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ON THE STABILITY OF THE POMPEIU FUNCTIONAL EQUATION

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Abstract. In this note, we examine the stability of the Pompeiu functional equation

$$f(x+y+xy) = f(x) + f(y) + f(x)f(y).$$

Given an operator T and a solution class $\{u\}$ with the property that $T(u) = 0$, when does $||T(v)|| \leqslant \varepsilon$ for an $\varepsilon > 0$ imply that $||u - v|| \leqslant \delta(\varepsilon)$ for some u and for some $\delta > 0$? This problem is called the stability of the functional transformation. A great deal of work has been done in connection with the ordinary and partial differential equations. If f is a function from a normed vector space into a Banach space, and $||f(x+y) - f(x) - f(y)|| \leqslant \varepsilon$, Hyers (1941) proved that there exists an additive map A such that $||f(x) - A(x)|| \leqslant \varepsilon$. If $f(x)$ is a real continuous function of x over , and $|f(x+y) - f(x) - f(y)| \leqslant \varepsilon$, it was shown by Hyers and Ulam (1952) that there exists a constant k such that $|f(x) - kx| \leqslant 2\varepsilon$. Taking these results into account, we say that the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is stable in the sense of Hyers and Ulam. A functional transformation is said to be superstable if given an operator T and a solution class $\{u\}$ with the property $T(u) = 0$, the inequality $||T(v)|| \leqslant \varepsilon$ for an $\varepsilon > 0$ imply $v \in \{u\}$. That is, T is superstable when a class of solution to the defining inequality coincides with the class of the original equation. The interested reader should refer to the book by Hyers, Isac and Rassias [4] for an indepth account on the subject of stability of functional equations.

Let denote the set of real numbers and $\mathbb{C}$ the set of complex numbers. Let $_\star = \setminus\{-1\}$, that is the set of real numbers except negative one. Further, the set of natural numbers will be denoted by $\mathbb{N}$.

If $\mathbf{G} =_\star$, then $(\mathbf{G}, \circ)$ is an abelian group where the group operation is defined as

$$x \circ y = x + y + xy.$$

A characterization of the homomorphisms of the group $(\mathbf{G}, \circ)$ can be obtained by solving the functional equation

$$f(x+y+xy) = f(x) + f(y) + f(x)f(y). \tag{PE}$$

This functional equation is known as the *Pompeiu functional equation* [5, 6, 7].

In this note, we study the stability of the Pompeiu functional equation.

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THEOREM 1. *If a function* $f \,:\, \to$ *satisfies the inequality*

$$|f(x+y+xy) - f(x) - f(y) - f(x)f(y)| \leqslant \varepsilon \tag{1}$$

for some $\varepsilon \geqslant 0$ *and for all* $x, y \in$ *, then* f *has the form*

$$f(x) = m(x+1) - 1$$

for all $x \in$ *, where* $m : \to$ *is either a multiplicative function or a bounded function.*

Proof. If we put $f(x) = g(x) - 1$ in (1), we then have

$$|g(x+y+xy) - g(x)g(y)| \leqslant \varepsilon$$

for any $x, y \in$. If we replace x by $x-1$ and y by $y-1$ in the last inequality, then we obtain

$$|g(xy-1) - g(x-1)g(y-1)| \leqslant \varepsilon \tag{2}$$

for $x, y \in$. Let us define a function $m : \to$ by

$$m(x) = g(x-1).$$

It then follows from (2) that

$$|m(xy) - m(x)m(y)| \leqslant \varepsilon \tag{3}$$

for any $x, y \in$.

Assume that m is unbounded. Then, we can choose a strictly monotone real sequence (y_n) with

$$\lim_{n\to\infty} |m(y_n)| = \infty.$$

Next, we let $y = y_n$ ($n \in \mathbb{N}$) in (3) and then divide the resulting inequality by $|m(y_n)|$ to get

$$\left|\frac{m(xy_n)}{m(y_n)} - m(x)\right| \leqslant \frac{\varepsilon}{|m(y_n)|}.$$

Now taking the limit as $n \to \infty$, we get

$$m(x) = \lim_{n\to\infty} \frac{m(xy_n)}{m(y_n)}. \tag{4}$$

Again, we replace y by yy_n in (3) and divide the resulting relation by by $|m(y_n)|$ to obtain

$$\left|\frac{m(xyy_n)}{m(y_n)} - m(x)\frac{m(yy_n)}{m(y_n)}\right| \leqslant \frac{\varepsilon}{|m(y_n)|}.$$

Thus, if we let $n \to \infty$ in the above inequality, the relation (4) implies that m is multiplicative. This completes the proof of the theorem.

The above theorem says that the Pompeiu functional equation is superstable. This superstability of the Pompeiu functional equation is caused by the fact that the operations on the left side of (PE) are addition and multiplication whereas the distance between the two sides of (PE) is measured by their difference. In the next theorem, we measure the distance between the two sides of (PE) by using a quotient and show that (PE) is not superstable.

THEOREM 2. *Given a cancellative abelian semigroup* $(G,\cdot)$, *assume that a function* $f : G \to \mathbb{C}\setminus\{-1\}$ *satisfies the inequality*

$$\left|\frac{f(x+y+xy)+1}{f(x)+f(y)+f(x)f(y)+1}-1\right| \leqslant \varepsilon \tag{5}$$

for some $\varepsilon \in [0,1)$ *and for all* $x,y \in G$. *Then, there exists a unique multiplicative function* $m : G \to \mathbb{C}\setminus\{0\}$ *such that*

$$\max\left\{\left|\frac{m(x+1)}{f(x)+1}-1\right|, \left|\frac{f(x)+1}{m(x+1)}-1\right|\right\} \leqslant \sqrt{1+\frac{1}{(1-\varepsilon)^2}-2\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}} \tag{6}$$

for any $x \in G$.

Proof. Replace x by $x-1$ and y by $y-1$ in (5) and define a function $g : G \to \mathbb{C}\setminus\{0\}$ by $g(x) := f(x-1)+1$ to get

$$\left|\frac{g(xy)}{g(x)g(y)}-1\right| \leqslant \varepsilon \tag{7}$$

for all $x,y \in G$. The inequality (7) is handled in a manner similar to the proof of Theorem 3.4 in [1].

If we express each complex number λ by

$$\lambda = |\lambda|\exp(i\arg\lambda)$$

with $-\pi < \arg\lambda \leqslant \pi$, then (7) implies

$$\left|\frac{|g(xy)|}{|g(x)|\,|g(y)|}\exp[i(\arg g(xy)-\arg g(x)-\arg g(y))]-1\right| \leqslant \varepsilon$$

for all $x,y \in G$. From this inequality we can obtain the following two relations:

$$1-\varepsilon \leqslant \frac{|g(xy)|}{|g(x)|\,|g(y)|} \leqslant 1+\varepsilon \tag{8}$$

and

$$\arg g(xy)-\arg g(x)-\arg g(y) \in 2\pi\mathbb{Z}+[-\sin^{-1}\varepsilon,\ \sin^{-1}\varepsilon]$$

for $x,y \in G$. Since we assumed $\varepsilon < 1$, it is clear that $\sin^{-1}\varepsilon < \pi/2$. According to Corollary 2.4 in [1], there exists a function $a : G \to$ such that

$$a(xy)-a(x)-a(y) \in 2\pi\mathbb{Z} \tag{9}$$

for $x,y \in G$ and

$$|a(x)-\arg g(x)| \leqslant \sin^{-1}\varepsilon \tag{10}$$

for $x \in G$.

It follows from (8) that

$$|\ln|g(xy)|-\ln|g(x)|-\ln|g(y)|| \leqslant -\ln(1-\varepsilon)$$

for any $x, y \in G$. Hence, a result of J. Rätz (see [8]) implies that there exists a homomorphism $h : (G, \cdot) \to (, +)$ such that

$$|h(x) - \ln|g(x)|\,| \leqslant -\ln(1-\varepsilon) \tag{11}$$

for $x \in G$.

Let us define a function $m : G \to \mathbb{C}\setminus\{0\}$ by

$$m(x) := \exp(h(x) + ia(x)).$$

Since h is a homomorphism, it follows from (9) that

$$\begin{aligned} m(xy) &= \exp(h(xy) + ia(xy)) \\ &= \exp(h(x) + h(y) + ia(x) + ia(y) + i2\pi k) \\ &= m(x)m(y) \end{aligned}$$

for any $x, y \in G$, where $k \in \mathbb{Z}$ is an appropriate constant.

Moreover, we have

$$\left|\frac{g(x)}{m(x)} - 1\right| = |\exp[\ln|g(x)| - h(x)]\exp[i(\arg g(x) - a(x))] - 1|$$

for $x \in G$. By (10) and (11) we see that the complex number $g(x)/m(x)$ belongs to the set

$$\Lambda = \{\lambda \in \mathbb{C} \,:\, 1-\varepsilon \leqslant |\lambda| \leqslant (1-\varepsilon)^{-1};\; -\sin^{-1}\varepsilon \leqslant \arg\lambda \leqslant \sin^{-1}\varepsilon\}.$$

It is not difficult to see

$$\begin{aligned} \sup\{|\lambda - 1| \,:\, \lambda \in \Lambda\} &= \left|(1-\varepsilon)^{-1}\exp(i\sin^{-1}\varepsilon) - 1\right| \\ &= \sqrt{1 + \frac{1}{(1-\varepsilon)^2} - 2\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}} \\ &=: \eta. \end{aligned}$$

Analogously, we may obtain the same upper bound η for $|m(x)/g(x) - 1|$, i.e.,

$$\left|\frac{g(x)}{m(x)} - 1\right| \leqslant \eta \quad \text{and} \quad \left|\frac{m(x)}{g(x)} - 1\right| \leqslant \eta \tag{12}$$

for $x \in G$. These complete the proof of (6).

Now, let $m_1 : G \to \mathbb{C}\setminus\{0\}$ be another multiplicative function satisfying the inequality (6) in place of m. Since m and m_1 are multiplicative, we have

$$m(x) = m(x^n)^{1/n} \quad \text{and} \quad m_1(x) = m_1(x^n)^{1/n}$$

for any $x \in G$ and $n \in \mathbb{N}$. Hence, these and (12) imply

$$\begin{aligned} \frac{m(x)}{m_1(x)} &= \left(\frac{m(x^n)}{g(x^n)}\right)^{1/n}\left(\frac{g(x^n)}{m_1(x^n)}\right)^{1/n} \\ &\leqslant (1+\eta)^{1/n}(1+\eta)^{1/n} \\ &\to 1 \quad \text{as } n \to \infty, \end{aligned}$$

which implies the uniqueness of m and the proof of the theorem is now complete.

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