\Phi(u_1)+\frac{1}{2}\Phi(u_2) = \Phi(u_1),$$

which is impossible.

6. Some final comments

In their famous paper [3] Alt and Luckhaus prove existence and uniqueness of variational solutions to a class of doubly non-linear parabolic problems of the form

$$(b(u))' - \operatorname{div}\,(a(x,t,Du)) = f \quad \text{in} \quad \Omega \times]0,T[\,. \tag{19}$$

Their proof is based on a new integration by parts formula and compactness arguments like the Minty lemma for monotone operators. In [12] Kacur extends the result of Alt and Luckhaus and proves existence and uniqueness for more general continuity and growth conditions in Orlicz-Sobolev spaces. The equation (19) contains many equations which are important in various applications. One example is the porous medium equation

$$u' - \Delta u^m = f \tag{20}$$

and its cousin, the p-parabolic equation

$$u' - \operatorname{div} |Du|^{p-2} Du = f\,. \tag{21}$$

The porous medium equation in fine structures is studied widely. For linear problems Darcy law-type asymptotics is well understood. The porous medium equation and the p-parabolic equation are subject to intensive studies. Theoretically the p-parabolic equation can be seen as a natural generalization to the L^p setting of the usual heat- or diffusion equation where one allows to play with the parameter p. It turns out that different regimes for the value of p corresponds to different physical situations which are described by (21). For example, the extreme case $p = 1$ together with $u' \equiv 0$ corresponds to the equation describing mean curvature, and the case $p = 2$ corresponds to usual linear heat distribution or linear diffusion

$$u' - \operatorname{div} Du = f\,. \tag{22}$$

The case $p = \infty$ appears e.g. in the study of growing sandpiles, see e.g. Aronsson et. al. [4]. By varying p one can also vary the physical properties in the problem of say a fluid or a fine structured composite, porous or stratified medium. In many situations it is very useful to consider a sequence of problems like e.g.

$$u_h' - \operatorname{div} A_h(x,t) |Du_h|^{p-2} Du_h = f\,. \tag{23}$$

This can be the case in homogenization, numerical analysis or e.g. control problems. It follows, by the general G-compactness results of Svanstedt [22], for nonlinear parabolic operators that the porous medium equation and the p-parabolic equation homogenize. For the p-parabolic equation there are also corrector results for the modeling error in the strong L^p-topology for the gradients and numerical schemes based on augmented Lagrangians available, see [23] and [24].

In the proof of convergence one make significant use of compact embedding properties like the Rellich embedding theorem in usual Sobolev spaces and the weak lower semicontinuity of the norm in L^p-spaces, $1 < p < \infty$. Together with appropriate structure conditions on the problem this guarantee existence and uniqueness of solution. But it is also one of the cornerstones in the theory of variational and operator convergence associated to these problems. By using the new compact embedding result Theorem 2 and existence result Theorem 4 one can now build up a theory for variational convergence analogous to the De Giorgi's Gamma-convergence or an operator convergence like the G-convergence for a large class of elliptic or parabolic operators now being defined on anisotropic Orlicz-Sobolev spaces. There are many advantages of such a development. Some work has already been done in this direction (see [15]).

The analysis in Orlicz-Sobolev spaces uses the properties like convexity and growth (Δ_2-property) in such a way that one can vary parameters with more flexibility than for usual Sobolev spaces. Therefore it is presumable that one should be able to study G- and Gamma-convergence for problems like (19) above in an Orlicz-Sobolev setting. In the periodic setting this means that one should be able to study the homogenization problem for a large class of elliptic-parabolic problems of the type (19). This would hopefully also give new important insights in simpler problems and special cases of (19) via the new results in the new function spaces exploiting their features.

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