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ON THE SERIES OF HAAR–FOURIER COEFFICIENTS

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Abstract. Sufficient conditions are given for the convergence of the series

$$\sum_{n=1}^{\infty} \lambda(n)\varphi(|c_n|),$$

where c_n are the Haar–Fourier coefficients of an integrable function, $\varphi(x)$ $(x \geqslant 0,\ \varphi(0) = 0)$ is an increasing and concave function, and $\lambda(x)$ $(x \geqslant 1)$ denotes a function satisfying certain easily achievable conditions.

1. Introduction

In a recent paper [4], among others, generalizing a result of N. Ogata [5] we gave sufficient condition for the convergence of the series

$$\sum_{n=1}^{\infty} n^{\delta}(\varphi(|a_n|) + \varphi(|b_n|)),$$

where a_n and b_n are Fourier coefficients, $\delta \geqslant 0$ and $\varphi(u)$ $(u \geqslant 0,\ \varphi(0) = 0)$ is an increasing and concave function.

In the now note we present a similar result for the convergence of the series

$$\sum_{n=1}^{\infty} \lambda(n)\varphi(|c_n|),$$

where c_n are the Haar–Fourier coefficients of an integrable function, and $\lambda(x)$ $(x \geqslant 1)$ denotes a function satisfying certain natural conditions. Plainly $\lambda(x) = x^{\delta}$ $(\delta \geqslant 0)$ will satisfy these condition.

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The orthonormal Haar system can be given in the interval $(0,1)$ as follows: $\chi_0^{(0)} := 1$ and for $n = 0,1,\dots$ and $k = 1,2,\dots,2^n$

$$\chi_n^{(k)}(x) := \begin{cases} 2^{n/2}, & \text{if} \quad x \in \left(\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right), \\ -2^{n/2}, & \text{if} \quad x \in \left(\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right), \\ 0, & \text{otherwise.} \end{cases}$$

We shall set

$$\chi_1(x) := \chi_0^{(0)}(x) \quad \text{and} \quad \chi_m(x) := \chi_n^{(k)}(x)$$

if $m = 2^n + k$ $(n = 0,1,\dots,;\ k = 1,2,\dots,2^n)$.

The Haar–Fourier coefficients of an integrable function f are

$$c_m := c_m(f) := \int_0^1 f(x)\chi_m(x)dx.$$

Near forty years ago several authors investigated the convergence of the series

$$\sum_{m=1}^{\infty} m^{\delta}|c_m|^{\beta} \quad (\beta > 0, \delta \geqslant 0),$$

e.g. Z. Ciesielski and I. Musielak [1], P. L. Ulyanov [6], B. I. Golubov [2] and myself [3].

In [3] we proved, among others, the following

THEOREM A. *Let* $\lambda(x)$ $(x \geqslant 1)$ *be a positive and monotone function with the property* $K\lambda(2^n) \geqslant \lambda(2^{n-1}) \geqslant K^{-1}\lambda(2^n)$ $(K \geqslant 1,\ n = 1,2,\dots)$. *If* $0 < \beta \leqslant 1$ *and*

$$\int_0^1 \frac{\lambda(1/x)}{x^{2-\beta/2}} \left(\int_0^{1-x} |f(x+t) - f(x)|dt \right)^{\beta} dx < \infty,$$

then

$$\sum_{m=1}^{\infty} \lambda(m)|c_m|^{\beta} < \infty. \tag{1.1}$$

If

$$\int_0^1 \frac{\lambda(1/x)x^{\alpha\beta}}{x^{2-\beta/2}} dx = \infty$$

then there exists a function $f(x) \in \operatorname{Lip}\alpha$ $(0 < \alpha \leqslant 1)$ *such that for its Haar–Fourier coefficients the series (1.1) diverges.*

Before formulating our result we present some definitions.

We shall say that a positive function $\gamma(x)$ $(x \geqslant 1)$ is *quasi* β*-power-monotone increasing (decreasing)* if there exists a constant $K := K(\beta,\gamma) \geqslant 1$ such that

$$Ku^{\beta}\gamma(u) \geqslant v^{\beta}\gamma(v) \quad (u^{\beta}\gamma(u) \leqslant Kv^{\beta}\gamma(v))$$

holds for any $u \geqslant v$ $(\geqslant 1)$.

A positive function $\gamma(x)$ $(x \geqslant 1)$ will be called *limitedly varying* if there exist two positive constants $K_1 := K_1(\gamma) \leqslant K_2 := K_2(\gamma)$ such that

$$K_1\gamma(x) \leqslant \gamma(2x) \leqslant K_2\gamma(x)$$

holds for any $x \geqslant 1$.

Here and further in the sequel, K and K_i will denote positive constants depending on the parameters concerned in the particular problem in which it appears. If we wish to express the dependence explicitly, we write K in the form $K(\alpha, \beta, \ldots)$. The constants are not necessarily the same at different occurrences.

In the present work we generalize the Theorem A such a way that we replace the function x^β, β is appearing in Theorem A as an exponent, by an increasing concave function $\varphi(x)$. Our new theorem will be deduced from a more general result to be proven here as Lemma 3.

2. Theorem

We prove the following result.

THEOREM. *Let $\varphi(u)$ $(u \geqslant 0,\ \varphi(0) = 0)$ be an increasing and concave function. Furthermore let $\lambda(x)$ $(x \geqslant 1)$ be a positive limitedly varying function and either quasi δ–power–monotone increasing with some $\delta \geqslant 0$, or quasi ρ–power–monotone decreasing with some $\rho \leqslant 0$.*

Then

$$\sum_{m=3}^{\infty} \lambda(m)\varphi(|c_m|)$$

$$\leqslant K(\varphi,\lambda) \int_0^1 \lambda\Big(\frac{1}{x}\Big)x^{-2}\varphi\Big(x^{1/2}\int_0^{1-x} |f(x+t) - f(t)|dt\Big)dx.$$

Furthermore if

$$\int_0^1 \lambda\Big(\frac{1}{x}\Big)x^{-2}\varphi(x^{\frac{1}{2}+\alpha})dx = \infty$$

then there exists a function $f_\alpha(x) \in \mathrm{Lip}\alpha$ $(0 < \alpha \leqslant 1)$, such that with its Haar–Fourier coefficients the series

$$\sum_{m=1}^{\infty} \lambda(m)\varphi(|c_m|)$$

diverges.

It is obvious that our new Theorem in the special case $\varphi(x) = x^\beta$ $(0 < \beta \leqslant 1)$ is also a mild generalization of Theorem A, namely the monotonicity–condition is replaced by a quasi power–monotonicity–condition.

3. Lemmas

The first two lemmas are known, and the third one is a general result implying our Theorem effortlessly.

LEMMA 1. *The function*

$$f(x) := f_\alpha(x) := \sum_{n=1}^{\infty} 2^{-\alpha n} \cos 2^{n+1}\pi x, \quad 0 < \alpha < 1,$$

belongs to the class $\mathrm{Lip}\,\alpha$.

Let N_n *denote the number of the indices* m *being between* 2^n *and* 2^{n+1}, *and simultaneously the Haar–Fourier coefficients* $c_m(f)$ *satisfy the following inequality*

$$|c_m(f)| \geqslant 2^{\alpha-3} 2^{-n(\alpha+1/2)}.$$

Then

$$N_n \geqslant K(\alpha) 2^n,$$

where $K(\alpha)$ *is a positive constant.*

The proof of the first statement of Lemma 1 can be found e.g. in [7, p. 47] and that of the second one in [2, p. 1284].

LEMMA 2. (Jensen's inequality). *Let* $\varphi(u)$ $(u \geqslant 0,\ \varphi(0) = 0)$ *be an increasing concave function. Then, for any finite sequence of nonnegative numbers* $x_1, x_2, \dots, x_m$ *and any finite sequence of positive numbers* $p_1, p_2, \dots, p_m$ *the following inequality*

$$\frac{\sum_{n=1}^{m} p_n \varphi(x_n)}{\sum_{n=1}^{m} p_n} \leqslant \varphi\Big(\frac{\sum_{n=1}^{m} p_n x_n}{\sum_{n=1}^{m} p_n} \Big)$$

holds.

LEMMA 3. *Let* $\gamma > 0$ *and* $\varphi(u)$ $(u \geqslant 0,\ \varphi(0) = 0)$ *be an increasing concave function. Furthermore let* $\lambda(x)$ $(x \geqslant 1)$ *be such a positive limitedly varying function that it is either quasi* δ*–power–monotone increasing with some* $\delta \geqslant 0$, *or quasi* ρ*–power–monotone decreasing with some* $\rho \leqslant 0$. *Then*

$$\sigma_{n,\ell} := \sum_{\nu=n}^{n+\ell} \Big(\sum_{m=2^\nu+1}^{2^{\nu+1}} \lambda(m)\varphi(|c_m|) \Big)^\gamma \tag{3.1}$$

$$\leqslant K(\varphi,\lambda,\gamma) \int_{2^{-n-\ell-3}}^{2^{-n-1}} \lambda^\gamma\Big(\frac{1}{x}\Big) x^{-\gamma-1} \varphi^\gamma\Big(x^{1/2} \int_0^{1-x} |f(x+t) - f(t)|\, dt\Big) dx$$

holds for any natural numbers n *and* ℓ.

Furthermore there exists a function $f(x) \in \mathrm{Lip}\alpha \ (0 < \alpha \leqslant 1)$, *whose Haar-Fourier coefficients satisfy the inequality*

$$\sigma_{n,\ell} \geqslant K(\varphi,\lambda,\gamma,\alpha) \int_{2^{-n-\ell}}^{2^{-n}} \lambda(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(x^{\alpha+\frac{1}{2}})dx. \tag{3.2}$$

Proof. Using the properties of the functions $\lambda(x)$ and the Jensen's inequality we obtain that

$$\sum_{m=2^n+1}^{2^{n+1}} \lambda(m)\varphi(|c_m|) \leqslant K\lambda(2^{n+1}) \sum_{m=2^n+1}^{2^{n+1}} \varphi(|c_m|) \tag{3.3}$$

$$\leqslant K\lambda(2^{n+1})2^n\varphi(2^{-n} \sum_{m=2^n+1}^{2^{n+1}} |c_m|).$$

By the definition of c_m

$$\sum_{m=2^n+1}^{2^{n+1}} |c_m| = 2^{n/2}\sum_{k=1}^{2^n}\Big|\int_{(2k-2)2^{-n-1}}^{(2k-1)2^{-n-1}} (f(t)-f(t+2^{-n-1}))dt\Big|$$

holds, and thus

$$\varphi\Big(2^{-n} \sum_{m=2^n+1}^{2^{n+1}} |c_m|\Big) \leqslant \varphi\Big(2^{-n/2}\int_0^{1-2^{-n-1}} |f(t)-f(t+2^{-n-1})|dt\Big).$$

Hence, by (3.3), we get that

$$\sigma_{n,\ell} \leqslant K_1\sum_{\nu=n}^{n+\ell} \lambda^{\gamma}(2^{\nu+1})2^{\nu\gamma}\varphi^{\gamma}(2^{-\nu/2}\int_0^{1-2^{-\nu-1}} |f(t)-f(t+2^{-\nu-1})|dt)$$

$$\leqslant K_2\sum_{\nu=n}^{n+\ell}\int_{2^{-\nu-3}}^{2^{-\nu-2}} \lambda^{\gamma}(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(2^{-\nu/2}\Big\{\int_0^{1-2^{-\nu-1}} |f(t)-f(t+x)|dt$$

$$+\int_0^{1-2^{-\nu-1}} |f(x+t)-f(t+2^{-\nu-1})|dt\Big\})dx$$

$$\leqslant K_3\sum_{\nu=n}^{n+\ell}\int_{2^{-\nu-3}}^{2^{-\nu-2}} \lambda^{\gamma}(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(x^{1/2}\int_0^{1-x} |f(t)-f(t+x)|dt)dx$$

$$+K_3\sum_{\nu=n}^{n+\ell}\int_{2^{-\nu-3}}^{2^{-\nu-2}} \lambda^{\gamma}(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(2^{-\nu/2}\int_0^{1-2^{-\nu-1}} |f(x+t)-f(t+2^{-\nu-1})|dt)dx.$$

Now an integration by substitution $t = u - x$ gives that

$$\int_0^{1-2^{-\nu-1}} |f(t+x)-f(t+2^{-\nu-1})|dt$$

$$= \int_{x}^{1-2^{-\nu-1}+x} |f(u) - f(u + 2^{-\nu-1} - x)|du,$$

and thus by $x = 2^{-\nu-1} - y$ we obtain that

$$\int_{2^{-\nu-3}}^{2^{-\nu-2}} \lambda^{\gamma}(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(2^{-\nu/2}\int_{0}^{1-2^{-\nu-1}} |f(x+t) - f(t + 2^{-\nu-1})|dt)dx$$

$$\leqslant K_4 \int_{2^{-\nu-2}}^{2^{-\nu-1}} \lambda^{\gamma}(\frac{1}{y})y^{-\gamma-1}\varphi^{\gamma}(2^{-\nu/2}\int_{2^{-\nu-1}-y}^{1-y} |f(u) - f(u+y)|du)dy =: I_\nu.$$

Finally with $y = x$ and $u = t$ we realize that

$$I_\nu \leqslant K_5 \int_{2^{-\nu-2}}^{2^{-\nu-1}} \lambda^{\gamma}(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(x^{1/2}\int_{0}^{1-x} |f(t) - f(x+t)|dt)dx.$$

Collecting our estimations we conclude that

$$\sigma_{n,\ell} \leqslant K_6 \sum_{\nu=n}^{n+\ell} \int_{2^{-\nu-3}}^{2^{-\nu-1}} \lambda^{\gamma}(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(x^{1/2}\int_{0}^{1-x} |f(t) - f(x+t)|dt)dx$$

holds, and hence (3.1) plainly follows.

In order to prove the statement (3.2) we distinguish two cases. If $0 < \alpha < 1$ then we consider the function $f(x)$ given in Lemma 1. The Haar–Fourier coefficients of this function satisfy the following inequality

$$\sum_{m=2^\nu+1}^{2^{\nu+1}} \lambda(m)\varphi(|c_m|) \geqslant K\lambda(2^\nu)2^\nu\varphi(2^{-\nu(\alpha+\frac{1}{2})}), \tag{3.4}$$

whence

$$\begin{aligned} \sigma_{n,\ell} &\geqslant K_1 \sum_{\nu=n}^{n+\ell} \lambda^{\gamma}(2^\nu)2^{\nu\gamma}\varphi^{\gamma}(2^{-\nu(\alpha+\frac{1}{2})}) \\ &\geqslant K_2 \int_{2^{-n-\ell}}^{2^{-n}} \lambda^{\gamma}(\frac{1}{x})x^{-\gamma-1}\varphi^{\gamma}(x^{\alpha+\frac{1}{2}})dx \end{aligned} \tag{3.5}$$

obviously follows considering the properties of the functions $\lambda(x)$ and $\varphi(x)$.

This yields (3.2) for $0 < \alpha < 1$.

If $\alpha = 1$ let $f(x) := 1 - 2x$. It is clear that $f \in \mathrm{Lip}1$. On the other hand an easy calculation shows that

$$c_m(f) = 2^{-1}2^{-3n/2} \quad \text{if} \quad 2^n < m \leqslant 2^{n+1}.$$

Now reproduce the estimations given in (3.4) and (3.5) with $\alpha = 1$ in place of α, $0 < \alpha < 1$, we can see that the inequality (3.2) with $\alpha = 1$ also holds.

Herewith the proof of Lemma 3 is complete.

4. Proof of Theorem

Applying Lemma 3 with $\gamma = 1$, $n = 1$ and letting $\ell \to \infty$ we clearly get the statements of our Theorem.

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