

ON A GENERALIZATION OF HILBERT'S DOUBLE SERIES THEOREM

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Abstract. By introducing three parameters A , B and λ , we give a generalization of Hilbert's inequality with a best constant factor and some particular results.

1. Introduction

The Hilbert's double series theorem is said: If $\{a_n\}$, $\{b_n\}$ are two sequences of real numbers, such that $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$, and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \pi \left(\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right)^{1/2}, \quad (1.1)$$

where the constant factor π is best possible. Inequality (1.1) is well known as Hilbert's inequality, which is important in analysis and its applications (cf. [1, Chap. 9]). In recent years, Hu [2], Yang [3], and Gao [4] have made three meaningful improvements of it. By introducing a parameter λ , and estimating the weight coefficient, Yang and Debnath [5] gave a new generalization of (1.1) as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^\lambda} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left[\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} a_n^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} b_n^2 \right]^{1/2} \quad (0 < \lambda \leq 2), \quad (1.2)$$

where the best constant factor relates a β function of λ as:

$$B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) = \int_0^\infty \frac{1}{(1+u)^\lambda u^{1-\lambda/2}} du.$$

Kuang and Debnath [6] continued doing this work.

The corresponding integral inequality of (1.1) is:

$$\int_{-1/2}^{\infty} \int_{-1/2}^{\infty} \frac{f(x)g(y)}{x+y+1} dx dy < \pi \left(\int_{-1/2}^{\infty} f^2(x) dx \int_{-1/2}^{\infty} g^2(y) dy \right)^{1/2}, \quad (1.3)$$

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where π is best possible (f and g are real functions, such that $0 < \int_{-1/2}^{\infty} f^2(x)dx < \infty$, and $0 < \int_{-1/2}^{\infty} g^2(y)dy < \infty$). If we make transform $x = X - 1/2$, $y = Y - 1/2$, and set $F(X) = f(X - 1/2)$, $G(Y) = g(Y - 1/2)$ in (1.3), we may get:

$$\int_0^{\infty} \int_0^{\infty} \frac{F(X)G(Y)}{X+Y} dXdY < \pi \left(\int_0^{\infty} F^2(X)dX \int_0^{\infty} G^2(Y)dY \right)^{1/2}, \tag{1.4}$$

which is the standard Hilbert’s integral inequality (cf. [1, Chap. 9]). It is obvious that inequalities (1.3) and (1.4) are equivalent. Recently, inequality (1.4) has been improved by [7], and generalized by [8,9].

By the same way of [5], Yang [18] gave a generalization of (1.4) as:

$$\int_0^{\infty} \int_0^{\infty} \frac{F(X)G(Y)}{(X+Y)^{\lambda}} dXdY < B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(\int_0^{\infty} X^{1-\lambda} F^2(X)dX \int_0^{\infty} Y^{1-\lambda} G^2(Y)dY \right)^{1/2} \tag{1.5}$$

$(\lambda > 0)$.

If we make another transform: $X = Ax + 1/2$, $Y = By + 1/2$ ($A, B > 0$), and set $f(x) = F(Ax + 1/2)$, $g(y) = G(By + 1/2)$ in (1.5), we may get:

$$\int_{\frac{-1}{2A}}^{\infty} \int_{\frac{-1}{2B}}^{\infty} \frac{f(x)g(y)}{(Ax + By + 1)^{\lambda}} dx dy < \frac{1}{(AB)^{\lambda/2}} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \times \left[\int_{\frac{-1}{2A}}^{\infty} \left(x + \frac{1}{2A} \right)^{1-\lambda} f^2(x) dx \int_{\frac{-1}{2B}}^{\infty} \left(y + \frac{1}{2B} \right)^{1-\lambda} g^2(y) dy \right]^{\frac{1}{2}} \tag{1.6}$$

$(\lambda > 0)$.

It is obvious that inequalities (1.5) and (1.6) are equivalent, and (1.6) is a generalization of (1.3).

Our main object of this paper is to give a generalization of (1.1) and (1.2), which correlates with (1.6). We find the ranges of A , B and λ in the new inequality become more limited than (1.6). That is:

THEOREM 1.1. *If $0 < A, B \leq 1$, $0 < \lambda \leq 2$, $\{a_n\}, \{b_n\}$ are sequences of real numbers, such that $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2A} \right)^{1-\lambda} a_n^2 < \infty$, and $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2B} \right)^{1-\lambda} b_n^2 < \infty$, then we have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(Am + Bn + 1)^{\lambda}} < \frac{1}{(AB)^{\lambda/2}} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \times \left[\sum_{n=0}^{\infty} \left(n + \frac{1}{2A} \right)^{1-\lambda} a_n^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2B} \right)^{1-\lambda} b_n^2 \right]^{1/2}, \tag{1.7}$$

where the constant factor $B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) / (AB)^{\lambda/2}$ is best possible. In particular, when

$\lambda = 1, 2$, we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{Am + Bn + 1} < \frac{\pi}{\sqrt{AB}} \left[\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right]^{1/2}; \tag{1.8}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(Am + Bn + 1)^2} < \frac{1}{AB} \left[\sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{-1} a_n^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{-1} b_n^2 \right]^{1/2}. \tag{1.9}$$

2. Some lemmas

LEMMA 2.1. *If $0 < A, B \leq 1$, and $0 < \lambda \leq 2$, $Q(n)$ is defined by*

$$Q(n) = \frac{1}{(An + 1/2)^{\lambda/2} B} \int_0^{1/(2An+1)} \frac{1}{(1+y)^{\lambda} y^{1-\lambda/2}} dy - \left[\frac{1}{2} + \frac{(2-\lambda)B}{12} \right] \frac{2^{1-\lambda/2}}{(An+1)^{\lambda}} - \frac{2^{1-\lambda/2} \lambda B}{12(An+1)^{\lambda+1}} \tag{2.1}$$

($n \in \mathbf{N}_0$, \mathbf{N}_0 is the set of nonnegative integers), then we have $Q(n) > 0$, for any $n \in \mathbf{N}_0$.

Proof. Integration by parts, we obtain

$$\begin{aligned} \int_0^{1/(2An+1)} \frac{1}{(1+y)^{\lambda} y^{1-\lambda/2}} dy &= \frac{2}{\lambda} \int_0^{1/(2An+1)} \frac{1}{(1+y)^{\lambda}} dy^{\lambda/2} \\ &= \frac{2}{\lambda} \frac{2^{\lambda/2}}{2^{\lambda} (An+1)^{\lambda}} \left(An + \frac{1}{2}\right)^{\lambda/2} + \frac{4}{2+\lambda} \int_0^{1/(2An+1)} \frac{1}{(1+y)^{\lambda+1}} dy^{1+\lambda/2} \\ &> \frac{2^{1-\lambda/2}}{\lambda (An+1)^{\lambda}} \left(An + \frac{1}{2}\right)^{\lambda/2} + \frac{2^{1-\lambda/2}}{(2+\lambda)(An+1)^{\lambda+1}} \left(An + \frac{1}{2}\right)^{\lambda/2}. \end{aligned} \tag{2.2}$$

By virtue of (2.1) and (2.2), we have

$$Q(n) > \left[\frac{1}{\lambda B} - \frac{1}{2} - \frac{(2-\lambda)B}{12} \right] \frac{2^{1-\lambda/2}}{(An+1)^{\lambda}} + \left[\frac{1}{(2+\lambda)B} - \frac{\lambda B}{12} \right] \frac{2^{1-\lambda/2}}{(An+1)^{\lambda+1}}. \tag{2.3}$$

Since $0 < B \leq 1$, and $0 < \lambda \leq 2$, we have $\frac{1}{\lambda B} - \frac{1}{2} - \frac{(2-\lambda)B}{12} \geq \frac{1}{\lambda} - \frac{1}{2} - \frac{(2-\lambda)}{12} \geq 0$, and

$$\frac{1}{(2+\lambda)B} - \frac{\lambda B}{12} = \frac{12 - \lambda(\lambda+2)B^2}{12(2+\lambda)B} > \frac{12 - 2(2+2)}{12(2+\lambda)B} = \frac{1}{3(2+\lambda)B} > 0.$$

In view of (2.3), and $A > 0$, we find $Q(n) > 0$ ($n \in \mathbf{N}_0$). The lemma is proved.

We need a known inequality as follows (cf. [6], (2.1)): Let f have its first four derivatives on $(0, \infty)$ and $(-1)^n f^{(n)}(x) \geq 0$ ($n = 0, 1, 2, 3, 4$), and $f(x), f'(x) \rightarrow 0, (x \rightarrow \infty)$, then the following inequality holds:

$$\sum_{m=0}^{\infty} f(m) \leq \int_0^{\infty} f(x) dx + \frac{1}{2} f(0) - \frac{1}{12} f'(0). \tag{2.4}$$

For fixed $0 < A, B \leq 1$, and $0 < \lambda \leq 2$, define $f_n(x)$ as

$$f_n(x) = \frac{1}{(An + Bx + 1)^\lambda (Bx + 1/2)^{1-\lambda/2}}, \quad x \in [0, \infty) \quad (n \in \mathbf{N}_0).$$

Setting $y = (Bx + 1/2)/(An + 1/2)$, we have

$$\begin{aligned} \int_0^{\infty} f_n(x) dx &= \frac{1}{(An + 1/2)^{\lambda/2} B} \int_{1/(2An+1)}^{\infty} \frac{1}{(1+y)^\lambda y^{1-\lambda/2}} dy \\ &= \frac{1}{(An + 1/2)^{\lambda/2} B} \left[B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) - \int_0^{1/(2An+1)} \frac{1}{(1+y)^\lambda y^{1-\lambda/2}} dy \right]. \end{aligned}$$

Since $f_n(0) = \frac{2^{1-\lambda/2}}{(An + 1)^\lambda}$, and $f'_n(0) = -\frac{\lambda 2^{1-\lambda/2} B}{(An + 1)^{\lambda+1}} - \frac{(2 - \lambda) 2^{1-\lambda/2} B}{(An + 1)^\lambda}$, by (2.4) and (2.1), we find

$$\begin{aligned} \sum_{m=0}^{\infty} f_n(m) &\leq \frac{1}{(An + 1/2)^{\lambda/2} B} \left[B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) - \int_0^{1/(2An+1)} \frac{1}{(1+y)^\lambda y^{1-\lambda/2}} dy \right] \\ &\quad + \frac{2^{1-\lambda/2}}{2(An + 1)^\lambda} + \frac{\lambda 2^{1-\lambda/2} B}{12(An + 1)^{\lambda+1}} + \frac{(2 - \lambda) 2^{1-\lambda/2} B}{12(An + 1)^\lambda} \\ &= \frac{1}{(An + 1/2)^{\lambda/2} B} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) - Q(n). \end{aligned} \tag{2.5}$$

LEMMA 2.2. If $0 < A, B \leq 1$, and $0 < \lambda \leq 2$, define the weight coefficient $\omega_\lambda(A, B, n)$ as

$$\omega_\lambda(A, B, n) = \sum_{m=0}^{\infty} \frac{1}{(An + Bm + 1)^\lambda} \left(\frac{An + 1/2}{Bm + 1/2} \right)^{1-\lambda/2} \quad (n \in \mathbf{N}_0), \tag{2.6}$$

then for $n \in \mathbf{N}_0$ we have

$$\omega_\lambda(A, B, n) < \frac{A^{1-\lambda}}{B} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(n + \frac{1}{2A} \right)^{1-\lambda}; \tag{2.7}$$

$$\omega_\lambda(B, A, n) < \frac{B^{1-\lambda}}{A} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(n + \frac{1}{2B} \right)^{1-\lambda}. \tag{2.8}$$

Proof. By (2.6) and (2.5), we have

$$\begin{aligned} \omega_\lambda(A, B, n) &= \left(An + \frac{1}{2} \right)^{1-\lambda/2} \sum_{m=0}^{\infty} f_n(m) \\ &\leq \frac{1}{B} B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(An + \frac{1}{2} \right)^{1-\lambda} - \left(An + \frac{1}{2} \right)^{1-\lambda/2} Q(n). \end{aligned} \tag{2.9}$$

By Lemma 2.1, we have (2.7), and by the symmetry of A and B , we have (2.8). The lemma is proved.

LEMMA 2.3. *If $0 < A \leq 1$, $0 < \lambda \leq 2$, and $0 < \varepsilon < \lambda/2$, then*

$$\int_0^\infty \left(x + \frac{1}{2A}\right)^{-(1+\varepsilon)} \int_0^{1/(2Ax+1)} \frac{1}{(1+u)^\lambda u^{(2-\lambda+\varepsilon)/2}} du dx = O(1) \quad (\varepsilon \rightarrow 0^+). \quad (2.10)$$

Proof. For $0 < A \leq 1$, $0 < \varepsilon < \lambda/2$, and $x \geq 0$, we have

$$\begin{aligned} \int_0^{1/(2Ax+1)} \frac{1}{(1+u)^\lambda u^{(2-\lambda+\varepsilon)/2}} du &< \int_0^{1/(2Ax+1)} \frac{1}{u^{(2-\lambda+\lambda/2)/2}} du \\ &= \frac{4}{\lambda} (2A)^{-\lambda/4} \left(x + \frac{1}{2A}\right)^{-\lambda/4}. \end{aligned}$$

Then we obtain

$$\begin{aligned} 0 &< \int_0^\infty \left(x + \frac{1}{2A}\right)^{-(1+\varepsilon)} \int_0^{1/(2Ax+1)} \frac{1}{(1+u)^\lambda u^{(2-\lambda+\varepsilon)/2}} du dx \\ &< \frac{4}{\lambda} (2A)^{-\lambda/4} \int_0^\infty \left(x + \frac{1}{2A}\right)^{-(1+\varepsilon)-\lambda/4} dx = \frac{4}{\lambda(\varepsilon + \lambda/4)} (2A)^\varepsilon < \frac{16}{\lambda^2} \cdot 2^{\lambda/2}. \end{aligned}$$

Expression (2.10) is valid. This proves the lemma.

3. Proof of Theorem 1.1 and some particular results

Proof of Theorem 1.1. By Cauchy's inequality and (2.6) we have

$$\begin{aligned} \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m b_n}{(Am + Bn + 1)^\lambda} &= \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m}{(Am + Bn + 1)^{\lambda/2}} \left(\frac{Am + 1/2}{Bn + 1/2}\right)^{(1-\frac{\lambda}{2})/2} \times \\ &\quad \times \frac{b_n}{(Am + Bn + 1)^{\lambda/2}} \left(\frac{Bn + 1/2}{Am + 1/2}\right)^{(1-\frac{\lambda}{2})/2} \\ &\leq \left[\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_m^2}{(Am + Bn + 1)^\lambda} \left(\frac{Am + 1/2}{Bn + 1/2}\right)^{1-\lambda/2} \times \right. \\ &\quad \left. \times \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{b_n^2}{(Am + Bn + 1)^\lambda} \left(\frac{Bn + 1/2}{Am + 1/2}\right)^{1-\lambda/2} \right]^{1/2} \\ &= \left\{ \sum_{m=0}^\infty \left[\sum_{n=0}^\infty \frac{1}{(Am + Bn + 1)^\lambda} \left(\frac{Am + 1/2}{Bn + 1/2}\right)^{1-\lambda/2} \right] a_m^2 \times \right. \\ &\quad \left. \times \sum_{n=0}^\infty \left[\sum_{m=0}^\infty \frac{1}{(Bn + Am + 1)^\lambda} \left(\frac{Bn + 1/2}{Am + 1/2}\right)^{1-\lambda/2} \right] b_n^2 \right\}^{1/2} \\ &= \left[\sum_{m=0}^\infty \omega_\lambda(A, B, m) a_m^2 \sum_{n=0}^\infty \omega_\lambda(B, A, n) b_n^2 \right]^{1/2}. \end{aligned} \quad (3.1)$$

Hance by (2.7) and (2.8), we have (1.7).

For $0 < \varepsilon < \frac{\lambda}{2}$, setting $\tilde{a}_n = \left(n + \frac{1}{2A}\right)^{-(2-\lambda+\varepsilon)/2}$, and $\tilde{b}_n = \left(n + \frac{1}{2B}\right)^{-(2-\lambda+\varepsilon)/2}$, we obtain

$$\begin{aligned} \frac{(2A)^\varepsilon}{\varepsilon} &= \int_0^\infty \frac{1}{[t + 1/(2A)]^{1+\varepsilon}} dt < \sum_{n=0}^\infty \frac{1}{[n + 1/(2A)]^{1+\varepsilon}} = \sum_{n=0}^\infty \left(n + \frac{1}{2A}\right)^{1-\lambda} \tilde{a}_n^2 \\ &= (2A)^{1+\varepsilon} + \sum_{n=1}^\infty \frac{1}{[n + 1/(2A)]^{1+\varepsilon}} < (2A)^{1+\varepsilon} + \int_0^\infty \frac{1}{[t + 1/(2A)]^{1+\varepsilon}} dt \\ &< 2^{1+\lambda/2} + \frac{(2A)^\varepsilon}{\varepsilon}, \end{aligned}$$

and

$$\sum_{n=0}^\infty \left(n + \frac{1}{2A}\right)^{1-\lambda} \tilde{a}_n^2 = \frac{(2A)^\varepsilon}{\varepsilon} + O(1) = \frac{1}{\varepsilon}(1 + o(1)) \quad (\varepsilon \rightarrow 0^+).$$

By the same way, we have

$$\sum_{n=0}^\infty \left(n + \frac{1}{2B}\right)^{1-\lambda} \tilde{b}_n^2 = \frac{1}{\varepsilon}(1 + o(1)).$$

Hence we get

$$\left[\sum_{n=0}^\infty \left(n + \frac{1}{2A}\right)^{1-\lambda} \tilde{a}_n^2 \sum_{n=0}^\infty \left(n + \frac{1}{2B}\right)^{1-\lambda} \tilde{b}_n^2 \right]^{1/2} = \frac{1}{\varepsilon}(1 + \beta) \quad (\beta \rightarrow 0, \text{ for } \varepsilon \rightarrow 0^+). \tag{3.2}$$

If the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)/(AB)^{\lambda/2}$ in (1.7) is not best possible, then there exists a positive number $K < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, such that (1.7) is valid by changing K for the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, for some parameters A, B and λ ($0 < A, B \leq 1, 0 < \lambda \leq 2$). In particular, we have

$$\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{\tilde{a}_m \tilde{b}_n}{(Am+Bn+1)^\lambda} < \frac{K}{(AB)^{\lambda/2}} \left[\sum_{n=0}^\infty \left(n + \frac{1}{2A}\right)^{1-\lambda} \tilde{a}_n^2 \sum_{n=0}^\infty \left(n + \frac{1}{2B}\right)^{1-\lambda} \tilde{b}_n^2 \right]^{1/2}. \tag{3.3}$$

Since $\int_0^\infty \frac{1}{(1+u)^\lambda u^{(2-\lambda+\varepsilon)/2}} du = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) + o(1)$ ($\varepsilon \rightarrow 0^+$), and when $\varepsilon(> 0)$ is as small as enough, we have $\pi + \alpha > K + K\beta$ ($a \rightarrow 0, \text{ for } \varepsilon \rightarrow 0^+$), then,

in this case, by (2.10) and (3.2), we find

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(Am + Bn + 1)^\lambda} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(Am + Bn + 1)^\lambda} \times \\
 &\times \left(m + \frac{1}{2A}\right)^{-(2-\lambda+\varepsilon)/2} \left(n + \frac{1}{2B}\right)^{-(2-\lambda+\varepsilon)/2} \\
 &> \int_0^\infty \left(x + \frac{1}{2A}\right)^{-(2-\lambda+\varepsilon)/2} \left[\int_0^\infty \frac{1}{(Ax + By + 1)^\lambda} \left(y + \frac{1}{2B}\right)^{-(2-\lambda+\varepsilon)/2} dy \right] dx \\
 &= \frac{1}{A^{(\lambda+\varepsilon)/2} B^{(\lambda-\varepsilon)/2}} \int_0^\infty \left(x + \frac{1}{2A}\right)^{-(1+\varepsilon)} \times \\
 &\times \int_{1/(2Ax+1)}^\infty \frac{1}{(1+u)^\lambda u^{(2-\lambda+\varepsilon)/2}} du dx \quad \left(\text{setting } u = \frac{By + 1/2}{Ax + 1/2}\right) \\
 &= \frac{1}{A^{(\lambda+\varepsilon)/2} B^{(\lambda-\varepsilon)/2}} \int_0^\infty \left(x + \frac{1}{2A}\right)^{-(1+\varepsilon)} \int_0^\infty \frac{1}{(1+u)^\lambda u^{(2-\lambda+\varepsilon)/2}} du dx \\
 &\quad - \frac{1}{A^{(\lambda+\varepsilon)/2} B^{(\lambda-\varepsilon)/2}} \int_0^\infty \left(x + \frac{1}{2A}\right)^{-(1+\varepsilon)} \int_0^{1/(2Ax+1)} \frac{1}{(1+u)^\lambda u^{(2-\lambda+\varepsilon)/2}} du dx \\
 &= \frac{(2A)^\varepsilon}{\varepsilon(AB)^{\lambda/2}} \left(B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) + o(1)\right) - O(1) = \frac{1}{\varepsilon(AB)^{\lambda/2}} \left(B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) + \alpha\right) \\
 &> \frac{1}{\varepsilon(AB)^{\lambda/2}} (K + K\beta) = \frac{K}{(AB)^{\lambda/2}} \left(\sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{1-\lambda} \tilde{a}_n^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{1-\lambda} \tilde{b}_n^2\right)^{1/2}. \tag{3.4}
 \end{aligned}$$

Since inequality (3.4) contradicts (3.3), it follows that the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)/(AB)^{\lambda/2}$ in (1.7) is best possible. Since $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$, and $B(1, 1) = 1$, we have (1.8) and (1.9). This proves the theorem.

Changing A/C for A , and B/C for B in Theorem 1.1, by simplification of (1.7), we have

COROLLARY 3.1. *If $0 < A, B \leq C, 0 < \lambda \leq 2, \{a_n\}, \{b_n\}$ are sequences of real numbers, such that $0 < \sum_{n=0}^{\infty} \left(n + \frac{C}{2A}\right)^{1-\lambda} a_n^2 < \infty$, and $0 < \sum_{n=0}^{\infty} \left(n + \frac{C}{2B}\right)^{1-\lambda} b_n^2 < \infty$, then we have*

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(Am + Bn + C)^\lambda} &< \frac{1}{(AB)^{\lambda/2}} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left[\sum_{n=0}^{\infty} \left(n + \frac{C}{2A}\right)^{1-\lambda} a_n^2 \times \right. \\
 &\times \left. \sum_{n=0}^{\infty} \left(n + \frac{C}{2B}\right)^{1-\lambda} b_n^2 \right]^{1/2}. \tag{3.5}
 \end{aligned}$$

where the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)/(AB)^{\lambda/2}$ is best possible.

In particular, for $1 = A = B \leq C$, $0 < \lambda \leq 2$, we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+C)^\lambda} < B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left[\sum_{n=0}^{\infty} \left(n + \frac{C}{2} \right)^{1-\lambda} a_n^2 \sum_{n=0}^{\infty} \left(n + \frac{C}{2} \right)^{1-\lambda} b_n^2 \right]^{1/2}. \quad (3.6)$$

REMARK 1. Inequality (3.5) is a generalization of (1.7). When $C = 2$, (3.6) reduce to

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{a_{m-1} b_{n-1}}{(m+n)^\lambda} < B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left[\sum_{n=1}^{\infty} n^{1-\lambda} a_{n-1}^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_{n-1}^2 \right]^{1/2}, \quad (3.7)$$

which correlates with (1.5), and has been improved by [10] and [11], for $\lambda = 1$, and extended by [12]. Inequality (1.7) is a generalization of (1.1) and (1.2); so is (3.6).

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