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HADAMARD'S INEQUALITY FOR A TRIANGLE, A REGULAR POLYGON AND A CIRCLE

A. McD. Mercer

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Abstract. Analogues of Hadamard's integral inequalities are found in two dimensions for the regions stated in the title.

1. Introduction

If the function f is convex in the interval $[a,b]$ then Hadamard's classical inequalities read:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

We shall refer to the right and left inequalities here as (1R) and (1L) and similarly for other equations. The inequalities mentioned in the title of this article refer to the analogues of these, in the various cases. *In all that follows* we shall assume that the function f, now a function of two variables, is continuous and convex over the region being considered at the time so that:

$$f\left(\sum \lambda_k \mathbf{x}_k\right) \leqslant \sum \lambda_k f(\mathbf{x}_k) \quad \text{when} \quad \lambda_k \in [0,1] \quad \text{and} \quad \sum \lambda_k = 1$$

2. The analogue of (1) for a triangle

Let A_1 be a triangular region in the plane and let A_1 also denote its area. It will be clear from the context which is which. We take the position vectors of the vertices of A_1 as **a**, **b**, **c** taken anticlockwise. Next let A_{k+1} denote the region (and its area) obtained by joining the mid points of the sides of A_k $(k = 1,2,\dots)$. It is a simple matter to see that this sequence of regions converges to the point $\frac{1}{3}[\mathbf{a}+\mathbf{b}+\mathbf{c}]$. Then with these notations and f convex over A_1 we have the following analogue of (1).

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THEOREM 1. *Writing*

$$I_k = \frac{1}{A_k} \iint_{A_k} f(\mathbf{x})dxdy$$

then

$$f\left(\frac{1}{3}[\mathbf{a}+\mathbf{b}+\mathbf{c}]\right) \leqslant \cdots \leqslant I_3 \leqslant I_2 \leqslant I_1 \leqslant \frac{1}{3}[f(\mathbf{a})+f(\mathbf{b})+f(\mathbf{c})]. \quad (2)$$

Proof. First we prove the last inequality above. When λ, μ, ν each lie in $[0,1]$ and $\lambda+\mu+\nu=1$ then any $\mathbf{x} \in A_1$ can be written as $\mathbf{x} = \lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c}$. Let us take λ and μ as independent variables, (so that $\nu = 1-\lambda-\mu$) and apply the change of variable

$$(x,y) \to (\lambda,\mu)$$

to the integral in I_1. It is easy to see that the Jacobean of this transformation is given by

$$\frac{\partial(x,y)}{\partial(\lambda,\mu)} = \begin{vmatrix} a_1-b_1 & b_1-c_1 \\ a_2-b_2 & b_2-c_2 \end{vmatrix}$$

so that

$$\left|\frac{\partial(x,y)}{\partial(\lambda,\mu)}\right| = 2A_1.$$

So, using the convexity of f and obtaining the λ, μ limits of the integral by consideration of the plane $\lambda+\mu+\nu=1$ in the (λ,μ,ν) space, we have:

$$\begin{aligned}\iint_{A_1} f(\mathbf{x})dxdy &= 2A_1 \int_0^1 \int_0^{1-\mu} f(\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c})d\lambda d\mu \\ &\leqslant 2A_1 \int_0^1 \int_0^{1-\mu} [\lambda f(\mathbf{a})+\mu f(\mathbf{b})+\nu f(\mathbf{c})]d\lambda d\mu \\ &= \frac{A_1}{3}[f(\mathbf{a})+f(\mathbf{b})+f(\mathbf{c})]. \quad (3)\end{aligned}$$

This concludes the proof of the last inequality in (2).

Next we consider the inequality $I_2 \leqslant I_1$. Denote by $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ the vertices of A_2 so that

$$\mathbf{p} = \frac{\mathbf{b}+\mathbf{c}}{2}, \quad \mathbf{q} = \frac{\mathbf{c}+\mathbf{a}}{2}, \quad \mathbf{r} = \frac{\mathbf{a}+\mathbf{b}}{2}. \quad (4)$$

Analogously to (3) we have

$$\frac{1}{A_2}\iint_{A_2} f(\mathbf{x})dxdy = 2\int_0^1 \int_0^{1-\mu} f(\lambda\mathbf{p}+\mu\mathbf{q}+\nu\mathbf{r})d\lambda d\mu.$$

Using (4) and the convexity of f this does not exceed

$$\int_0^1 \int_0^{1-\mu} f(\lambda\mathbf{b}+\mu\mathbf{c}+\nu\mathbf{a})d\lambda d\mu + \int_0^1 \int_0^{1-\mu} f(\lambda\mathbf{c}+\mu\mathbf{a}+\nu\mathbf{b})d\lambda d\mu. \quad (5)$$

Each of these integrals equals

$$\frac{1}{2A_1}\iint_{A_1} f(\mathbf{x})dxdy$$

and this concludes the proof that $I_2 \leqslant I_1$. The proof that $I_{k+1} \leqslant I_k \ (k = 2, 3 \dots)$ proceeds in exactly the same way.

Finally, since the sequence of regions A_k converges to the point $\frac{1}{3}[\mathbf{a}+\mathbf{b}+\mathbf{c}]$ the first inequality in (2) follows from the mean value theorem for double integrals.

NOTE. If, instead of forming the sequence of regions A_k by bisecting the sides of the previous member, we divide the sides of the previous member in the ratio $t:\ 1-t$ with $0 < t < 1$ (following cyclic order) the result (2) continues to hold with these new regions. The proof of this is almost identical to the above, differing only at (5) because in that case we have, for example,

$$\mathbf{p} = (1-t)\mathbf{b} + t\mathbf{c} \quad \text{etc.}$$

3. The analogues of (1) for a regular polygon and a circle

THEOREM 2. *If* $\mathbf{p}_k \ (k = 1, 2, \dots n)$ *are the vertices of a regular polygonal region* P *(whose area is also denoted by* P*) then*

$$f\left(\frac{1}{n}\sum \mathbf{p}_k\right) \leqslant \frac{1}{P}\iint_P f(\mathbf{x})dxdy \leqslant \frac{1}{n}\sum f(\mathbf{p}_k). \tag{6}$$

THEOREM 3. *If* D *is a closed circular disc of radius* r*, boundary* C *and centre* $\mathbf{c}$ *then:*

$$f(\mathbf{c}) \leqslant \frac{1}{\pi r^2}\iint_D f(\mathbf{x})dxdy \leqslant \frac{1}{2\pi r}\int_C f(\mathbf{x})ds. \tag{7}$$

NOTE. The result (7) is not new (see [1]) but the present proof, which is an immediate consequence of (6), is quite different from the earlier one.

We now proceed to the proofs of these two theorems.

Proof of (6R). If $P_k \ (k = 1, 2, \dots n)$, are the vertices whose position vectors are $\mathbf{p}_k$ let us draw the diagonals $P_1P_k \ (k = 3, 4, \dots, n-1)$. The areas of the triangles $P_1P_kP_{k+1}$ so formed will be denoted by $\triangle_k \ (k = 2, 3 \dots, n-1)$. For this construction let us call P_1 "the preferred vertex".

When we apply the last inequality in (2) to each of the triangles $\triangle_k$ and sum over them all we get the result:

$$\iint_p f(\mathbf{x})dxdy \leqslant \sum_2^{n-1} \frac{\triangle_k}{3}[f(\mathbf{p}_1) + f(\mathbf{p}_k) + f(\mathbf{p}_{k+1})]$$

$$= \frac{P}{3}f(\mathbf{p}_1) + \frac{\triangle_2}{3}(\mathbf{p}_2) + \sum_3^{n-1}\left[\frac{\triangle_k}{3} + \frac{\triangle_{k-1}}{3}\right]f(\mathbf{p}_k) + \frac{\triangle_{n-1}}{3}f(\mathbf{p}_n).$$

This is a Hadamard–type inequality using the preferred vertex P_1. We now repeat this process taking, in turn, the vertices P_2, P_3, ..., P_n as preferred vertices. Adding all of these we get:

$$n\iint_p f(\mathbf{x})dxdy \leqslant P\sum_1^n f(\mathbf{p}_k) \tag{8}$$

and this is the required result. It should be noted that although the last inequality in (2) holds for *any* triangle the polygon result (6R) has been proved only for a *regular* polygon. This is because it is essential to the proof that the subregions corresponding to each preferred vertex P_k be congruent.

Proof of (7R). To obtain this result one inscribes a regular polygon of n sides in the circle C. Following a familiar process the result (7R) follows from (8) on taking the limit as $n \to \infty$. We leave the details to the reader.

NOTE. For the proofs of (6L) and (7L) there is clearly no loss of generality in supposing that the centers of the polygonal and circular regions have the origin $\mathbf{0}$ as their centre.

It is now convenient to prove the following lemma.

LEMMA. *Let* Ω *be a closed convex region in the plane whose boundary is* Γ *and suppose that* Γ *is centrally symmetric. That is, if its polar coordinate equation is* $r = \Psi(\theta)$ *then* $\Psi(\theta+\pi) = \Psi(\theta)$ *for all* θ. *If we denote the area of* Ω *also by* Ω *then:*

$$f(\mathbf{0}) \leqslant \frac{1}{\Omega}\iint_\Omega f(\mathbf{x})dxdy.$$

Proof of the Lemma. By convexity we have

$$2f(\mathbf{0}) \leqslant f(\rho,\theta) + f(\rho,\theta+\pi) \quad \text{for} \quad 0 \leqslant \rho \leqslant \Psi(\theta).$$

Carrying out the integration indicated by

$$\int_0^{2\pi}\int_0^{\Psi(\theta)} \dots \rho d\rho d\theta$$

and noting, by the periodicity of f with respect to θ, that the two integrals which appear on the right are the same, we get the required result.

Proof of (6L) (n even) and (7L). As mentioned above there no loss of generality if we take the centre of each of these regions to be the origin. The two results now follow at once from the Lemma.

Of course when n is odd the Lemma is not applicable and for this case a separate proof is needed.

Proof of (6L) (n odd). Let $n = 2k+1$ and let the centre of the polygon be the origin. If $r = \Psi(\theta)$ is the polar equation of the polygon then Ψ will be periodic with period $\dfrac{2\pi}{n}$. In particular we wil have

$$\Psi(\theta) = \Psi\Big(\theta + \frac{2k\pi}{n}\Big) = \Psi\Big(\theta - \frac{2k\pi}{n}\Big): \quad \text{for all} \quad \theta.$$

Now the triangle with vertices

$$(\varrho, \Psi(\theta)), \quad \left(\varrho, \Psi\left(\theta + \frac{2k\pi}{n}\right)\right), \quad \left(\varrho, \Psi\left(\theta - \frac{2k\pi}{n}\right)\right)$$

($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ say) has the origin $\mathbf{0}$ in its interior so that

$$\mathbf{0} = \lambda \mathbf{u} + \mu \mathbf{v} + \nu \mathbf{w}$$

for some $(\lambda, \mu, \nu) \in [0, 1]$ and $\lambda + \mu + \nu = 1$.

A simple calculation shows that

$$\lambda : \mu : \nu = \sin\left(\frac{2\pi}{n}\right) : \sin\left(\frac{2k\pi}{n}\right) : \sin\left(\frac{2k\pi}{n}\right)$$

so that λ, μ, ν are seen to be independent of θ.

Hence by convexity:

$$f(\mathbf{0}) \leqslant \lambda f(\varrho, \Psi(\theta)) + \mu f\left(\varrho, \Psi\left(\theta + \frac{2k\pi}{n}\right)\right) + \nu f\left(\varrho, \Psi\left(\theta - \frac{2k\pi}{n}\right)\right).$$

Carrying out the integration indicated by

$$\int_0^{2\pi} \int_0^{\Psi(\theta)} \ldots \rho d\rho d\theta$$

and using periodicity we get the result

$$Pf(\mathbf{0}) \leqslant \iint_P f(\mathbf{x}) dxdy$$

which concludes the proof of (6L) for the case of n odd.

REFERENCES

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Department of Mathematics and Statistics
University of Guelph
Ontario, N1G 2W1, Canada
e-mail: amercer@reach.net