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ON THE INEQUALITY FOR THE ARITHMETIC AND GEOMETRIC MEANS

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(*communicated by J. Marshall Ash*)

In this note we give an elementary proof by induction for the classical inequality

$$G_n = \sqrt[n]{a_1 a_2 \dots a_n} \leqslant \frac{a_1 + a_2 + \cdots + a_n}{n} = A_n, \text{ for } n \geqslant 2, \tag{1}$$

between the Geometric mean G_n and the Arithmetic mean A_n of n positive real numbers $a_1, a_2, \dots, a_n$, where exact equality is possible if and only if $a_1 = a_2 = \cdots = a_n$.

The proof is based on the following special case of (1), which is equivalent to a simple identity and serves as the focal point of the exposition in this paper.

$$\sqrt[m]{p^{m-1}q} \leqslant \frac{(m-1)p + q}{m}, \quad for \quad p > 0, \; q > 0, \; m \geqslant 2, \tag{2}$$

where exact equality is possible if and only if $p = q$.

This approach to (1) is different from those found in the literature in the references that are based on calculus or convexity. The novelty of this approach is that it handles the delicate part of the transition from n to $n+1$ in the inductive argument for (1), as well as the equally important extremal case of equality, that is not treated adequately in the literature, directly via the inequality in (2). This makes the proof of (1) accessible to any class of students who are acquainted with the principle of mathematical induction. We conclude with a discussion of examples which confirm the versatility of the inequality in (2) as a tool for assessing the convergence of some well-known numerical algorithms.

We may rewrite (2), after dividing both sides by p, setting $x = q/p$ and rearranging the terms, in the following equivalent form:

$$x - 1 \geqslant m\left(\sqrt[m]{x} - 1\right), \quad for \quad x \geqslant 0, \; m \geqslant 2, \tag{3}$$

where exact equality occurs if and only if $x = 1$. Consider the identity

$$x - 1 = \left(\sqrt[m]{x} - 1\right)\left[1 + \sqrt[m]{x} + \left(\sqrt[m]{x}\right)^2 + \cdots + \left(\sqrt[m]{x}\right)^{m-1}\right] = \left(\sqrt[m]{x} - 1\right) f(x), \tag{4}$$

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where $f(x) \geqslant m$ *for* $x \geqslant 1$, $f(x) \leqslant m$ *for* $0 \leqslant x \leqslant 1$, *and* $f(x) = m$ *if and only if* $x = 1$. Then $x - 1 \geqslant m\left(\sqrt[m]{x} - 1\right)$ for $x \geqslant 1$, $1 - x \leqslant m\left(1 - \sqrt[m]{x}\right)$ for $0 \leqslant x \leqslant 1$, and $x - 1 = m\left(\sqrt[m]{x} - 1\right)$ *if and only if* $x = 1$, proving (3).

Proof of (1) by induction. Clearly, (1) is true for $n = 2$, since $(a_1 + a_2)^2 \geqslant 4a_1a_2$. Next, suppose $G_n \leqslant A_n$ for some positive integer n. Then, from (2) for $p = G_n$, $q = a_{n+1}$, $m = n + 1$, we obtain

$$G_{n+1} = \sqrt[n+1]{(G_n)^n a_{n+1}} \leqslant \frac{nG_n + a_{n+1}}{n+1} \leqslant \frac{nA_n + a_{n+1}}{n+1} = A_{n+1}. \tag{5}$$

Hence, if (1) holds for some positive integer n then it also holds for $n + 1$, proving that the assertion of inequality in (1) is true for all positive integers $n \geqslant 2$.

Clearly, the equality: $a_1 = a_2 = \cdots = a_n$ is a sufficient condition for exact equality in (1). It remains to show that it is also a necessary condition. That is

$$G_n = A_n \Longrightarrow a_1 = a_2 = \cdots = a_n, \quad for \quad n \geqslant 2. \tag{6}$$

For $n = 2$, (6) is true, because $(a_1 + a_2)^2 = 4a_1a_2$ implies $a_1 = a_2$. Next, suppose (6) is true for some positive integer n, and $G_{n+1} = A_{n+1}$. Then the term on the far left side is equal to the term on the far right side in (5), implying exact equality throughout in (5), that is

$$G_{n+1} = \sqrt[n+1]{(G_n)^n a_{n+1}} = \frac{nG_n + a_{n+1}}{n+1} = \frac{nA_n + a_{n+1}}{n+1} = A_{n+1} \tag{7}$$

Therefore, $G_n = A_n$, which implies that $a_1 = a_2 = \cdots = a_n = G_n$ by assumption. On the other hand, from the exact equality,

$$\sqrt[n+1]{(G_n)^n a_{n+1}} = \frac{nG_n + a_{n+1}}{n+1},$$

in (7) and (2), for $p = G_n$, $q = a_{n+1}$, $m = n + 1$, it follows that $G_n = a_{n+1,}$ concluding that $a_1 = a_2 = \cdots = a_n = a_{n+1}$. This shows that, if (6) is true for some positive integer n, then it is also true with $n + 1$ in place of n, and proves assertion (6) by induction. This completes the proof of (1).

The assertion in (1) requires equality of all the terms in the inequality as a necessary and sufficient condition for the possibility of exact equality. Consequently, the product of a finite set of positive real numbers with a fixed sum is maximum when they are equal, and the sum of a finite set of positive real numbers with a fixed product is minimum when they are equal. This is a well known principle of optimization in multivariate calculus, where it is derived using Lagrangian multipliers.

Many inequalities in modern analysis for sequences or integrals have their origin or can be traced to the inequality between the arithmetic mean and the geometric mean of a finite set of positive real numbers. They have received an extensive treatment in the classic treatise “Inequalities” by Hardy, Littlewood, and Polya [5], in Beckenbach and Bellman [3], and elsewhere. Inequalities play an important role as tools for the study of convergence of analytical or numerical algorithms. We discuss three such examples that relate to the inequalities in (2) and (1).

EXAMPLE 1. Heron's Recursive Algorithm for Roots. Let $a > 0$ be a positive real number, $m \geqslant 2$ be a positive integer and consider the iteration defined by

$$x_n = \frac{1}{m}\left((m-1)x_{n-1} + \frac{a}{(x_{n-1})^{m-1}}\right), \quad \text{for } n \geqslant 1, \tag{8}$$

where x_0 an initial value larger or smaller than $\sqrt[m]{a}$. For $m = 2$, the iteration is the square root algorithm which is attributed to Heron of Alexandria (60 A. D.). The algorithm is a special case of the Newton-Raphson iteration for the solution of the equation $x^m - a = 0$. It has the following properties:

(i) *the sequence* x_n *is decreasing for* $n \geqslant 1$ *and convergent to* $\sqrt[m]{a}$;

(ii) *the sequence* $\frac{a}{(x_n)^{m-1}}$ *is increasing for* $n \geqslant 1$ *and convergent to the* $\sqrt[m]{a}$.

These follow easily from (2). The expression in (8) that defines x_n, for $n \geqslant 1$, is simply the right hand side of (2), where

$$p = x_{n-1}, \; q = \frac{a}{(x_{n-1})^{m-1}}, \; m = m.$$

Hence, $x_n > \sqrt[m]{a}$ for $n \geqslant 1$. On the other hand, for $n \geqslant 2$, we may rewrite (8) in the form

$$x_{n-1} - x_n = \frac{1}{m}\left(\frac{(x_{n-1})^m - a}{(x_{n-1})^{m-1}}\right) > 0,$$

since $x_n > \sqrt[m]{a}$. Thus x_n is a decreasing sequence that is bounded from below by $\sqrt[m]{a}$. Hence, x_n is a convergent sequence, whose limit say is x. On taking limits of both sides in (8) and solving the resulting equation for x, we conclude that $x^m = a$, proving the first assertion concerning the monotone convergence of the iteration x_n to the mth root of a. The second assertion follows from the observation that the sequence $(x_n)^{m-1}$ is decreasing and convergent to $\frac{a}{\sqrt[m]{a}}$ by the first assertion.

EXAMPLE 2. The following properties of monotone convergence of the sequence $(1 + \frac{x}{n})^n$ to the exponential function e^x are a direct consequence of (2) :

(i) $(1 + \frac{x}{n})^n$ *is an increasing sequence for* $n > -x$ *and* $-\infty < x < +\infty$;

(ii) $(1 + \frac{x}{n})^{n+1}$ *is a decreasing sequence for* $n > -x$ *and* $-\infty < x \leqslant 1$.

From (2), for $p = 1 + \frac{x}{n}$, $q = 1$, $m = n + 1$, $n + x > 0$, we get

$$\sqrt[n+1]{\left(1 + \frac{x}{n}\right)^n} < \frac{n\left(1 + \frac{x}{n}\right) + 1}{n+1} = 1 + \frac{x}{n+1},$$

proving (i). On the other hand, for $p = 1/(1 + \frac{x}{n})$, $q = 1$, $m = n + 2$, $n + x > 0$, we get

$$\sqrt[n+2]{\left(\frac{1}{1 + \frac{x}{n}}\right)^{n+1}} < \frac{(n+1)\left(\frac{1}{1+\frac{x}{n}}\right) + 1}{n+2} \leqslant \frac{1}{1 + \frac{x}{n+1}},$$

provided $(n + x + 1)\left(n^2 + 2n + x\right) \leqslant (n+1)(n+2)(n+x)$ or, equivalently, $-\infty < x \leqslant 1$, proving (ii).

EXAMPLE 3. From (3), (4) and (1), we obtain the inequalities

$$\ln x \leqslant \frac{x-1}{\sqrt{x}} \leqslant x-1, \quad for \quad x>1, \tag{9}$$

$$\frac{x-1}{\sqrt{x}} \leqslant \ln x \leqslant x-1, \quad for \quad 0<x<1. \tag{10}$$

If we let $m \to \infty$ in (3) we get $\ln x \leqslant x-1$ for $x>0$. Consider the expression $f(x)$ defined in (4). Then, using (1) with $n=m$, $a_j = (\sqrt[m]{x})^{j-1}$, for $j=1,2,\ldots,m$, we get

$$\frac{f(x)}{m} \geqslant \sqrt[m]{(\sqrt[m]{x})^{1+2+\cdots+(m-1)}} = x^{\frac{m-1}{2m}}.$$

But, $m(\sqrt[m]{x}-1) = (x-1)/(\frac{f(x)}{m})$ by (4). Hence, $m(\sqrt[m]{x}-1) \leqslant (x-1)/x^{\frac{m-1}{2m}}$, for $x>1$, and on taking limits as $m \to \infty$ this yields the inequality $\ln x \leqslant \frac{x-1}{\sqrt{x}}$ in (9) for $x>1$ On the other hand , when $0<x<1$ if we replace x by $1/x$ in the preceding inequality we obtain the inequality $\frac{x-1}{\sqrt{x}} \leqslant \ln x$ in (9) for $0<x<1$. As x varies from 1 to 2 the accuracy of the numerical approximation of $\ln x$ by $\frac{x-1}{\sqrt{x}}$ varies from almost four decimal places to almost one decimal place, and exceeds that of the usual approximation of $\ln x$ by $x-1$ for $x>1$. We also note that from (9) it follows that $\ln x < \sqrt{x}$ for $x>1$.

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