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STURMIAN COMPARISON METHOD: THE VERSION FOR FIRST ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper the Sturmian Comparison Method is developed for the first order neutral differential equation of the type

$$l(y) := [y(t+1) - P(t)y(t)]' + Q(t)y(t+1-\sigma) = 0, \quad \sigma \geqslant 0. \tag{1}$$

Using this method, a general theorem is proved on the location of zeros of (1), which is then applied to obtain two concrete results. The first one of them turns out to be the best possible in the case where P and Q are constants. The second one is concerned, for the first time, with the oscillation theory of first order neutral differential equations, in the case where the coefficient $Q(t)$ is oscillatory.

1. Introduction

In the present paper we investigate oscillation properties of the first order neutral differential equation of the type

$$l(y) := [y(t+1) - P(t)y(t)]' + Q(t)y(t+1-\sigma) = 0, \quad \sigma \geqslant 0, \tag{1}$$

where $P : [a,b] \longmapsto R$ and $Q : [a,b] \longmapsto R$ are continuous.

Throughout the paper we will denote

$$\rho := \max\{1, \sigma - 1\}, \quad \rho_0 := \max\{0, \sigma - 1\} \tag{2}$$

and will suppose that $b - a > \rho$. Under a solution of Eq. (1) it will be understood a continuous function $y : [a - \rho_0, b+1] \longmapsto R$ such that $[y(t+1) - P(t)y(t)]$ is continuously differentiable and Eq. (1) is fulfilled on $[a,b]$.

The main tool in our investigation will be the Sturmian Comparison Method which, as it is well known, was the starting point for all the oscillation theory. The two classical Sturmian theorems for second order ordinary differential equations read as follows.

THEOREM A. (The Fundamental Sturm Comparison Theorem) *Let a, b be two adjacent zeroes of a solution y of the equation*

$$y'' + p(t)y = 0, \tag{3}$$

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and let

$$q(t) \geqslant p(t) \text{ on } [a,b].$$

Then any solution of the equation

$$z'' + q(t)z = 0 \tag{4}$$

has at least one zero on $[a,b]$.

COROLLARY A1. (The Sturm Oscillation Comparison) *If* $q(t) \geqslant p(t)$ *on* $[t_0, +\infty)$ *and all solutions of Eq.* (3) *are oscillatory, then all solutions of Eq.* (4) *are also oscillatory.*

Note that these statements radically differ from each other. Theorem 1., describing properties of solutions on a finite interval, has local character while Corollary 1. refers to global properties of solutions on the half–axis. It should be noted that just the latter is known in the literature under the name of Sturm comparison theorem and it is generalized in many directions (for nonlinear differential equations, higher order equations, delay differential equations etc.). Corollary A1 was generalized for neutral differential equations as well (see, for example, [1], Section 3.4). It should be noted, however, that Corollary A1 and its generalizations, unlike Theorem 1., cannot give any information about location of zeros of solutions.

The first generalization of Theorem 1. for delay differential equations appeared in [2]. The method elaborated therein was called Sturmian Comparison Method and was stated in detail in [3], Chapter 4. First generalizations of Theorem A for neutral differential equations were published in [4] (for Eq. (1) with $\sigma = 0$) and in [5] (for second order neutral differential equations). The complete exposition of [5] can be found in [6], Section 3.5, as well. In the present paper we continue the investigations in the direction begun in [4].

2. Sturmian comparison method for neutral differential equations

The object to compare Eq. (1) with is the "testing" inequality

$$\tilde{l}(z)(t) := -z'(t-1) + P(t)z'(t) + \tilde{Q}(t-1+\sigma)z(t-1+\sigma) \geqslant 0 \text{ on } [a,b], \tag{5}$$

where $\tilde{Q} : [a-1+\sigma, b-1+\sigma] \longmapsto R$ is continuous.

A solution of Ineq. (5) is a continuously differentiable function $z : [a-1, b+\rho_0] \longmapsto R$ satisfying it everywhere on $[a,b]$.

Let now the functions $y : [a-\rho_0, b+1] \longmapsto R$ and ${}_0] \longmapsto R$ be any solutions of Eq. (1) and Ineq. (5), respectively. Multiply Eq. (1) by $z(t)$ and Ineq. (5) by $y(t)$, subtract and integrate over $[a,b]$. Using integration by parts, we obtain the following identities. For $0 \leqslant \sigma < 1$:

$$\int_a^b [ly](t)z(t)dt = \int_a^b [\tilde{l}z](t)y(t)dt + \int_a^{b-(1-\sigma)} [Q(t)-\tilde{Q}(t)]y(t+1-\sigma)z(t)dt$$
$$+\int_{a-1}^{a} y(t+1)z'(t)dt - \int_{b-1}^{b} y(t+1)z'(t)dt + z(b)[y(b+1)-P(b)y(b)]$$
$$-z(a)[y(a+1)-P(a)y(a)] - \int_{a-(1-\sigma)}^{a} \tilde{Q}(t)y(t+1-\sigma)z(t)dt$$
$$+\int_{b-(1-\sigma)}^{b} Q(t)y(t+1-\sigma)z(t)dt, \tag{6}$$

for $\sigma > 1$:

$$\int_a^b [ly](t)z(t)dt = \int_a^b [\tilde{l}z](t)y(t)dt + \int_{a+(\sigma-1)}^{b} [Q(t)-\tilde{Q}(t)]y(t+1-\sigma)z(t)dt$$
$$+\int_{a-1}^{a} y(t+1)z'(t)dt - \int_{b-1}^{b} y(t+1)z'(t)dt + z(b)[y(b+1)-P(b)y(b)]$$
$$-z(a)[y(a+1)-P(a)y(a)] - \int_{b}^{b+(\sigma-1)} \tilde{Q}(t)y(t+1-\sigma)z(t)dt$$
$$+\int_{a}^{a+(\sigma-1)} Q(t)y(t+1-\sigma)z(t)dt, \tag{7}$$

and for $\sigma = 1$:

$$\int_a^b [ly](t)z(t) = \int_a^b [\tilde{l}z](t)y(t) + \int_a^b [Q(t)-\tilde{Q}(t)]y(t)z(t)dt$$
$$+\int_{a-1}^{a} y(t+1)z'(t)dt - \int_{b-1}^{b} y(t+1)z'(t)dt + z(b)[y(b+1)-P(b)y(b)]$$
$$-z(a)[y(a+1)-P(a)y(a)]. \tag{8}$$

These identities play a crucial role in proving the following statement in the form of Theorem A. It combines three cases: $0 \leqslant \sigma < 1$, $\sigma > 1$ and $\sigma = 1$.

THEOREM 1. *Let the functions $Q : [a,b] \longmapsto R$ and $\tilde{Q} : [a+\sigma-1, b+\sigma-1] \longmapsto R$ be such that*

$$\begin{array}{lll} 1^\circ. & \text{for } 0 \leqslant \sigma < 1: & \tilde{Q}(t) \geqslant 0 \quad \text{on} \quad [a-(1-\sigma), a], \\ & \text{for } \sigma > 1: & \tilde{Q}(t) \geqslant 0 \quad \text{on} \quad [b, b+(\sigma-1)], \\ & \text{for } \sigma = 1: & \text{no restriction on the sign of } \tilde{Q}; \end{array} \tag{9}$$

$$\begin{array}{lll} 2^\circ. & \text{for } 0 \leqslant \sigma \leqslant 1: & Q(t) \geqslant \begin{cases} \tilde{Q}(t) & \text{on } t \in [a, b-(1-\sigma)], \\ 0 & \text{on } t \in [b-(1-\sigma), b]; \end{cases} \\ & \text{for } \sigma > 1: & Q(t) \geqslant \begin{cases} 0 & \text{on } t \in [a, a+(\sigma-1)], \\ \tilde{Q}(t) & \text{on } t \in [a+(\sigma-1), b]; \end{cases} \end{array} \tag{10}$$

3° : *Ineq.* (5) *has at least one solution* z *defined on* $[a-1, b+\rho_0]$ *such that*

$$\text{(i)} \quad z(a) = z(b) = 0, \;\; z(t) > 0 \quad \text{on } (a,b); \tag{11}$$

$$\text{(ii)} \quad \begin{cases} \text{for } 0 \leqslant \sigma < 1 : z(t) \leqslant 0 & \text{on } \; [a-(1-\sigma), a], \\ \text{for } \sigma > 1 : z(t) \leqslant 0 & \text{on } \; [b, b+(\sigma-1)], \\ \text{for } \sigma = 1 : & \text{no restriction;} \end{cases} \tag{12}$$

$$\text{(iii)} \quad z'(t) > 0 \text{ on } [a-1, a] \text{ and } z'(t) < 0 \text{ on } [b-1, b]. \tag{13}$$

Then Eq. (1) *has no positive solution on* $(a-\rho_0, b+1)$. *In other words, any solution of Eq.* (1) *has a zero on* $(a-\rho_0, b+1)$.

Proof. Suppose that on $[a-\rho_0, b+1]$ there exists a positive solution y of (1) and for y and z write out (6), (7) or (8). The left-hand side is zero while all the terms in the right-hand side are nonnegative, at least two of them being strictly positive. This contradiction proves Theorem 1.

3. Constructing of the "testing" inequalities

Theorem 2., like any comparison theorem, is uneffective since such is the expression "If Ineq. (5) has a solution z satisfying ... ". The problem now is to formulate, *in terms of the coefficients* P *and* $\tilde{Q}$ effective conditions for the existence of such a z. Lemma 3. and Lemma 3. below are examples of such conditions.

LEMMA 1A. *Let* $\sigma \neq 1$, $\varphi : [a-1, b+\rho_0] \longmapsto]0, +\infty[$ *be an arbitrary continuous function such that*

$$\int_a^b \varphi(t)dt = \pi, \qquad \left| \int_t^{t+\sigma-1} \varphi(t)dt \right| < \pi, \;\; t \in [a-1, b], \tag{14}$$

and let a continuous function $k : [a-1, b+\rho_0] \longmapsto R$ *satisfy*

$$\begin{cases} k(t) < \varphi(t) \cot \displaystyle\int_t^a \varphi(s)ds \;\; \text{on } [a-1, a), \\ k(t) < \varphi(t) \cot \displaystyle\int_t^b \varphi(s)ds \;\; \text{on } [b-1, b). \end{cases} \tag{15}$$

Let, moreover, $P : [a,b] \longmapsto R$ *satisfy*

$$R(t,\sigma) sgn(\sigma - 1) \geqslant 0 \;\; \text{on } [a,b] \quad \text{in case } \; \sigma \neq 1, \tag{16}$$

where

$$R(t,\sigma)=\exp\left(-\int_{t-1}^{t}k(s)ds\right)\left[\varphi(t-1)\cos\int_{t-1}^{t+\sigma-1}\varphi(s)ds\right.$$
$$\left.-k(t-1)\sin\int_{t-1}^{t+\sigma-1}\varphi(s)ds\right]$$
$$-P(t)\left[\varphi(t)\cos\int_{t}^{t+\sigma-1}\varphi(s)ds-k(t)\sin\int_{t}^{t+\sigma-1}\varphi(s)ds\right]\ \textit{for}\ t\in[a,b],\quad(17)$$

and let $\tilde{Q}:[a+\sigma-1,b+\sigma-1]\longmapsto R$ *be defined by*

$$\tilde{Q}(t+\sigma-1):=\operatorname{cosec}\left(\int_{t}^{t+\sigma-1}\varphi(s)ds\right)\left\{\exp\left(-\int_{t-1}^{t+\sigma-1}k(s)ds\right)\times\right.$$
$$\times\left[\varphi(t-1)\cos\int_{t-1}^{t}\varphi(s)ds-k(t-1)\sin\int_{t-1}^{t}\varphi(s)ds\right]$$
$$\left.-P(t)\varphi(t)\exp\left(-\int_{t}^{t+\sigma-1}k(s)ds\right)\right\}\ \textit{for}\ t\in[a,b].\qquad(18)$$

Then the function $z(t)=\exp\left(\int_a^t k(s)ds\right)\sin\int_a^t\varphi(s)ds$ *is a solution of Ineq.* (5) *on* $[a,b]$ *satisfying* (11)–(13). *If, in addition,*

$$L(t)\operatorname{sign}(\sigma-1)\geqslant 0\begin{cases}\textit{for}\ t\in[a,a+1-\sigma] & \textit{in case}\ 0\leqslant\sigma<1\\ \textit{for}\ t\in[b-(\sigma-1),b] & \textit{in case}\ \sigma>1\end{cases},\qquad(19)$$

where

$$L(t):=\exp\left(-\int_{t-1}^{t}k(s)ds\right)\times$$
$$\times\left[\varphi(t-1)\cos\int_{t-1}^{t}\varphi(s)ds-k(t-1)\sin\int_{t-1}^{t}\varphi(s)ds\right]-P(t)\varphi(t),\qquad(20)$$

then Ineq. (5) *can serve as a "testing" inequality in the sense of Th.* 1.

Proof. We have

$$z'(t)=\exp\left(\int_a^t k(s)ds\right)\left[\varphi(t)\cos\int_a^t\varphi(s)ds+k(t)\sin\int_a^t\varphi(s)ds\right]$$
$$\text{for}\ \ t\in[a-1,b+\rho_0],$$

$$z(t+\sigma-1)=\exp\left(\int_a^t k(s)ds\right)\exp\left(\int_t^{t+\sigma-1}k(s)ds\right)\times$$
$$\times\left[\sin\int_t^{t+\sigma-1}\varphi(s)ds\cos\int_a^t\varphi(s)ds+\cos\int_t^{t+\sigma-1}\varphi(s)ds\sin\int_a^t\varphi(s)ds\right]$$
$$\text{for}\ \ t\in[a,b],$$

$$z'(t-1) = \exp\left(\int_a^t k(s)ds\right)\exp\left(-\int_{t-1}^t k(s)ds\right)\times$$
$$\times\left\{\left[\varphi(t-1)\sin\int_{t-1}^t \varphi(s)ds + k(t-1)\cos\int_{t-1}^t \varphi(s)ds\right]\sin\int_a^t \varphi(s)ds\right.$$
$$\left.+\left[\varphi(t-1)\cos\int_{t-1}^t \varphi(s)ds - k(t-1)\sin\int_{t-1}^t \varphi(s)ds\right]\cos\int_a^t \varphi(s)ds\right\}$$
$$\text{for } t\in[a,b].$$

Substituting all this into (5), we see that the following inequality must be fulfilled

$$A(t)\sin\int_a^t \varphi(s)ds + B(t)\cos\int_a^t \varphi(s)ds \geqslant 0 \ \text{ for } t\in[a,b], \tag{21}$$

where

$$A(t) := \tilde{Q}(t+\sigma-1)\cos\int_t^{t+\sigma-1}\varphi(s)ds + P(t)k(t)\exp\left(-\int_t^{t+\sigma-1}k(s)ds\right)$$
$$-\exp\left(-\int_{t-1}^{t+\sigma-1}k(s)ds\right)\left[\varphi(t-1)\sin\int_{t-1}^t\varphi(s)ds\right.$$
$$\left.+k(t-1)\cos\int_{t-1}^t\varphi(s)ds\right] \quad \text{for } t\in[a,b], \tag{22}$$

$$B(t) := \tilde{Q}(t+\sigma-1)\sin\int_t^{t+\sigma-1}\varphi(s)ds + P(t)\varphi(t)\exp\left(-\int_t^{t+\sigma-1}k(s)ds\right)$$
$$-\exp\left(-\int_{t-1}^{t+\sigma-1}k(s)ds\right)\left[\varphi(t-1)\cos\int_{t-1}^t\varphi(s)ds\right.$$
$$\left.-k(t-1)\sin\int_{t-1}^t\varphi(s)ds\right] \quad \text{for } t\in[a,b]. \tag{23}$$

By (18) we have $B(t)\equiv 0$ and therefore Ineq. (21) is equivalent to

$$A(t)\sin\int_a^t\varphi(s)ds \geqslant 0 \ \text{ for } t\in[a,b]. \tag{24}$$

Using (22), (18) and (17), after some calculations we get

$$A(t)\sin\int_t^{t+\sigma-1}\varphi(s)ds = R(t,\sigma)\exp\left(-\int_t^{t+\sigma-1}k(s)ds\right) \quad \text{for } t\in[a,b],$$

so in view of (14) the inequality (24) follows from (16). Therefore z satisfies (5) on $[a,b]$.

The conditions (11) and (12) obviously follow from (14). As to (13), it can be easily checked that the conditions $z'(t)>0$ on $[a-1,a)$ and $z'(t)<0$ on $[b-1,b)$ coincide with (15).

In view of (15) and (19) we obtain (9) for $\tilde{Q}$ defined by (18). Therefore Ineq. (5) can serve as a "testing" inequality in Th.1.

LEMMA 2. *Let* $\sigma = 1$, $\varphi : [a-1, b] \longmapsto (0, +\infty)$ *be an arbitrary continuous function with* $\int_a^b \varphi(t)dt = \pi$ *and let a continuous function* $k : [a, b] \longmapsto R$ *be such that the equality*

$$L(t) = 0 \text{ for } t \in [a, b] \tag{25}$$

holds with L *defined by (20). Let, moreover,* (15) *hold and* $\tilde{Q}$ *in Ineq.* (5) *satisfy*

$$\tilde{Q}(t) \geqslant \exp\left(-\int_{t-1}^{t} k(s)ds\right) \times \\ \times \left[\varphi(t-1)\sin\int_{t-1}^{t}\varphi(s)ds - k(t-1)\cos\int_{t-1}^{t}\varphi(s)ds\right] \\ -P(t)k(t) \text{ for } t \in [a, b]. \tag{26}$$

Then the statement of Lemma 1 is true and Ineq. (5) *can serve as a "testing" inequality in Th.* 1.

Proof. Analogously to Lemma 1 we obtain (21) with

$$A(t) = \tilde{Q}(t) + P(t)k(t) - \exp\left(-\int_{t-1}^{t} k(s)ds\right) \times \\ \times \left[\varphi(t-1)\sin\int_{t-1}^{t}\varphi(s)ds - k(t-1)\cos\int_{t-1}^{t}\varphi(s)ds\right]$$

and $B(t) = -L(t) = 0$.

In view of (25) and (26) Ineq. (21) yields (24). In view of (9) no additional condition is needed for the case $\sigma = 1$.

REMARK 1. Theorem 1 and Lemma 1 for the case $\sigma = 0$ were formulated and proved in [4] (see also [2], Chapter 2).

Now we formulate the main oscillation theorem and its corollary.

THEOREM 2. *Suppose that there exist continuous functions* $\varphi : [a-1, b+\rho_0] \longmapsto]0, +\infty[$ *and* $k : [a-1, b+\rho_0] \longmapsto R$ *such that the conditions* (14) - (20) *for* $\sigma \neq 1$ $((25)-(26)$ *for* $\sigma = 1)$ *are satisfied. Let, moreover, Cond.* 2° *of Theorem* 2. *hold. Then any solution of Eq.* (1) *has at least one zero on* $(a-\rho_0, b+1)$.

COROLLARY 1. *Let there exist a sequence* $[a_n, b_n]$ *of disjoint intervals such that the conditions of Theorem 2 hold on each of them and* $a_n \to +\infty$ *as* $n \to \infty$. *Then all solutions of Eq.* (1) *are oscillatory and any solution has at least one zero on each* $(a_n - \rho_0, b_n + 1)$.

4. Some effective criteria

Everywhere in this section we will assume that $\sigma \neq 1$.

1°. Let some constants k and ν satisfy

$$\begin{cases} \dfrac{\nu}{\tan(1-\sigma)\nu} < k < \dfrac{\nu}{\tan\nu} & \text{if } 0 \leqslant \sigma < 1, \\ k < \dfrac{\nu}{\tan\nu} & \text{if } 1 \leqslant \sigma \leqslant 2, \\ k < \dfrac{\nu}{\tan(\sigma-1)\nu} & \text{if } \sigma > 2 \end{cases} \tag{27}$$

and

$$0 < \nu\rho < \pi, \tag{28}$$

where ρ is defined by (2). If we put $b = a + \pi/\nu$, $k(t) \equiv k$, $\varphi(t) \equiv \nu$, then the conditions (14) and (15) are evidently satisfied. Further, define

$$M[k,\nu] := \frac{\nu\cos\sigma\nu - k\sin\sigma\nu}{\nu\cos(\sigma-1)\nu - k\sin(\sigma-1)\nu} e^{-k}, \tag{29}$$

$$N[k,\nu;P(t)] := \frac{\nu\cos\nu - k\sin\nu - \nu e^{k}P(t)}{e^{k\sigma}\sin(\sigma-1)\nu}. \tag{30}$$

Theorem 3. implies the following

COROLLARY 2. *Let* $\sigma \neq 1$ *and* (27) *and* (28) *be fulfilled. Assume*

$$(P(t) - M[k,\nu])\,\mathrm{sign}(\sigma-1) \leqslant 0 \text{ on } \left[a, a+\frac{\pi}{\nu}\right], \tag{31}$$

$$Q(t-\sigma+1) \geqslant N[k,\nu;P(t)] \text{ on } \left[a, a+\frac{\pi}{\nu}\right], \tag{32}$$

where $M[k,\nu]$ *and* $N[k,\nu;P(t)]$ *are defined by* (29) *and* (30), *respectively. Then any solution of Eq.* (1) *has a zero on the interval* $(a-\rho_0, a+\frac{\pi}{\nu}+1)$.

Proof. As we have already noted, (14) and (15) take the form (27) and (28). Further, (16) takes the form (31), and (18) yields

$$\tilde{Q}(t-\sigma+1) \equiv N[k,\nu;P(t)].$$

As to (19), in our case it is equivalent to

$$\left[P(t) - e^{-k}\left(\cos\nu - \frac{k}{\nu}\sin\nu\right)\right]\mathrm{sign}(\sigma-1) \leqslant 0 \text{ on } \left[a, a+\frac{\pi}{\nu}\right],$$

and follows from (31) since

$$\left[\frac{\nu\cos\sigma\nu - k\sin\sigma\nu}{\nu\cos(\sigma-1)\nu - k\sin(\sigma-1)\nu} - \left(\cos\nu - \frac{k}{\nu}\sin\nu\right)\right]\mathrm{sign}(\sigma-1) \leqslant 0.$$

Indeed, in view of (27), it is easy to see that the last inequality is equivalent to

$$-\frac{\nu^2+k^2}{\nu}\sin(\sigma-1)\nu\sin\nu\,\mathrm{sign}(\sigma-1) \leqslant 0,$$

which, by (28), is obviously true. Therefore all the conditions of Theorem 3. are fulfilled, and the proof is complete.

Let now P and Q be defined in a neighbourhood of $+\infty$ and k satisfy

$$\begin{cases} \dfrac{1}{1-\sigma} < k < 1 & \text{if } 0 \leqslant \sigma < 1 \\ k < 1 & \text{if } 1 < \sigma \leqslant 2, \\ k < \dfrac{1}{\sigma - 1} & \text{if } \sigma > 2 \end{cases} \tag{33}$$

and

$$M[k,0] := \lim_{\nu \to 0} M[k,\nu] = \frac{1-\sigma k}{1-(\sigma-1)k} e^{-k}, \tag{34}$$

$$N[k,0] := \lim_{\nu \to 0} N[k,\nu; M[k,\nu]] = \frac{k^2 e^{-\sigma k}}{1-(\sigma-1)k}. \tag{35}$$

REMARK 2. The statement of Cor.2 is *sharp* in the following sence: Eq. (1) with $P(t) := M[k,0]$ and $Q(t) := N[k,0]$ has an eventually positive solution $y(t) = e^{-kt}$.

2°. Now suppose that $0 < \sigma < 1$ and let be $k(t) := c \sin \pi t$ with $0 < c < 1$, so $k(t)$ is a 1-antiperiodic function, that is $k(t+1) \equiv -k(t)$. Let $\varphi(t) := \nu$ be sufficiently small. It can be easily checked that the conditions (14) and (15) are satisfied. According to (17) and (18), define

$$\begin{aligned} P(t) :&= \frac{\nu \cos \sigma\nu - k(t-1) \sin \sigma\nu}{\nu \cos(1-\sigma)\nu + k(t) \sin(1-\sigma)\nu} \exp\left(-\int_{t-1}^{t} k(s)ds\right) \\ &= \frac{\nu \cos \sigma\nu + c \sin \sigma\nu \sin \pi t}{\nu \cos(1-\sigma)\nu + c \sin(1-\sigma)\nu \sin \pi t} \exp\left(\frac{2c}{\pi} \cos \pi t\right), \end{aligned} \tag{36}$$

$$\begin{aligned} &\tilde{Q}(t-1-\sigma) := \\ &= \frac{\left[\nu^2 - k^2(t)\right] \sin \nu - 2\nu k(t) \cos \nu}{\nu \cos(1-\sigma)\nu + c \sin(1-\sigma)\nu \sin \pi t} \exp\left(-\int_{t-1}^{t-1-\sigma} k(s)ds\right) \\ &= \frac{\nu \cos \nu - c \sin \pi t \left[2 \cos \nu + c \sin \pi t \frac{\sin \nu}{\nu}\right]}{\cos(1-\sigma)\nu + c \sin \pi t \frac{\sin(1-\sigma)\nu}{\nu}} \exp\left(\frac{2c}{\pi} \sin \frac{\pi\sigma}{2} \sin(\pi t + \frac{\pi\sigma}{2})\right). \end{aligned} \tag{37}$$

The uniform limit of $\tilde{Q}(t-1-\sigma)$ as $\nu \to 0$ equals

$$\frac{-c \sin \pi t\,(2 + c \sin \pi t)}{1 + c(1-\sigma) \sin \pi t},$$

so we see that if ν is sufficiently small, the condition $\tilde{Q}(t) \geqslant 0$ holds on $[a-1+\sigma, a]$ for $a = 2n+1, n \in N$ (not for any a !). As for the condition $Q(t) \geqslant 0$ on $[a + \frac{\pi}{\nu} - 1 + \sigma, a + \frac{\pi}{\nu}]$ (under supposition $Q \geqslant \tilde{Q}$), it can be ensured by choosing an appropriate ν, namely, $\nu = \frac{\pi}{2k+1}$, $k \in N$. Therefore all conditions of Theorem 2 are valid. So we see that the following statement is true

COROLLARY 4. *Let* $0 < \sigma < 1$, $a = 2n + 1$, $\nu = \frac{\pi}{2k+1}$, *where* n, $k \in N$ *with* k *sufficiently large. Let, moreover,* P *and* $\tilde{Q}$ *be defined by* (36) *and* (37), *respectively. Then every solution of Eq.* (1) *with* $Q(t) \geqslant \tilde{Q}(t)$ *on* $[a, a + \frac{\pi}{\nu}]$ *has at least one zero on* $(a, a + \frac{\pi}{\nu} + 1)$.

REMARK 4. Note that $Q(t) \equiv \tilde{Q}(t)$ does not preserve sign on the whole $(a, a + \frac{\pi}{\nu})$. As far as we know, such a result with an oscillating coefficient is the first one in the oscillation theory for neutral differential equations.

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